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Abstract. In this paper, we provide complete classification of classes of bipartite graphs defined by a single forbidden induced bipartite subgraph with respect to bounded/unbounded clique-width.

1 Introduction

Clique-width is a graph parameter which is of primary importance in algorithmic graph theory, as many graph problems that are NP-hard in general admit polynomial-time solutions on graphs of bounded clique-width [5]. This parameter is known to be unbounded in general and in many restricted graph families, such as split [14], planar or bipartite graphs [9]. In this paper, we focus on bipartite graphs. As the clique-width of general bipartite graphs is unbounded, we study this parameter restricted to bipartite graphs in special classes. Without loss of generality we assume that all our classes are closed under taking induced subgraphs, because the clique-width of an induced subgraph of a graph G cannot be larger than the clique-width of G [6]. Classes of graphs closed under taking induced subgraphs are called *hereditary*. It is known that a class of graphs X is hereditary if and only if X can be characterized in terms of forbidden induced subgraphs. The class of graphs containing no induced subgraphs in a set M will be denoted $Free(M)$. If M contains only one graph, we say that $Free(M)$ is a *monogenic* class. In this paper, we study subclasses of bipartite graphs defined by a single forbidden induced *bipartite* subgraph and provide complete classification of graph classes in this family according to bounded/unbounded clique-width.

The paper is organized as follows. In the rest of this section, we introduce general notations. In the next section, we give necessary definitions and auxiliary results. Section 3 proves the main result, i.e., classification of monogenic classes of bipartite graphs with respect to bounded/unbounded clique-width. This classification is based on some previously known results, as well as on three new results presented in the subsequent sections.

Most notations we use are customary: $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. The neighborhood of a vertex $v \in V$, denoted $N(v)$, is the subset of vertices adjacent to v. Given a subset $U \in V(G)$, we denote by $G[U]$ the subgraph of G induced by U and by $G - U$ the subgraph of G induced by $V(G) - U$. Also, $N_U(v) = N(v) \cap U$. For two disjoint subsets $U, W \subset V$, we say that U is *disconnected from* W if there are no edges between U and W, and that U, W form a join if if there are all possible edges between U and W. As usual, C_n is the chordless cycle, P_n is the chordless path, and K_n is the complete graph on n vertices. Also, $K_{n,m}$ is the complete bipartite graph with parts of size n and m, a *claw* is the graph $K_{1,3}$, $2K_2$ is the disjoint union of two copies of K_2. By $S_{i,j,k}$ we denote the graph represented in Figure 1 and by $\mathcal{S}$ the class of graphs every connected component of which has the form $S_{i,j,k}$.

An *independent set* is a subset of pairwise non-adjacent vertices.

2 Preliminaries

The clique-width of a graph was introduced by Courcelle, Engelfriet and Rozenberg in [4] as the minimum number of labels needed to construct a graph by means of the following four operations: create a new vertex v with label i (denoted $i(v)$), take the disjoint union of two labeled graphs G and H (denoted $G \oplus H$), connect vertices with specified labels i and j (denoted $\eta_{i,j}$), rename label i to label j (denoted $\rho_{i \to j}$). Every graph can be built by a

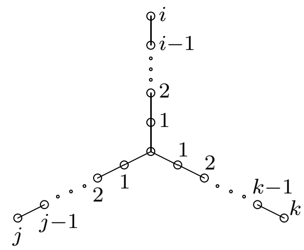

Figure 1: The graph $S_{i,j,k}$

sequence of these four operations, which can be described by an algebraic expression. For instance, the cycle C_5 on vertices a, b, c, d, e (listed in the cyclic order) can be defined by the following expression:

$$\eta_{4,1}(\eta_{4,3}(4(e) \oplus \rho_{4\to 3}(\rho_{3\to 2}(\eta_{4,3}(4(d) \oplus \eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))))). \quad (1)$$

An expression built from the above four operations is called a *k-expression* if it uses k different labels. The clique-width of a graph G, denoted $cwd(G)$, is the minimum k such that there exists a k-expression defining G.

Alternatively, any k-expression τ defining G can be represented by a rooted tree $tree(\tau)$, whose leaves correspond to the operations of vertex creation, while the internal nodes correspond to the other three operations. An example of the tree representing the expression (1) is depicted in Figure 2.

The above example shows that $cwd(C_5) \le 4$. Moreover, it is not hard to see that the clique-width of any cycle is at most 4. In a similar way, one can show that the clique-width of any path is at most 3. With some extra work, the same conclusion can be made for any tree, and even more generally, for any distance-hereditary graph [9]. As we mentioned in the introduction, this is important because many problems being NP-hard for general graphs admit polynomial time solutions when restricted to graphs of bounded clique-width. More precisely, for every fixed k, every decision or optimization graph problem expressible in monadic second-order logic with quantifiers over vertices can be solved in linear time, if a k-expression defining G is given as input [5]. Moreover, as shown recently by Oum and Seymour [15], a polynomial-time solution is guaranteed even if only a graph G of bounded clique-width is given as input: an expression of bounded width defining G can be computed in polynomial time.

Proving boundedness or unboundedness of the clique-width is generally a nontrivial task. To name a few results of this type, let us mention that the clique-width is unbounded in classes of split [14], unit interval, permutation [9] and even bipartite permutation graphs [3]. Among classes of bounded clique-width we distinguish *complement reducible graphs* (*cographs* for short), which is precisely the class $Free(P_4)$. This is a key result that motivated many more research directions in the theory of clique-width. For instance, the clique-width has been shown to be bounded in some classes of graphs containing "few" P_4's [14]. Another line of research deals with the bipartite analog of cographs introduced in [8] under the name

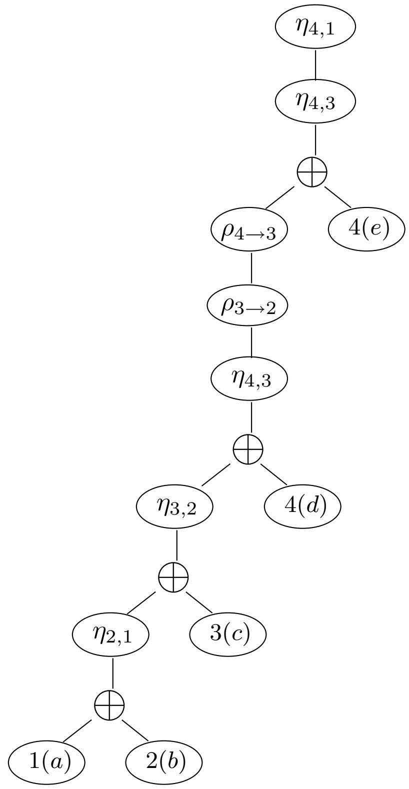

Figure 2: The tree representing the expression (1) defining a C_5

bi-complement reducible graphs. The clique-width turned out to be bounded for these graphs and some of their extensions [7, 10, 11]. The latter paper is of particular interest, since it deals with a class of bipartite graphs defined by a single forbidden induced *bipartite* subgraph. We call such classes *monogenic*. They are of special interest, because any hereditray class of graphs is the intersection of a (possibly infinite) series of monogenic classes.

