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On the Control of a Dynamical System Defined by a Decreasing One-Dimensional Set-Valued Function

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RRR 17-2007, April, 2007

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Abstract. The uncertainity of the price is modeled by set-valued function in a single product market.

1 Introduction

This paper is devoted to the analysis of certain dynamical systems defined by one-dimensional set-valued functions. The main motivation is the potential application in economics with special emphasis on agricultural markets.

The future price of a commodity is always uncertain. In most of the cases uncertainity is modeled by probability, see e.g. [1]. Sometimes the system is changing fast and therefore it is not possible to learn its probabilistic properties if they exist at all. This is typical for agricultural markets where the parameters of the system do not remain the same from one harvest to the other one because of the ever improving technology. Therefore in modeling such a system one may use mathematical tools different from probability. One such tool is the set-valued function.

The basic dynamics of the market from one year to the next one is discussed in general in the frame called cobweb model. The harvested quantity is known for producers and customers in each year. This is the quantity to appear on the market until the next harvest. The equilibrium price of the market closely related to the harvested quantity. The higher the quantity is the lower the price is. Assume that the farmer can produce several different crops. The producer estimates the market prices of the next year and according to the estimations he/she decides on the use of the arable land. That means at the same time a decision on the quantities produced in the next year. The higher the estimated price is, the larger area is used for a crop. We can conclude in a similar way further on: the larger the area is used for a crop, the higher the quantity will be produced, and the higher quantity appears on the market, the smaller the equilibrium price is. The final conlusion is that the market price is a decreasing function of the estimated price.

In agriculture there are commodities such that it is very easy both to enter and leave their markets. The condition for it is that the production must have a simple technology. One typical example is potato. We assume that the producer has a simple behaviour, i.e. if he/she expects high prices, then he/she increases the production, therefore a high quantity appears on the market and the price drops, and vice versa, if the expected price is low, then the production is low as well, thus only a small quantity is sold on he market causing high price. The price expectations using prices of the past lead typically to such phenomena, e.g. the adaptive price expectation of Nerlove [5] and the extrapolative expectation of [6] are such ones.

There are indications that the farmers do not estimate future prices numerically [3], or make significant and skewed errors [2]. The research on price expectations is still going on. Both theory and applications are in progress. Especially many papers deal with African issues, see e.g. [4].

In this paper another approach is discussed, which more or less faces with all of the aforementioned difficulties: no stochastic distribution is known although the system is uncertain, the estimation is not a well-defined single numeric value. The authors concluded that a possible model of a single commodity market is a dynamical system defined by a decreasing set-valued function.

2 Basic assumptions

It is assumed that the dynamical system can be controled. The first four assumptions are describing the system in general and the next two ones concern to the control, and the last one excludes some degenerate cases.

- **(A1)** The state of the system is a real number.
- **(A2)** The time is discrete and it takes its value t from the set $\mathbb{Z}_+ = \{0, 1, 2, ...\}$.
- **(A3)** Assume that the state of the system is x_t at time t. Then the state at time $t+1$ is an element of the set $G(x_t)$ if the control is switched off. $G : \mathbb{R} \Rightarrow \mathbb{R}$ is a set-valued function and for all $x \in \mathbb{R} : G(x) = [a(x), b(x)]$, where $a(\cdot)$, and $b(\cdot)$ are real functions.
- **(A4)** The functions $a(\cdot)$, and $b(\cdot)$ satisfy the conditions
  1. $\forall\, x \in \mathbb{R} :\ a(x) \ \le\ b(x)$.
  2. Both $a(\cdot)$, and $b(\cdot)$ are continuous on the whole real line.
  3. Both $a(\cdot)$, and $b(\cdot)$ are strictly decreasing on the whole real line.
- **(A5)** The control is an additive term, say q_{t+1} in time t. Thus the state of the system in the case when control is switched on is

$$x_{t+1} \ = \ x'_{t+1} \ + \ q_{t+1} \ \in \ G(x_t) \ + \ q_{t+1}, \tag{1}$$

  i.e. x'_{t+1} an element of $G(x_t)$, becomes known only in time period t.
- **(A6)** The value of the control may vary from period to period, but it must belong to the interval $[d_1, d_2]$, where $d_1 \ < \ 0 \ < \ d_2$.
- **(A7)** The functions $g(x) = b(a(x) + d_1) + d_2$, and $h(x) = a(b(x) + d_1) + d_2$ have only finite many fixed points.

