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Vertex- and edge-minimal and locally minimal graphs. [a]

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Vertex- and edge-minimal and locally minimal graphs. [1]

Endre Boros and Vladimir Gurvich

Abstract. Given a family of graphs $\mathcal{G}$ and a graph $G \in \mathcal{G}$, standardly, G is called **edge-minimal** (respectively, **vertex-minimal**) graph of $\mathcal{G}$ if $G' \notin \mathcal{G}$ for every subgraph (respectively, for every induced subgraph) G' of G; furthermore, G is called **locally edge-minimal** (respectively, **locally vertex-minimal**) graph of $\mathcal{G}$ if $G' \notin \mathcal{G}$ whenever G' is obtained from G by deleting of a vertex (respectively, of an edge). Similarly, the same concepts are introduced for the directed graphs (digraphs).

By the Strong Perfect Graph Theorem, the vertex-minimal graphs with $\chi > \omega$ are *odd holes and anti-holes.* In contrast, the **locally** vertex-minimal graphs with $\chi > \omega$ are *partitionable* graphs. It is also known that the (locally) **edge-minimal** imperfect graph are odd holes. Hence, by the perfect graph theorem, the odd anti-holes are the only (locally) edge-**maximal** imperfect graphs and C_5 is a unique imperfect graph that is (locally) edge-minimal and edge-maximal simultaneously. Similarly, we analyze the graphs with $\chi = \omega$ and also perfect graphs.

Somewhat surprisingly, there exist non-trivial perfect graphs that are **locally** edge-minimal and edge-maximal simultaneously. In other words, such a graph is perfect but it becomes imperfect after an edge is added to or deleted from it.

In this paper we consider vertex- and edge-minimal and locally minimal graphs in the following families: (i) perfect and imperfect graphs, (ii) graphs with $\chi = \omega$ and with $\chi > \omega$, (iii) digraphs than have a kernel and kernel-free digraphs, and finally, (iv) the complementary connected d-graphs.

Key words: Rotterdam graphs, odd holes, odd anti-holes, perfect graphs, imperfect graphs, locally edge-minimal, locally vertex-minimal, kernel, complementary connected

1 Introduction

1.1 Minimal and locally minimal Boolean vectors

Let us recall that x is a *minimal true vector* of a Boolean function f if $f(x) = 1$ and $f(x') = 0$ for every predecessor x' of x, that is, whenever $x' < x$. Furthermore, x is a *locally minimal* true vector of f if $f(x) = 1$ and $f(x') = 0$ for every immediate predecessor of x, that is, for each x' such that $x - x' = e_i$. Clearly, each minimal true vector is locally minimal and these two concepts are equivalent for the monotone Boolean functions, yet, not only for them, as we shall see below.

If a Boolean function is given by a polynomial oracle then local minimality of a given (true) vector can be tested in polynomial time. In contrast, verifying minimality can be NP-hard.

In this paper we consider Boolean functions related to the families of graphs mentioned in the abstract.

1.2 Vertex- and edge-minimal and locally minimal imperfect graphs

The Strong Perfect Graph Theorem (SPGT, conjectured by Berge in 1961 and proved by Chudnovsky, Robertson, Seymour, and Thomas in 2002, [8]) claims that every imperfect graph $G = (V, E)$ contains an odd hole or odd anti-hole. In other words, the latter are the only vertex-minimal imperfect graphs. To justify this reformulation we assign a Boolean variable to each vertex $v \in V$ and introduce a Boolean function $f : 2^V \to \{0, 1\}$ such that $f(W) = 1$ (true) if and only if the subgraph $G[W]$ induced by the subset $W \subseteq V$ is imperfect. Since an induced subgraph of a perfect graph is perfect, the Boolean function f is monotone in this example. Hence, the sets of its minimal and locally minimal true vectors coincide and we obtain the following characterization.

Theorem 1 *([8]). Odd holes and odd anti-holes are the only vertex-minimal and also the only locally vertex-minimal imperfect graphs.* □

Yet, a (not induced) subgraph of a perfect graph might be imperfect. For example, edges (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_1), and (v_1, v_3) form a perfect graph but by deleting the last edge we obtain C_5, which is imperfect. In other words, the corresponding Boolean function (whose variables are assigned to the edges rather than vertices) is not monotone. Nevertheless, the edge-minimal and locally edge-minimal imperfect graphs coincide. In 1972 Olaru proved that they both are odd holes.

Theorem 2 *([20, 21]). If $G = (V, E)$ is imperfect and $G_e = (V, E \setminus \{e\})$ is perfect for every $e \in E$ then G is an odd hole.* □

Let us note that odd anti-holes, except C_5, are not locally edge-minimal imperfect graphs, since one can delete an edge of an odd anti-hole and get a graph with an odd hole.

Let us also remark that in Theorem 2 we can substitute *kernel-solvability* for perfectness.

Theorem 3 *([6]). If $G = (V, E)$ is not kernel-solvable and $G_e = (V, E \setminus \{e\})$ is kernel-solvable for every $e \in E$ then G is an odd hole.* □

This is not surprising. Indeed, as it was conjectured in 1983 by Berge and Duchet [2, 3] a graph is perfect if and only if it is kernel-solvable. The "only if part" was proved in [5], while the "if part" follows from SPGT. However, the proof in [6] does not depend on SPGT.