As we shall see later, the result from [11] is unimprovable within the family of monogenic classes of bipartite graphs. More generally, we shall provide complete classification of this family with respect to bounded/unbounded clique-width. To this end, let us introduce some auxiliary results related to the clique-width of bipartite graphs.

A systematic investigation of the clique-width in classes of bipartite graphs defined by forbidding induced subgraph has been initiated in [13], where the following result has been proved:

Theorem 1. *[13] Let X be a class of bipartite graphs defined by a finite set F of forbidden induced bipartite subgraphs. If F contains neither a graph in $\mathcal{S}$ nor a graph the bipartite complement of which is in $\mathcal{S}$, then the clique-width of graphs in X is unbounded.*

In the above theorem, the bipartite complement of a bipartite graph $G = (W, B, E)$ is a bipartite graph $\widetilde{G}$ defined as follows: $\widetilde{G} := (W, B, (W \times B) - E)$.

Lemma 1. *[12] If G is a bipartite graph, then $cwd(\widetilde{G}) \leq 4cwd(G)$.*

Observe that throughout the paper we shall assume that a bipartite graph $G = (W, B, E)$ is given *together* with a bipartition of its vertex set into a subset W of white vertices and a subset B of black vertices. Several particular bipartite graphs will be of special interest in our paper. One of them is a skew star $S_{1,2,3}$ (see Figure 1).

Theorem 2. *[11] The clique-width of a $S_{1,2,3}$-free bipartite graph is at most 5.*

Four other graphs that are of particular importance in this paper are represented in Figure 3.

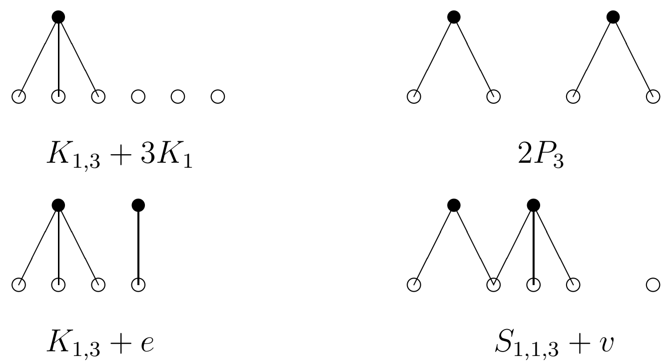

Figure 3: Four critical bipartite graphs

Theorem 3. *[13] The clique-width of a $K_{1,3} + 3K_1$-free bipartite graph is bounded.*

A connected bipartite graph is *well-orderable* if there is an ordering of the vertices $x_1, \ldots, x_n$ such that

1. $N_{\{x_2,\ldots,x_n\}}(x_1) = \{x_2\}$ in G or in $\widetilde{G}$.
2. For $1 < i < n$, if $N_{\{x_i,\ldots,x_n\}}(x_{i-1}) = \{x_i\}$ in G then $N_{\{x_{i+1},\ldots,x_n\}}(x_i) = \{x_{i+1}\}$ in $\widetilde{G}$
3. For $1 < i < n$, if $N_{\{x_i,\ldots,x_n\}}(x_{i-1}) = \{x_i\}$ in $\widetilde{G}$ then $N_{\{x_{i+1},\ldots,x_n\}}(x_i) = \{x_{i+1}\}$ in G.

Lemma 2. *[16] The clique-width of a well-orderable graph is at most 4.*

Clealy when we study the notion of clique-width we can be restricted to connected graphs. More generally, without loss of generality we may consider only graphs that are prime with respect to *modular decomposition*. Given a subset of vertices U and a vertex $v \notin U$, we say that v *distinguishes* U if v has both a neighbor and a non-neighbor in U. In a graph, a subset of vertices U indistinguishable by the vertices outside U is called a *module*. A graph every module of which is a singleton is said to be *prime*. Alternatively, we can say that a connected bipartite graph G is prime if any two distinct vertices of G have different neighborhoods. The importance of this notion is due to the following lemma, which allows us to consider only prime graphs in any hereditary class.

Lemma 3. *[6]* $cwd(G) = \max\{cwd(H) \mid H \text{ is a prime induced subgraph of } G\}$.

To introduce one more helpful result, let us partition the vertex set of a bipartite graph G arbitrarily into two parts and denote the graphs induced by these parts by G_1 and G_2. For $i = 1, 2$, we denote by B_i the set of black vertices and by W_i the set of white vertices of G_i. Finally, we denote by b_1, w_2 the number of black and white modules (subsets of vertices with the same neighborhood) in the subgraph $G[B_1 \cup W_2]$, and by b_2, w_1 the number of black and white modules in the subgraph $G[B_2 \cup W_1]$.

Lemma 4. $cwd(G) \le \max\{(b_1 + w_1)cwd(G_1), (b_2 + w_2)cwd(G_2), (b_1 + w_1 + b_2 + w_2)\}$.

Proof. While creating the graph G_1, we add a prefix to the label of each vertex indicating the module in the subgraph $G[B_1 \cup W_2]$ this vertex belongs to. Therefore, G_1 will be created by means of a set of $(b_1 + w_1)cwd(G_1)$ extended labels. After completion of creation of G_1, we can keep only prefix of each label, which leaves us with a set of $b_1 + w_1$ different labels. Independently, we build G_2 by means of a set of $(b_2 + w_2)cwd(G_2)$ labels and keep only $b_2 + w_2$ different labels after the graph is build. Without loss of generality we assume that the sets of labels present in G_1 and G_2 after their creation are disjoint. Then we complete the construction by joining the two graphs together with the $\oplus$ operation and connecting, where necessary, the vertices from different modules of G_1 and G_2 with the help of $b_1 + w_1 + b_2 + w_2$ labels. □

In the special case when $b_1 = w_1 = b_2 = w_2 = 1$, the above claim can be strengthened as follows.

Corollary 1. *If* $b_1 = w_1 = b_2 = w_2 = 1$, *then* $cwd(G) \le \max\{cwd(G_1), cwd(G_2), 4\}$.

Applying this corollary recursively to graphs G_1 and G_2, we reduce the problem of determining the clique-width to graphs which

(a) are connected,

(b) have connected bipartite complement,

(c) admit no partition into a complete bipartite graph and an independent set.

Bipartite graphs satisfying (a), (b) and (c) will be called *indecomposable with respect to canonical decomposition*. The idea of *canonical decomposition* has been introduced in [7], where the reader can be assisted with the proof of the above corollary.

We conclude the section with the following important result which is valid for any graphs, not necessarily bipartite.

Lemma 5. *[1] For a class of graphs X and an integer ρ, let $[X]_\rho$ denote the class of graphs G such that $G - U$ belongs to X for some subset $U \subseteq V(G)$ of cardinality at most ρ. If X is a class of graphs of bounded clique-width, then so is $[X]_\rho$.*

3 Main result

Theorem 4. *Let X be a class of H-free bipartite graphs. If H is an induced subgraph of one of the graphs $S_{1,2,3}$, $K_{1,3}+3K_1$, $K_{1,3}+e$, $S_{1,1,3}+v$, then the clique-width of H-free graphs is bounded. Otherwise, it is unbounded.*

Proof. To prove the theorem, we will show that either

(1) H is an induced subgraph of one of the graphs $S_{1,2,3}$, $K_{1,3}+3K_1$, $K_{1,3}+e$, $S_{1,1,3}+v$,

in which case the clique-width of H-free bipartite graphs is bounded by Theorems 2, 3, 6 and 7 or

(2) H or $\widetilde{H}$ contains either a cycle or a $K_{1,4}$ or a $2P_3$,

in which case the clique-width of H-free bipartite graphs is unbounded by Theorems 1 and 5.