Equation (1) does not define the control uniquely. It is important to distinguish two types of control. It is called *a priori* if the control, i.e. q_{t+1}, is determined first and after that x'_{t+1} becomes known. The opposite case, when q_{t+1} can be determined after that x'_{t+1} becomes known is called *a posteori*. Notice that the controller cannot affect x'_{t+1} in any way. In this paper the main emphasize is given to the a posteori case. The value x'_{t+1} is called the *realization of the system at time period* $t+1$.

It is also assumed the starting point is known and it is x_0. The aim of the control is to reach the target interval $[V, W]$, and keep the system in it. We suppose that

$$x_0 \ < \ V \ < W. \tag{2}$$

The following notation is used. Let f be an arbitrary real function. Then $f^+(x) = f(x) + d_2$, and $f^-(x) = f(x) + d_1$. Then the functions g, and h in Assumption (A7) can be written in the following short form: $g(x) = b^+(a^-(x))$, and $h(x) = a^+(b^-(x))$. It is easy to see, that both g and h are strictly increasing.

3 On the existence of trajectories

There are cases when the goal of the control cannot be achieved. Therefore the first issue to be discussed is the existence of the desired trajectories.

Theorem 1. *No controled trajectory can enter into the $[V, +\infty)$ interval in finite many steps if and only if $b^+(x_0) < V$ and there exists a point $y \in [\min\{x_0, b^+(x_0)\}, V]$ such that $g(y) \leq y$.*

Proof The necessity of the inequality $b^+(x_0) < V$ is obvious. The possible maximal value of a trajectory in step 1 is $b^+(x_0)$. Thus if the inequality does not hold then any trajectory, which has this value at step 1 enters the target interval.

In general we determine the possible maximal and minimal values of the trajectories at step t.

(a) If $t = 0$ then the only possible value is $x_0 < V$.

(b) If $t = 1$ then $x_1 \in G(x_0) + [d_1, d_2] = [a(x_0) + d_1, b(x_0) + d_2] = [a^-(x_0), b^+(x_0)]$.

(c) Let $t \geq 2$. At time t the system has the state x_t. The minimal(maximal) value where the trajectory can step from it is $a^-(x_t)(b^+(x_t))$. Hence it follows from the strictly monotone decreasing property of the functions $a(\cdot)$, and $b(\cdot)$ that we obtain the possible absolute minimal(maximal) value for x_t if x_{t-1} is the possible absolute maximal(minimal) value at step $t-1$. It means that if

$$x_2 \in [a^-(b^+(x_0)), b^+(a^-(x_0))] = [a^-(b^+(x_0)), g(x_0)]$$

and

$$x_3 \in [a^-(g(x_0)), b^+(a^-(b^+(x_0)))] = [a^-(g(x_0)), g(b^+(x_0))]$$

and in general for even t, say $t = 2k$,

$$x_{2k} \in [a^-(g^{k-1}(b^+(x_0))), g^k(x_0)]$$

and for odd t, say $t = 2k + 1$,

$$x_{2k+1} = [a^-(g^k(x_0)), g^k(b^+(x_0))].$$

For even values of t the possible absolute maximal values of the trajectories are $x_0, g(x_0), ..., g^k(x_0), ...$ This sequence is defined by the recursive equation

$$u_0 = x_0, \quad u_k = g(u_{k-1}). \tag{3}$$

First assume that the function $g(\cdot)$ does not have a fixed point. If for all real number z the inequality $z > g(z)$ holds then the sequence is strictly decreasing and converges to $-\infty$. Thus the trajectory is unable to enter the target interval, and at the same time the necessary and sufficient condition given in the statement holds. If the opposite inequality, i.e. $z < g(z)$, holds for all real numbers, then the sequence is strictly increasing and converges to $+\infty$. Thus the trajectory enters the target interval but the necessary and sufficient condition does not hold.

If $g(\cdot)$ has at least one fixed point then there are the following cases:

(i) x_0 itself is a fixed point and the sequence is constant.

(ii) All of the fixed points are strictly less than x_0. If $g(x_0) < x_0$ then the sequence decreasing and converges to the maximal fixed point $x_f^{\max} < x_0$ and never enters the target interval. At the same time x_0 is appropriate for the value y of the statement. If $g(x_0) > x_0$ then the sequence is increasing and converges to $+\infty$ but appropriate y does not exists.

(iii) All fixed points are strictly greater than x_0. If $g(x_0) < x_0$ then the sequence decreasing and converges to $-\infty$. Again $y = x_0$ is a good choice. If $g(x_0) > x_0$ then the sequence is increasing and converges to the minimal fixed point $x_f^{\min} > x_0$. If $x_f^{\min} > V$ then the sequence enters the target interval and appropriate y exists. If $x_f^{\min} \leq V$ then the sequence does not reach the target interval in finite many steps and $y = x_f^{\min}$.

(iv) x_0 is between two fixed points, say x_f^l and x_f^u, such that $x_f^l < x_0 < x_f^u$ and there is no other fixed point in the interval $[x_f^l, x_f^u]$. Then the trajectory converges to x_f^l if $g(x_0) < x_0$ and $y = x_0$ is a good choice. If $g(x_0) > x_0$ then the sequence converges to x_f^u. If $x_f^u \leq V$ then the trajectory does not enter the target interval in finite many cases and $y = x_f^u$ is a good choice. Otherwise, i.e. if $x_f^u > V$, the trajectory enters the target interval and no appropriate y exists.

Hence one can obtain the conclusion that the trajectory cannot enter the target interval in even numbered steps if and only if the statement is true.

For odd values of t the sequence is $b^+(x_0), g(b^+(x_0)), ..., g^k(b^+(x_0)), ...$ If $b^+(x_0) \geq V$ then there are trajectories entering the target interval. If $b^+(x_0) < V$ then the proof is similar to the case of even t's. Q.E.D.

Theorem 2. *In the a posteori case it is possible to choose the control for all trajectories such that the trajectory enters the* $[V, +\infty]$ *interval and it stays there forever if and only if*

$$\forall\, y \in [x_0, V] : \; h(y) > y \quad \text{and} \quad V \;\leq\; a^+(V). \tag{4}$$

Proof First assume that (4) holds. Then an appropriate control is

$$q_{t+1} \;=\; \begin{cases} \min\{d_2,\, V - x'_{t+1}\} & \text{if } x'_{t+1} \leq V \\ \max\{d_1,\, V - x'_{t+1}\} & \text{if } x'_{t+1} > V. \end{cases} \tag{5}$$

Notice that q_{t+1} is determined such that if $V - d_2 \;\leq\; x'_{t+1} \;\leq\; V - d_1$ then x_{t+1} becomes V. Outside the interval q_{t+1} controls the trajectory with maximal possible speed toward V.

It follows from the strictly decreasing property of function $a(\cdot)$, and the condition (4), and the inequality $x_0 < V$ that

$$V \le a^+(V) < a^+(x_0).$$

Hence starting from x_0 the first point of the trajectory, i.e. x_1, is at least V. It can be greater than V only if $x_1^{'} > V - d_1$, but in that case (5) requires the application of the possible minimal control. Thus x_1 must be an element of the interval $[V, \max\{V, b^-(x_0)\}]$.