Asssume that condition (2) fails. Then each part of H contains at most six vertices, since otherwise either H or $\widetilde{H}$ contains a $K_{1,4}$. This simple observation reduces the proof to finitely many cases that can be analysed by direct inspection.

Case 1: neither H nor its complement has a vertex of degree three, i.e., every connected component of H is a path. Denote by H_1 the largest of the components. Then H_1 contains at most six vertices, since otherwise a $2P_3$ arises. If H_1 has six vertices, then $H = H_1$ (else $\widetilde{H}$ contains a vertex of degree three), in which case H is an induced subgraph of $S_{1,2,3}$. Similarly, if H_1 has five vertices, then H is an induced subgraph of $S_{1,1,3}+v$, and if H_1 has at most 4 vertices, then H is an induced subgraph of $S_{1,2,3}$.

Case 2: without loss of generality, H has a black vertex of degree three, say a. Then, since (2) fails, H has no other vertices of degree 3, and there are at most 2 additional black vertices.

Case 2.1: H has no black vertices except a. Then H is an induced subgraph of $K_{1,3}+3K_1$.

Case 2.2: H has exactly one additional black vertex, say b. Then, taking into account that (2) fails, we conclude that a has at most two white non-neighbors. If the number of such non-neighbors is 2, then H coincides with $S_{1,1,3}+v$. If a has less than two white non-neighbors, then either H is an induced subgraph of $S_{1,1,3}+v$ (if b has a neighor among the neighbors of a) or H is an induced subgraph of $K_{1,3}+e$.

Case 2.3: H contains two black vertices except a, say b and c. Then each of b and c has exactly one neighbor among the neighbors of a, and a has at most one white non-neighbor, in which case H is an induced subgraph of $S_{1,2,3}$. □

4 $2P_3$-free bipartite graphs

In this section, we show that $2P_3$-free bipartite graphs can have arbitrarily large clique-width. This result also provides an affirmative answer to the so far open question of the unboundedness of the clique-width of P_7-free bipartite graphs. Note that, in contrast with this result, in the class of P_6-free bipartite (and more generally (P_6, K_3)-free) graphs the clique-width is bounded [2].

To prove the unboundedness of the clique-width in the class of $2P_3$-free bipartite graphs, let us define a sequence of bipartite graphs $\{G_n\}_{n\geq 1}$ as follows.

Consider an $(n+1) \times (n+1)$ square grid with two vertices $w_{i,j}$ and $b_{i,j}$ in row i and column j, where $i, j \in \{0, \ldots, n\}$. Delete vertices $w_{0,i}$ and $b_{i,0}$ for $i \in \{0, \ldots, n\}$. Let $B_1 = \{b_i := b_{0,i} \mid i \in \{1, \ldots, n\}\}$, $W_1 = \{w_i := w_{i,0} \mid i \in \{1, \ldots, n\}\}$, $B_2 = \{b_{i,j} \mid i, j \in \{1, \ldots, n\}\}$, $W_2 = \{w_{i,j} \mid i, j \in \{1, \ldots, n\}\}$. Then define:

$E_1 := \{w_{i,j} b_{i,j} \mid i, j \in \{1, \ldots, n\}\}$,
$E_2 := \{w_i b_j \mid i, j \in \{1, \ldots, n\}\}$,
$E_3 := \{w_i b_{j,k} \mid i, j, k \in \{1, \ldots, n\}, j \leq i\}$, and
$E_4 := \{b_i w_{j,k} \mid i, j, k \in \{1, \ldots, n\}, j \leq i\}$.

For each integer $n \geq 1$, the bipartite graph with vertex set $B_1 \cup B_2 \cup W_1 \cup W_2$ and edge set $E_1 \cup E_2 \cup E_3 \cup E_4$ will be denoted G_n. By definition, each of the vertex sets B_1, B_2, W_1, and W_2 is an independent set, $|B_1| = |W_1| = n$, $|B_2| = |W_2| = n^2$, $B_1 \cup W_1$ induces a complete bipartite graph, each of $B_1 \cup W_2$ and $W_1 \cup B_2$ induces a $2K_2$-free bipartite graph (i.e., $N_{W_2}(b_i) \subseteq N_{W_2}(b_j)$ and $N_{B_2}(w_i) \subseteq N_{B_2}(w_j)$ for all $1 \leq i < j \leq n$), and $B_2 \cup W_2$ induces $n^2 K_2$ (i.e., the disjoint union of n^2 copies of K_2).

Proposition 1. *G_n is $2P_3$-free.*

Proof. Without loss of generality, let $x \in B_1 \cup B_2$ be the middle vertex of an induced P_3, denoted A. By contradiction, let y be the middle vertex of another P_3, denoted B, such that the set $A \cup B$ induces a $2P_3$.

First, assume that $x \in B_2$. Then, by definition of G_n, at least one edge of A belongs to $G_n[B_2 \cup W_1]$. Then no vertex in B can belong to B_1, since B_1 is completely adjacent to W_1. Therefore, as before, at least one edge of B belongs to $G_n[B_2 \cup W_1]$. But then $G_n[B_2 \cup W_1]$ contains a $2K_2$, a contradiction.

So $x \in B_1$, and hence y does not belong to W_1 (else there is an edge between x and y). To rule out the cases $y \in B_2$ and $y \in W_2$, which are symmetric to the one considered above, we conclude that $y \in B_1$. Therefore, all edges of $A \cup B$ belong to $G_n[B_1 \cup W_2]$, which is not possible since this is a $2K_2$-free graph. ☐

Theorem 5. *$cwd(G_n) \geq n/3$.*

Proof. Let $cwd(G_n) = t$ and let τ be a t-expression defining G_n. The subtree of *tree*(τ) rooted at a node x will be denoted *tree*(x, τ). This subtree corresponds to a subgraph of G_n which will be denoted $G_n(x)$. We shall say that the subgraph $G_n(x)$ contains a full row if there is an i such that $G_n(x)$ contains all white vertices of the row i (i.e., $w_{i,0}, w_{i,1}, \ldots, w_{i,n}$), all black vertices of the row i (i.e., $b_{i,1}, \ldots, b_{i,n}$) and all the edges of the form $w_{i,j} b_{i,j}$ for $j = 1, \ldots, n$. We define the notion of $G_n(x)$ containing a full column in a similar way.

Let x be a lowest $\oplus$ node in *tree*(τ) such that $G_n(x)$ contains a full row or a full column. Denote the children of x in *tree*(τ) by y and z. Let us color all vertices in $G_n(y)$ blue and all vertices in $G_n(z)$ red, and the remaining vertices of G_n yellow. We will use the term non-yellow to mean a vertex that is either blue or red. The color of a vertex v will be denoted $color(v)$.