The minimal value of x_2 is obtained if x_1 has its maximal value, i.e. $\max\{V, b^-(x_0)\}$, and the $x_2^{'}$ realization is minimal when the maximal control can be applied. Hence

$$x_2 \ge a^+(\max\{V, b^-(x_0)\}) = \min\{a^+(V), h(x_0)\} \ge \min\{V, h(x_0)\}.$$

Similarly the maximal value of x_2 is obtained from the minimal value of x_1 with the maximal realization. Then it follows from the (5) control that $x_2 \le \max\{V, b^-(V)\}$. In a similar way one can conclude that $x_3 \ge \min\{V, h(V)\}$. In general

$$x_{2k} \ge \min\{V, h^k(x_0)\} \quad \text{and} \quad x_{2k+1} \ge \min\{V, h^k(V)\}.$$

If $h(y) > y$ holds for all $y \in [x_0, V]$ then any trajectory of the form $y, h(y), ..., h^k(y), ...$ must converge either to $+\infty$ or a fixed point of $h(\cdot)$ greater than V. Hence the trajectory with control (5) enters the the target interval $[V, +infty]$ in finite many steps.

At time t the system is at state x_t. There is a unique control such that the system will have the greatest possible state at time $t+2$: at time $t+1$ the smallest possible value must be applied, i.e. d_1, and at time $t+2$ the greatest possible one, i.e. d_2. The worst case of the realizations is $b(x_{t+1})$ at time $t+1$, and $a(x_{t+2})$. Thus the greatest state what can be guaranteed at time $t+2$ is $h(x_t)$. If there is a $y \in [x_0, V]$ such that $h(y) \le y$ then it follows from the previous theorem that the $[V, +\infty)$ interval cannot be reached from x_0 in finite many steps. Q.E.D.

Theorem 3. *In the a posteori case it is possible to choose the control for all trajectories such that the trajectory enters the $[V, W]$ interval and it stays there forever if and only if there exist a value $V_0 \in [V, W]$ such that*

$$\forall\, y \in [x_0, V_0] : h(y) > y \quad \text{and} \quad V_0 \le a^+(V_0) \quad \text{and} \quad b^-(V_0) \le W. \tag{6}$$

Proof It follows from the previous theorem that if (6) holds then all trajectories can be controled such that they enter the $[V_0, +\infty)$ interval and stays in it forever. Assume that control (5) is applied with $V = V_0$ and the trajectory just entered the interval $[V_0, +\infty)$, i.e. $x_k < V_0$ if $k < t$. Then it follows from the decreasing property of function $b(\cdot)$ and the definition of control (5) that x_{t+1} is at most $b^-(V_0)$.

Assume in an indirect way that V_0 does not exists but for every trajectory there is a control such that the trajectory enters the interval $[V, W]$ and stays there forever. Then it follows from the previous theorem that for all $y \in [x_0, V]$ the inequalities $h(y) > y$, and $V \le a^+(V)$ hold. Then the indirect assumption is only true if $b^-(V) > W$. As the trajectory does not leave the interval $[V, W]$ for every $x_t \in [V, W]$ and for every realization of $G(x_t)$ there must be a control that $x_{t+1} \le W$. As $b(\cdot)$ is decreasing it is necessary that $b^-(W) \le W$. Hence there is a unique $W_{b^-} \in [V, W]$ such that $b^-(W_{b^-}) = W$ and $W_{b^-} > V$. Here we distiguish two cases:

Case 1: $\forall\, y \in [V, W_{b^-}] : h(y) > y$. Hence

$$W_{b^-} < h(W_{b^-}) = a^+(b^-(W_{b^-})) = a^+(W) \le a^+(W_{b^-}).$$

Then it follows from the equation $b^-(W_{b^-}) = W$ that W_{b^-} satisfies the property that are claimed from V_0 in the statement.

Case 2: $h(W_{b^-}) \le W_{b^-}$. Then there is at least one fixed point of function $h(\cdot)$ in the interval $(V, W_{b^-}]$. Let $x_{h,f}$ be the minimal among these fixed points. The function $h(\cdot)$ gives the possible maximal value of the state in every second steps. Let t be iteration when the state of the trajectory is at least V. Let us consider the sequence $\{h^{-1}(x_t), x_t, x_{t+2} = h(x_t), x_{t+4} = h(x_{t+2}), ...\}$. It is the trajectory of a dynamic system defined by the (single-valued) function $h(\cdot)$ and started from $h^{-1}(x_t)$. The starting point exist as function $h(\cdot)$ is strictly increasing. A trajectory cannot stay in a fixed point unless it starts from the fixed point. Hence the elements of the sequence are never equal to $x_{h,f}$ but only converge to it. Then the trajectory may have a state strictly greater than W in every second step if the realization are provided alternatively from function $a(\cdot)$, and $b(\cdot)$. Q.E.D.

A trajectory cannot reach the whole line. In certain cases the trajectories determine a stripped structure of the line as the following theorem shows.

Theorem 4. *Assume that $x_0 \in [h_a, h_f]$, where the values h_a and h_f are two consecutive fixed points of function h. Let $p_a = b^-(h_a)$ and $p_f = b^-(h_f)$. Assume that $h_a < h_f < p_f < p_a$. Then there is a control for all trajectories that the trajectory does not leave the interval $[h_a, p_a]$.*

Remarks. It follows from the relation $h_a < h_f$ and the decreasing property of the functions that $p_f < p_a$. The equations $h_a = a^+(p_a)$ and $h_f = a^+(p_f)$ are immediate consequences of the assumption that h_a and h_f are fixed points of the function h. Similarly p_a, and p_f are fixed points of function p.

Proof The realization x_1' is in the interval $[a(x_0), b(x_0)]$. It follows from the decreasing property of the functions that $b(x_1') \le b(h_a)$ and $a(x_1') \ge a(h_f)$. Notice that

$$p_f = \left(b^-\right)^{-1}(h_f).$$

As function $h(\cdot)$ is strictly monotone increasing we have the inequality:

$$h(x_0) \le h(h_f) = h_f.$$

Hence

$$a^+(b^-(x_0)) \leq h(h_f) = a^+(b^-(h_f)) = h_f.$$

By using again the monotone decreasing property of function $a^+(\cdot)$ one obtains that

$$b^-(x_0) \geq b^-(h_f) = p_f.$$

Similarly

$$h_a = h(h_a)) = a^+(b^-(h_a)) \leq h((x_0) = a^+(b^-(x_0)).$$

It follows from here that

$$b^-(h_a) = p_a \geq b^-(x_0).$$

Thus x_1 can be kept in the interval $[h_a, p_a]$. Then it can be proven with the same method that x_2 can be kept in the interval $[h_a, p_a]$, etc. Q.E.D.

If the trajectory enters the interval $[h_f, p_f]$ then it can be kept even in this interval. Otherwise in every odd step it is in $[p_f, p_a]$ and in every even step in$[h_f, g_f]$.

The theorem does not exclude that the trajectory enters the interval $[h_f, p_f]$. If it does so and within the interval there is a similar structure then the trajectory can kept in an even narrower region. On the other hand if the realization x'_t is always the worst possible, i.e. $a(x_{t-1})$ if t is even and $b(x_{t-1})$ if t is odd then there are two cases. If h_f is an attractive fixed point of function $h(\cdot)$, i.e. for all $z \in (h_a, h_f)$ the inequality $h(z) > z$ holds, then trajectory with the control used in the proof converges to the interval $[h_f, p_f]$ in the sense that

$$\lim_{k\to\infty} x_{2k+1} = p_f, \quad \text{and} \quad \lim_{k\to\infty} x_{2k} = h_f.$$

In the case if h_f is a repelling fixed point, i.e. for all $z \in (h_a, h_f)$ the inequality $h(z) < z$ holds, then

$$\lim_{k\to\infty} x_{2k+1} = p_a, \quad \text{and} \quad \lim_{k\to\infty} x_{2k} = h_a.$$

If the condition $h_f < p_f$ does not hold then there are several cases and their discussion is omitted.

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