Also, for $1 \leq i, j \leq n$, we denote $c_{i,j}$ the pair of vertices $w_{i,j}$,$b_{i,j}$. We color $c_{i,j}$ red (blue) if and only if $b_{i,j}$ and $w_{i,j}$ are red (blue) and there is an edge connecting them in $G_n(x)$, otherwise we color $c_{i,j}$ yellow.

We note that edges of G_n between different colored vertices are not present in $G_n(x)$. By the choice of x, $G_n(x)$ contains a non-yellow row or column, but none of its rows and columns is completely red or blue. Without loss of generality, assume that $G_n(x)$ contains a non-yellow column. We denote this column j and define three disjoint sets of rows as follows:

- $S_1 := \{i \mid w_i := w_{i,0}$ and $b_{i,j}$ have different colors$\}$
- If $i \notin S_1$, there exists $j' \neq j$ such that $c_{i,j}$ and $c_{i,j'}$ have different colors (otherwise i is monochromatic). Then
  - $S_2 := \{i \mid w_i := w_{i,0}$ and $b_{i,j'}$ have different colors$\}$
  - $S_3 := \{i \mid w_i := w_{i,0}$ and $b_{i,j'}$ have the same color$\}$

Claim. For $1 \leq i \leq 3$, $G_n(x)$ contains $|S_i|$ vertices with pairwise different labels.

Case 1: Let $i, i' \in S_1$, $i < i'$. Since $color(w_i) \neq color(b_{i,j})$ and $color(w_{i'}) \neq color(b_{i',j})$, neither of the edges $w_i b_{i,j}$ and $w_{i'} b_{i',j}$ is present in $G_n(x)$. On the other hand, in G_n vertex $b_{i',j}$ is connected to $w_{i'}$ but not to w_i, therefore w_i and $w_{i'}$ must have different labels in $G_n(x)$.

Case 2 is similar to Case 1.

Case 3: Consider a row $i \in S_3$ and a column j' such that $color(w_i) = color(b_{i,j}) = color(b_{i,j'})$ and $color(c_{i,j}) \neq color(c_{i,j'})$. Then the vertex $b_{i,j'}$ is not adjacent to the vertex $w_{i,j'}$ in $G_n(x)$. Indeed, if they are adjacent, then they have the same non-yellow color, which is the color of the pair $c_{i,j'}$. But then $color(c_{i,j}) = color(c_{i,j'})$, a contradiction. If $|S_3| > 1$, consider one more row $i' \in S_3$ and a column j'' such that $color(w_{i'}) = color(b_{i',j}) = color(b_{i',j''})$ and $color(c_{i',j}) \neq color(c_{i',j''})$. In $G_n(x)$, $w_{i,j'}$ is adjacent neither to $b_{i,j'}$ nor to $b_{i',j''}$, while in G_n, $w_{i,j'}$ is adjacent to $b_{i,j'}$ but not to $b_{i',j''}$. Therefore, in $G_n(x)$, $b_{i,j'}$ and $b_{i',j''}$ must have different labels.

Then by using the pigeonhole principle with the above claim, we conclude that $cwd(G_n) = t \geq n/3$. □

5 $(K_{1,3} + e)$-free bipartite graphs

In the present section we prove that the clique-width of $(K_{1,3} + e)$-free bipartite graphs is bounded. A partial result on this topic can be found in [16], which proves that the clique-width is bounded for $(K_{1,3} + e, 2P_3)$-free bipartite graphs. We now extend this result to the entire class of $(K_{1,3} + e)$-free bipartite graphs.

Theorem 6. *The cliquewidth of* $(K_{1,3}+e)$*-free bipartite graphs is bounded by a constant.*

Proof. Let G be a $(K_{1,3}+e)$-free bipartite graph. Without loss of generality we assume that G is prime and indecomposable with respect to canonical decomposition. Let let H be a maximum well-orderable induced subgraph of G, with the corresponding ordering $x_1, \dots, x_p$ of its vertex set. Without loss of generality, assume x_1 is adjacent to x_2. Since the first 7 vertices of H induce an $S_{1,2,3}$, we may suppose that $p \geq 7$, otherwise G is $S_{1,2,3}$-free in which case the clique-width of G is at most 5 (Theorem 2).

Let H_1 and H_2 denote the sets of odd- and even-indexed vertices of H, respectively. Let $V_1 \cup V_2$ be the bipartition of $V(G)$ such that $H_i \subseteq V_i$ for $i = 1, 2$. Define T_1 to be the set of vertices in $V_1 \backslash H$ adjacent to every vertex of H_2, I_1 to be the set of vertices in $V_1 \backslash H$ disconnected from H_2, and P_1 to be the set of vertices in $V_1 \backslash H$ with a neighbor and a non-neighbor in H_2. T_2, P_2, and I_2 are defined analogously. Observe that in the bipartite complement to G the roles of T_j and I_j change.

Proposition 2. *If* $x \in P_1$ *(resp.* $x \in P_2$*) and* x *is adjacent to* $x_i \in H$ *and* $2 \leq i \leq p-3$ $(5 \leq i \leq p-1)$*, then* x *is adjacent to* x_{i+2} (x_{i-2})*.*

Proof. Assume $x \in P_1$ is adjacent to $x_i \in H_2$ and not to x_{i+2}, then the graph induced by $\{x, x_i, x_{i-1}, x_{i+3}, x_{i+1}, x_{i+2}\}$ forms a $K_{1,3}+e$, a contradiction. Similarly, if $x \in P_2$ is adjacent to $x_i \in H_1$ and not to x_{i-2}, then the graph induced by $\{x, x_i, x_{i+1}, x_{i-3}, x_{i-1}, x_{i-2}\}$ forms a $K_{1,3}+e$, a contradiction. □

Proposition 3. T_1 *and* T_2 *form a join.* I_1 *and* I_2 *are disconnected.*

Proof. If a vertex $x \in T_1$ is not adjacent to a vertex $y \in T_2$, then the graph induced by $\{x_1, x_3, x_7, y, x_6, x\}$ forms a $K_{1,3}+e$. The second part follows by complementary arguments. □

Keeping in mind these propositions, as well as the fact that G is indecomposable with respect to canonical decomposition, we now show that either G is a well-orderable graph or it contains finitely many vertices. The proof is devided into 3 general cases.

Case 1: $p \geq 10$. Consider a vertex $x \in P_1$. Then

- x is not adjacent to x_2. Indeed, if x is adjacent to x_2, then p is even, since otherwise x is adjacent to every vertex of H_2 by Proposition 2, which contradicts the definition of P_1. If p is even, then x is adjacent to $\{x_4, \dots, x_{p-2}\}$ but not to x_p. But now $\{x_1, x_2, \dots, x_p, x\}$ forms a well-orderable graph larger than H, a contradiction.

- x is not adjacent to any vertex x_i in H_2 with $i \geq 10$, since otherwise the graph induced by $\{x_1, x_2, x_5, x_7, x, x_i\}$ forms a $K_{1,3}+e$.

- x is not adjacent to any vertex x_i in H_2 with $4 \leq i \leq p-3$, since otherwise the graph induced by $\{x_1, x_2, x, x_i, x_{i-1}, x_{i+3}\}$ forms a $K_{1,3}+e$.

- x is not adjacent to any vertex x_i in H_2 with $8 \leq i \leq p-1$, since otherwise the graph induced by $\{x, x_i, x_{i-7}, x_{i-6}, x_{i-3}, x_{i+1}\}$ forms a $K_{1,3}+e$.

But then x has no neighbors in H_2, which means that P_1 must be empty. In a similar way, we can show that P_2 must be empty too. But then $T_1 = T_2 = I_1 = I_2 = \emptyset$, since otherwise G is decomposable with respect to canonical decomposition. Therefore, $G = H$ is a well-orderable graph, and by Lemma 2 the clique-width of G is at most 4.

Case 2: $p = 8$ or $p = 9$. We will show that P_1 and T_2 form a join (P_2 and T_1 form a join) and P_1 is disconnected from I_2 (P_2 is disconnected from I_1).

Assume the contrary: a vertex $x \in P_1$ is not adjacent to a vertex $y \in T_2$. By Proposition 2, x is adjacent to either x_6 or x_8. Then either the graph induced by $\{x, x_6, y, x_1, x_3, x_7\}$ or $\{x, x_8, y, x_1, x_3, x_5\}$ forms $K_{1,3}+e$, a contradiction.

Now let a vertex $x \in P_2$ be non-adjacent to a vertex $y \in T_1$. If x is adjacent to either x_3 or x_1, we arrive at a contradiction similarly as above. If x is adjacent to neither x_3 nor x_1, then $p = 9$ and x is adjacent only to x_9. But then $\{x_1, x_2, \ldots, x_p, x\}$ induce a well-orderable graph larger than H, again a contradiction.

Now assume that a vertex $x \in P_2$ is adjacent to a vertex $y \in I_1$. A $K_{1,3}+e$ can be easily found if x is not adjacent to x_9 or x_7. If x is adjacent both to x_9 (in case $p = 9$) and to x_7, then by Proposition 2, x is adjacent to x_5 and x_3 and hence the graph induced by $\{x, x_1, x_2, \ldots, x_p\}$ forms a well-orderable graph larger than H. This contradiction shows that P_2 and I_1 are disconnected.

Finally, let $x \in P_1$ be adjacent to $y \in I_2$. In case $p = 8$, the arguments are symmetric to those presented before. If $p = 9$ and x is adjacent to x_2, we know it must be adjacent to x_4, x_6 and x_8, which contradicts the definition of P_1. If x is not adjacent to x_2, the graph induced by $\{x, y, x_1, x_2, x_5, x_7\}$ forms a $K_{1,3}+e$.

From the above discussion we conclude that if at least one of the sets T_1, T_2, I_1, or I_2 is not empty, then G is decomposable with respect to canonical decomposition, which contradiscts our initial assumption. To complete the proof, we will show that P_1 and P_2 are of bounded size.

For a subset of vertices $S \subset H_2$, we define $P_1(S) := \{x \in P_1 \mid N(x) \cap H_2 = S\}$. Remember that $P_1(\emptyset)$ and $P_1(H_2)$ are empty by definition. For any other $S \subset H_2$, if $P_1(S)$ contains at least two vertices, then any $y \in S$ is the center of a claw, and the reader can easily find an edge in H that forms a $K_{1,3}+e$ together with that claw. Therefore, for any $S \subset H_2$, $P_1(S)$ contains at most 1 vertex and hence P_1 is of bounded size. Similarly, we can show that P_2 is of bounded size. But then the size of G, and hence the clique-width of G, is bounded by a constant.

Case 3: $p = 7$. Similar to the previous case, we can show that P_1 and P_2 are of bounded size. Also, by analogy with Case 2 we can deduce that P_1 and T_2 form a join, while P_1 and I_2 are disconnected. However, we cannot make the same observations about P_2. To overcome this difficulty, we partition P_2 with respect to its relationship with I_1 and T_1 as follows:

- P_2^1 is the set of vertices in P_2 that have a neighbor in I_1 and form a join with T_1;

- P_2^2 is the set of vertices in P_2 that are disconnected from I_1 and have a non-neighbor in T_1;
- P_2^3 is the set of vertices in P_2 that are disconnected from I_1 and form a join with T_1.

Let us show that $P_2^1 \cup P_2^2 \cup P_2^3$ is a partition of P_2. To this end, consider a vertex $x \in P_2$ and assume x is not an element of P_2^3. Then either x has a neighbor $y \in I_1$ or a non-neighbor $z \in T_1$. Notice that if x is adjacent to $y \in I_1$, then x is adjacent to each vertex of T_1, since otherwise the graph induced by $\{x, y, z, x_2, x_4, x_6\}$ forms a $K_{1,3} + e$. Therefore, if x has a neighbor in I_1, it belongs to P_2^1. Similarly, if x has a non-neighbor in T_1, then x is disconnected from I_1, in which case $x \in P_2^2$.

We shall say that a vertex $v \in P_2$ is adjacent to a subset $S \subset H_1$ if $N(v) \cap H_1 = S$. Observe that v cannot be adjacent to $\{x_7, x_5, x_3\}$ or $\{x_7\}$, for then the set $\{x_1 \ldots, x_p, v\}$ would induce a well-orderable graph larger than H. A simple case analysis shows that v can only be adjacent to the sets $S_1 := \{x_5, x_3, x_1\}$, $S_2 := \{x_5, x_3\}$, $S_3 := \{x_3, x_1\}$, $S_4 := \{x_1\}$, $S_5 := \{x_7, x_3\}$ or $S_6 := \{x_7, x_1\}$.

Now let us show that either P_2^1 or P_2^2 is empty. Assume otherwise: $x \in P_2^1$ and $y \in P_2^2$. By definition, there must exist a vertex $w \in T_1$ non-adjacent to y and a vertex $z \in I_1$ adjacent to x. We then observe the following:

- y is not adjacent to S_4, S_2 or S_3, since otherwise the graph induced, respectively, by $\{w, x_4, x_3, x_7, y, x_1\}$, $\{w, x_1, x_2, x_7, y, x_3\}$, and $\{w, x_2, x_5, x_7, y, x_3\}$ forms a $K_{1,3} + e$.
- y is not adjacent to S_1, since otherwise G contains a $K_{1,3} + e$ induced by
  - $\{w, x_4, x_6, x, y, x_1\}$ if x is adjacent to S_2 or S_5,
  - $\{w, x_2, x_6, x, y, x_3\}$ if x is adjacent to S_4 or S_6,
  - $\{x, z, y, x_5, x_2, x_6\}$ if x is adjacent to S_3.
- y is not adjacent to S_5, since otherwise G contains a $K_{1,3} + e$ induced by
  - $\{x_7, y, x_2, x_4, x, z\}$ if x is adjacent to S_1 or to S_2,
  - $\{x_4, x_7, x, y, x_5, x_6\}$ if x is adjacent to S_6,
  - $\{x_1, w, x, z, y, x_7\}$ if x is adjacent to S_3 or to S_4.
- y is not adjacent to S_6, since otherwise G contains a $K_{1,3} + e$ induced by
  - $\{x_7, y, x_3, x_5, x, w\}$ if x is adjacent to S_1 or S_2,
  - $\{x_1, x_2, x, y, x_3, x_4\}$ if x is adjacent to S_4,
  - $\{x, x_3, z, w, y, x_7\}$ if x is adjacent to S_3,
  - $\{x_7, x, y, x_4, x_5, x_6\}$ if x is adjacent to S_5.

Thus we conclude that either P_2^2 or P_2^1 is empty. Without loss of generality, we may assume that P_2^2 is empty, since otherwise we can switch to the bipartite complement of G. Since G is indecomposable with respect to canonical decomposition, we must conclude that T_1 and I_2 are empty.

If I_1 is empty, then T_2 is empty too, since otherwise G is decomposable with respect to canonical decomposition. But then G, and hence the clique-width of G, is of bounded size.

Assume now I_1 is not empty. By definition, any vertex $x \in P_2^1$ has a neighbor $y \in I_1$. Therefore, x cannot be adjacent to S_1, S_3, S_5, or S_6, since otherwise the graph induced by $\{x_1, x_5, x, y, x_7, x_4\}$, $\{x_3, x_1, x, y, x_6, x_5\}$, $\{x_7, x_3, x, y, x_6, x_5\}$, or $\{x_7, x_1, x, y, x_6, x_5\}$ respectively forms a $K_{1,3} + e$.

If T_2 is non-empty, then, to avoid an induced $K_{1,3} + e$, we conclude that T_2 forms a join with the subset of vertices of I_1 that have a neighbor in P_2^1. But then G is can be partitioned into a biclique and an independent set, a contradiction. Therefore, T_2 is empty, which implies that $|I_1|$ is bounded. Indeed, any vertex $v \in I_1$ has a neighbor in P_2^1, else v is isolated in G. On the other hand, every vertex of P_2^1 is adjacent to at most 2 vertices of I_1, since otherwise a $K_{1,3} + e$ arises. But then $|I_1| \leq 2|P_2^1|$, and therefore G is of bounded size. This completes the proof of the theorem. □

6 $(S_{1,1,3} + v)$-free bipartite graphs

Theorem 7. *The clique-width of $(S_{1,1,3} + v)$-free bipartite graphs is bounded by a constant.*

Proof. The proof of this theorem follows the same strategy as that of Theorem 6. In particular, we assume that G is a prime $(S_{1,1,3} + v)$-free bipartite graph indecomposable with respect to canonical decomposition, and H is a maximum well-orderable induced subgraph of G. We keep the same notation H_1, H_2, P_1, P_2, I_1, I_2, T_1, and T_2 for subsets of vertices of G, and assume that G is indecomposable with respect to canonical decomposition.

As before, we observe that any 7 consequtive vertices of H induce an $S_{1,2,3}$. Therefore, we assume that $p \geq 7$, since otherwise G is $S_{1,2,3}$-free and the clique-width of G is bounded by Theorem 2. We also conclude that $p < 9$, else the vertices x_3 through x_9 induce an $S_{1,2,3}$ which together with x_1 creates an $S_{1,2,3}+v$. The rest of the proof is partitioned into 2 general cases according to the number of vertices of H.

Case 1: $p = 8$. This case is symmetric in the sense that any statement on an odd-indexed subset of G follows by symmetry for a respective even-indexed subset. Therefore, we analyze odd-indexed subsets only.

First, we observe that I_1 is empty, since otherwise the vertices $x_2, x_1, x_5, x_7, x_4, x_3, x$ with $x \in I_1$ induce an $S_{1,1,3} + v$.

Next we notice that T_1 is empty, since otherwise the vertices $x_4, x_3, x_7, x, x_6, x_5, x_1$ with $x \in T_1$ induce an $S_{1,1,3} + v$.

Now we analyze the structure of P_1 and P_2. Let $P_i(S)$ denote the set of vertices in P_i adjacent to the set S in H. The same proof that T_1 is empty can be used to show that $P_1(x_2, x_4, x_6)$, $P_1(x_4, x_6)$, $P_1(x_4, x_6, x_8)$ are empty. Now we identify those subsets S for which

$P_1(S)$ contains at most one vertex. To this end, we assume $x, y \in P_1(S)$ and conclude by contradiction that $|P_1(S)| \leq 1$ in case

- $S = \{x_6\}$ or $S = \{x_6, x_8\}$, since otherwise $\{x_6, x, y, x_5, x_2, x_7, x_3\}$ induces an $S_{1,1,3} + v$.
- $S = \{x_8\}$ or $S = \{x_4, x_8\}$, since otherwise $\{x_6, x, y, x_7, x_2, x_5, x_3\}$ induces an $S_{1,1,3} + v$.
- $S = \{x_2, x_8\}$ or $S = \{x_2, x_6, x_8\}$, since otherwise $\{x_8, x, y, x_7, x_4, x_3, x_1\}$ induces an $S_{1,1,3} + v$.

By Lemma 5, the sets $P_1(S)$ with $|P_1(S)| \leq 1$ can be neglected. So we are left with $P_1(x_2)$, $P_1(x_4)$, $P_1(x_2, x_4)$, $P_1(x_2, x_6)$, and $P_1(x_2, x_4, x_8)$. By symmetry we are restricted to the following subsets of P_2: $P_2(x_7)$, $P_2(x_5)$, $P_2(x_7, x_5)$, $P_2(x_7, x_3)$, and $P_2(x_7, x_5, x_1)$. In what follows we analyze the structure of bipartite graphs induced by $P_1(S)$ and $P_2(S')$ for various subsets $S \subset H_2$ and $S' \subset H_1$. The results of the analysis are summarized in the following table.

	$P_2(x_7)$	$P_2(x_5)$	$P_2(x_7, x_5)$	$P_2(x_7, x_3)$	$P_2(x_7, x_5, x_1)$
$P_1(x_2)$	*deg* ≤ 1	edgeless	edgeless	edgeless	complete
$P_1(x_4)$	edgeless	edgeless	complete	complete	edgeless
$P_1(x_2, x_4)$	edgeless	complete	complete	edgeless	complete
$P_1(x_2, x_6)$	edgeless	complete	edgeless	edgeless	complete
$P_1(x_2, x_4, x_8)$	complete	edgeless	complete	complete	*deg* ≤ 1

Due to the symmetry, it is enough to consider only half of the table.

- *$P_1(x_2)$ and $P_2(x_7)$ induce a bipartite graph of vertex degree at most 1*: if a vertex $x \in P_1(x_2)$ would have two neighbors $y, z \in P_2(x_7)$, then $\{x, y, z, x_2, x_7, x_4, x_6\}$ would form an $S_{1,1,3} + v$. By symmetry, no vertex of $P_2(x_7)$ can have more than one neighbor in $P_1(x_2)$.
- *$P_1(x_2)$ and $P_2(x_7, x_5)$ induce an edgeless graph*: if a vertex $x \in P_1(x_2)$ is adjacent to a vertex $y \in P_2(x_7, x_5)$, then $\{y, x, x_5, x_7, x_4, x_3, x_1\}$ induces an $S_{1,1,3} + v$.
- *$P_1(x_2)$ and $P_2(x_7, x_5, x_1)$ induce a complete bipartite graph*: if $x \in P_1(x_2)$ and $y \in P_2(x_7, x_5, x_1)$ are not adjacent, then $\{y, x_1, x_5, x_7, x_4, x_3, x\}$ induces an $S_{1,1,3} + v$.
- *$P_1(x_2)$ and $P_2(x_5)$ induce an edgeless graph*: if $x \in P_1(x_2)$ and $y \in P_2(x_5)$ are adjacent, then $\{x_7, x_8, x_4, x_2, x, y, x_6\}$ induces an $S_{1,1,3} + v$
- *$P_1(x_2)$ and $P_2(x_7, x_3)$ induce an edgeless graph unless $|P_2(x_7, x_3)| = 1$*: First we show that any vertex $x \in P_1(x_2)$ has at most one neighbor in $P_2(x_7, x_3)$. Indeed, if x is adjacent to $y, z \in P_2(x_7, x_3)$, then $\{x, y, z, x_2, x_5, x_6, x_4\}$ induces an $S_{1,1,3} + v$. Now we show that if $|P_2(x_7, x_3)| > 1$, then $x \in P_1(x_2)$ cannot have neighbors in $P_2(x_7, x_3)$. Assume the contrary: x is adjacent to $y \in P_2(x_7, x_3)$ and non-adjacent to $z \in P_2(x_7, x_3)$. Then $\{x_3, x_4, z, y, x, x_2, x_6\}$ induces an $S_{1,1,3} + v$, a contradiction.

- *$P_1(x_4)$ and $P_2(x_5)$ induce an edgeless graph*: if $x \in P_1(x_4)$ and $y \in P_2(x_5)$ are adjacent, then $\{x_4, x_7, x_3, x, y, x_5, x_1\}$ induces an $S_{1,1,3} + v$.

- *$P_1(x_4)$ and $P_2(x_7, x_5)$ induce a complete bipartite graph*: if $x \in P_1(x_4)$ and $y \in P_2(x_7, x_5)$ are not adjacent, then $\{x_4, x_3, x, x_7, y, x_5, x_1\}$ induces an $S_{1,1,3} + v$.

- *$P_1(x_4)$ and $P_2(x_7, x_3)$ induce complete bipartite graph*: if $x \in P_1(x_4)$ and $y \in P_2(x_7, x_3)$ are not adjacent, then $\{x_2, x_1, x_5, x_7, y, x_3, x\}$ induces an $S_{1,1,3} + v$.

- *$P_1(x_4)$ and $P_2(x_7, x_5, x_1)$ induce an edgeless graph*: if $x \in P_1(x_4)$ and $y \in P_2(x_7, x_5, x_1)$ are adjacent, then $\{x_5, x_2, x_6, y, x, x_4, x_8\}$ induces an $S_{1,1,3} + v$.

- *$P_1(x_2, x_4)$ and $P_2(x_7, x_5)$ induce a complete bipartite graph*: if $x \in P_1(x_2, x_4)$ and $y \in P_2(x_7, x_5)$ are not adjacent, then $\{x_5, x_6, y, x_2, x, x_4, x_8\}$ induces an $S_{1,1,3} + v$.

- *$P_1(x_2, x_4)$ and $P_2(x_7, x_3)$ induce an edgeless graph*: if $x \in P_1(x_2, x_4)$ and $y \in P_2(x_7, x_3)$ are adjacent, then $\{x, y, x_4, x_2, x_5, x_6, x_8\}$ induces an $S_{1,1,3} + v$.

- *$P_1(x_2, x_4)$ and $P_2(x_7, x_5, x_1)$ induce a complete bipartite graph*: if $x \in P_1(x_2, x_4)$ and $y \in P_2(x_7, x_5, x_1)$ are not adjacent, then $\{x_5, y, x_6, x_2, x, x_4, x_8\}$ induces an $S_{1,1,3} + v$.

- *$P_1(x_2, x_6)$ and $P_2(x_7, x_3)$ induce an edgeless graph*: if $x \in P_1(x_2, x_6)$ and $y \in P_2(x_7, x_3)$ are adjacent, then $\{x, x_2, x_6, y, x_3, x_4, x_8\}$ forms $S_{1,1,3} + v$.

- *$P_1(x_2, x_6)$ and $P_2(x_7, x_5, x_1)$ induce a complete bipartite graph*: if $x \in P_1(x_2, x_6)$ and $y \in P_2(x_7, x_5, x_1)$ are not adjacent, then $\{y, x_1, x_5, x_7, x_4, x_3, x\}$ induces an $S_{1,1,3} + v$.

- *$P_1(x_2, x_4, x_8)$ and $P_2(x_7, x_5, x_1)$ induce a bipartite graph of vertex degree at most 1*: if $y \in P_2(x_7, x_5, x_1)$ is adjacent to $x, z \in P_1(x_2, x_4, x_8)$, then $\{x_8, x, w, x_7, y, x_1, x_3\}$ induces an $S_{1,1,3} + v$. By symmetry no vertex of $P_1(x_2, x_4, x_8)$ can have more than one neighbor in $P_2(x_7, x_5, x_1)$.

Since G is a prime graph, every set of vertices with the same neighborhood has size 1. From this we conclude that $P_1(x_4)$, $P_1(x_2, x_6)$, $P_1(x_2, x_4)$, $P_2(x_5)$, $P_2(x_7, x_5)$, and $P_2(x_7, x_3)$ are singletons.

By deleting finitely many vertices we are left with the graph induced by H, $P_1(x_2)$, $P_1(x_2, x_4, x_8)$, $P_2(x_7)$, and $P_2(x_7, x_5, x_1)$. By deleting eight more vertices, we can rid ourselves of H as well. The bipartite complement of the remaining graph has two connected components, each of which is $S_{1,2,3}$-free. Therefore, the remaining graph has bounded clique-width. By Lemma 5, this implies that G is of bounded clique-width too.

Case 2: $p = 7$. This case is not symmetric, but in this case H is self-complementary. Remember that $S_{1,1,3} + v$ is self-complementary too and hence the bipartite complement of G is again $(S_{1,1,3} + v)$-free, which allows us to simplify the proof by using complementary arguments, where applicable.

As in the case when $p = 8$, we know that T_1 and I_1 are empty. Also, we conclude that either T_2 or I_2 is empty. Indeed, if $x \in T_2$ and $y \in I_2$, then the graph induced by $\{x_5, x_6, x_2, x, x_3, x_4, y\}$ forms $S_{1,1,3} + v$. Without loss of generality, we may assume that T_2 is empty, since otherwise we can consider the bipartite complement of G.

If we do an analysis similar to that done when $p = 8$, we find that $P_i(S)$ contains at most one vertex, except for $P_1(x_2)$, $P_1(x_4)$, $P_1(x_2, x_4)$, $P_1(x_2, x_6)$, $P_2(x_5)$, $P_2(x_7)$, $P_2(x_5, x_7)$, $P_2(x_3, x_7)$, $P_2(x_1, x_5, x_7)$, $P_2(x_3, x_5, x_7)$.

Exactly as in case $p = 8$, we conclude that $P_1(x_2)$ and $P_2(x_7, x_5)$ induce an edgeless graph, while $P_1(x_4)$ and $P_2(x_7, x_5)$ induce a complete bipartite graph. Observe that the set $P_1(x_2)$ plays the same role in G as the set $P_1(x_2, x_4)$ in the complement to G. A similar relationship exists between $P_1(x_4)$ and $P_1(x_2, x_6)$. But the role of $P_2(x_7, x_5)$ does not change in the complement of G. This discussion leads to the conclusion that the vertices in $P_2(x_7, x_5)$ have the same neighborhood and hence $|P_2(x_7, x_5)| = 1$. Analogously, we derive that $|P_2(x_7, x_3)| = 1$. For the remaining subsets we create a table similar to that in case $p = 8$. We also include in this table I_2.

	$P_2(x_7)$	$P_2(x_5)$	$P_2(x_7, x_5, x_1)$	$P_2(x_7, x_5, x_3)$	I_2
$P_1(x_2)$	$deg \leq 1$	Claim 1	complete	Claim 2	Claim 5
$P_1(x_4)$	edgeless	edgeless	Claim 3	Claim 4	Claim 6
$P_1(x_2, x_4)$	*	*	*	*	Claim 7
$P_1(x_2, x_6)$	*	*	*	*	Claim 8

Taking into account the relationship between $P_1(x_2)$ and $P_1(x_2, x_4)$, as well as between $P_1(x_4)$ and $P_1(x_2, x_6)$, we complete only the first two lines of the table, the rest follows by complementary arguments. Where applicable, the information in this table is borrowed from the respective table in the previous case. To fill in the remaining entries, we provide a number of claims.

Claim 1. *The graph induced by $P_1(x_2)$ and $P_2(x_5)$ either*

- *has no vertices of degree more than one, or*
- *has at most one non-trivial connected component, which is a $K_{1,n}$ with central vertex in $P_2(x_5)$.*

Proof. If a vertex $x \in P_1(x_2)$ has two neighbors $y, z \in P_2(x_5)$, then the graph induced by $\{x, y, z, x_2, x_7, x_4, x_6\}$ forms $S_{1,1,3} + v$. Therefore, any vertex of $P_1(x_2)$ has at most one neighbor in $P_2(x_5)$.

Now assume that $y \in P_2(x_5)$ has at least two neighbors $x, y \in P_1(x_2)$. If the graph induced by $P_1(x_2)$ and $P_2(x_5)$ has at least one more non-trivial connected component with vertices $a \in P_2(x_5)$ and $b \in P_1(x_2)$, then the set $\{x, y, z, x_5, a, b, x_3\}$ induces an $S_{1,1,3} + v$. This contradiction completes the proof. □

Claim 2. *No vertex of $P_1(x_2)$ distinguishes $P_2(x_3, x_5, x_7)$*

Proof. If $x \in P_1(x_2)$ is adjacent to $y \in P_2(x_3, x_5, x_7)$ but not $z \in P_2(x_3, x_5, x_7)$, then the set $\{x_3, z, x_4, y, x, x_2, x_6\}$ induces an $S_{1,1,3} + v$. □

Claim 3. *No vertex of $P_1(x_4)$ distinguishes $P_2(x_1, x_5, x_7)$.*

Proof. If $x \in P_1(x_4)$ is adjacent to $y \in P_2(x_1, x_5, x_7)$ but not $z \in P_2(x_1, x_5, x_7)$, then the set $\{x_1, z, x_2, y, x, x_4, x_6\}$ induces an $S_{1,1,3} + v$. □

Claim 4. *No vertex of $P_2(x_3, x_5, x_7)$ distinguishes $P_1(x_4)$.*

Proof. If a vertex $x \in P_2(x_3, x_5, x_7)$ is adjacent to $y \in P_1(x_4)$ and non-adjacent to $z \in P_1(x_4)$, then the set $\{x, y, x_3, x_7, x_2, x_1, z\}$ induces an $S_{1,1,3} + v$. □

Claim 5. *No vertex of $P_1(x_2)$ has more than two neighbors in I_2.*

Proof. If $y \in P_1(x_2)$ is adjacent to $x, w \in I_2$, then the set $\{y, x, w, x_2, x_5, x_6, x_4\}$ induces an $S_{1,1,3} + v$. □

Claim 6. *No vertex of $P_1(x_4)$ has more than two neighbors in I_2.*

Proof. If $y \in P_1(x_4)$ is adjacent to $x, w \in I_2$, then the set $\{y, x, w, x_4, x_7, x_2, x_6\}$ induces an $S_{1,1,3} + v$. □

Claim 7. *No vertex of $P_1(x_2, x_4)$ distinguishes I_2.*

Proof. If $y \in P_1(x_2, x_4)$ is adjacent to $x \in I_2$ but not $w \in I_2$, then the set $\{y, x, x_4, x_2, x_5, x_6, w\}$ induces an $S_{1,1,3} + v$. □

Claim 8. *No vertex of $P_1(x_2, x_6)$ distinguishes I_2.*

Proof. If $y \in P_1(x_2, x_6)$ is adjacent to $x \in I_2$ but not $w \in I_2$, then the set $\{y, x, x_6, x_2, x_7, x_4, w\}$ induces an $S_{1,1,3} + v$. □

From the above series of claims we know that no vertex of $P_2(x_7) \cup P_2(x_1, x_5, x_7) \cup P_2(x_3, x_5, x_7)$ distinguishes $P_1(x_2, x_6)$. On the other hand, we know that no vertex of $P_1(x_2, x_6)$ distinguishes $P_2(x_5)$ or I_2. Therefore, $P_1(x_2, x_6)$ consists of at most 4 modules and hence $|P_1(x_2, x_6)| \leq 4$. This allows us to exclude $P_1(x_2, x_6)$ from further considerations. The rest of the proof is devided into 2 subcases.

Case 2.1: I_2 is nonempty. Then $P_2(x_5)$ and $P_2(x_7)$ are empty (if $x \in P_2(x_5) \cup P_2(x_7)$ and $y \in I_2$, then either $\{x_5, x, x_6, x_2, x_7, x_4, y\}$ or $\{x_7, x, x_4, x_2, x_5, x_6, y\}$ induces an $S_{1,1,3} + v$). We now partition the graph $G - H$ into two induced subgraphs G_1 and G_2 as follows: $G_1 = G[P_1(x_2) \cup P_1(x_4) \cup I_2]$ and $G_2 = G[P_1(x_2, x_4) \cup P_2(x_1, x_5, x_7) \cup P_2(x_3, x_5, x_7)]$. In G_1, every connected component is a star $K_{1,n}$. In the complement to G_2, every component is a path with at most 3 vertices. Therefore, the clique-width of G_1 and G_2 is bounded. Consequently, by Lemma 4, the clique-width of $G - H$ and hence of G is bounded.

Case 2.2: I_2 is empty. Then $P_1(x_4)$ consists of finitely many modules and hence the size of $P_1(x_4)$ is bounded. We now partition the graph $G-(H\cup P_1(x_4))$ into two induced subgraphs G_1 and G_2 as follows: $G_1 = G[P_1(x_2)\cup P_2(x_5)\cup P_2(x_7)]$ and $G_2 = G[P_1(x_2,x_4)\cup P_2(x_1,x_5,x_7)\cup P_2(x_3,x_5,x_7)]$. As before, we conclude that the clique-width of G_1 and G_2, and hence of G (Lemmas 4, 5), is bounded.

□

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