Furthermore, by the Lovasz perfect graph theorem [18, 19], the complement to a perfect graph is perfect. This and Theorem 2 imply that both the edge-maximal and locally edge-maximal imperfect graphs are odd anti-holes. Hence, C_5 is a unique graph that is (locally) edge-minimal and -maximal simultaneously.

1.3 Locally edge-minimal and locally edge-maximal perfect graphs; Rotterdam graphs

Now let us consider the similar concepts for perfect rather than for imperfect graphs. Clearly, there is a unique (locally) vertex-minimal perfect graph, which consists of a single vertex. Indeed, by definition, an induced subgraph of a perfect graph is perfect.

It is also clear that the edge-minimal perfect graphs are edge-free. Indeed, an edge-free graph is perfect and it can be obtained from every graph (in particular, from a perfect one), defined on the same vertex set, just by deleting all edges. Respectively, the edge-maximal perfect graphs are complete. Yet, somewhat surprisingly, there exist non-trivial **locally** edge-minimal and -maximal (simultaneously) perfect graphs.

Remark 1 *This "paradox" can be easily resolved. Although perfect and imperfect graphs form two complementary families, and also, by the perfect graph theorem, the complement of a perfect (respectively, imperfect) graph is perfect (respectively, imperfect), yet, these two types of complementation are different.*

Let $k \geq 2$ be an integer and C_{2k} be a simple $2k$-cycle. Take two vertex-disjoint copies of it: $C = C_{2k}$ and $C' = C'_{2k}$ whose edges we denote by (v_i, v_{i+1}) and (v'_i, v'_{i+1}), respectively, where $i = 0, \ldots, 2k-1$, and we standardly assume that the sum is taken $\mod 2k$, that is, $i + 1 = 0$ if $i = 2k - 1$. Then, let us introduce $2k$ more vertices u_i, connect u_i to $v_i, v_{i+1}, v'_i, v'_{i+1}$, for $i = 0, \ldots, 2k-1$, and denote the obtained graph by G_k. We will show that G_k is a locally edge-minimal and simultaneously locally-edge-maximal perfect graph.

Theorem 4 *Graph G_k is perfect but it becomes imperfect whenever we delete from it or add to it any edge.*

It is assumed, however, that we can only add an edge that connects two already existing vertices, otherwise the obtained graph would remain perfect.

We will prove this theorem in Section 2. Now let us say a few more words about graph G_k. It has $6k$ vertices and $12k$ edges. Furthermore, G_k is planar, since it can be represented as the graph of vertices and edges of the following 3-dimensional polytope P_k.

Let us locate $2k$ squares in 3-dimensional space, so that one diagonal of each square is vertical, while the other one is horizontal and these $2k$ horizontal diagonals form perfect $2k$-gon. We get $6k$ vertices laying in three parallel planes: upper, lower, and middle, $2k$ in each. Let us note that the upper $2k$-gon is exactly above the lower one but the middle one is slightly larger than these two and it is turned with respect to them by angle $\pi/2k$. We define P_k as the convex hull of these $6k$ vertices. It is easy to see that P_k has $6k$ vertices, $12k$ edges, and $6k+2$ facets: $2k$ squares, $4k$ triangles, and two $2k$-gons. Let us also notice that $4k$ triangles have no common edges; in fact, every edge of P_k belongs to a triangle and even facet: a square or $2k$-gon.

Remark 2 *Although in the world there is no building of this shape, yet, Cubic Houses of Rotterdam are somewhat similar. So we will call P_k the* Rotterdam polytopes *and G_k* Rotterdam graphs*. We shall prove that the latter are perfect but become imperfect whenever a line is added to or removed from them - an attractive feature for architecture.*

1.4 Minimal and locally minimal graphs satisfying $\chi > \omega$ and $\chi = \omega$

Given a graph $G = (V, E)$, standardly $\chi = \chi(G)$ and $\omega = \omega(G)$ denote its chromatic and clique numbers, respectively. Recall that χ is the minimum number of colors in a proper coloring of G and ω is the number of vertices of a maximum clique of G. Obviously, $\chi(G) \geq \omega(G)$ for every G. However, unlike perfectness, the equality $\chi = \omega$ (as well as the inequality $\chi > \omega$) is not hereditary: it may hold for G and fail for its induced subgraph.

Theorem 5 *([8]). Every graph with $\chi > \omega$ contains an odd hole or anti-hole as an induced subgraph.* □

In other words, the vertex-minimal graphs with $\chi > \omega$ are the odd holes and anti-holes. This is just another reformulation of SPGT. However, the corresponding family of the locally minimal graphs is wider.

Theorem 6 *([4, 7]). The locally vertex-minimal graphs with $\chi > \omega$ are partitionable graphs.* □

In other words, $\chi(G) > \omega(G)$ and $\chi(G[V \setminus \{v\}]) = \omega(G)$ for each $v \in V$ if and only if G is partitionable. This is just one of many equivalent characterizations of partitionable graphs. The claim follows easily from the results of [4] and it is explicit in [7].

Remark 3 *The above characterization of minimality is based on SPGT, which is very difficult, while in case of local minimality the characterization is simple. In contrast, for many problems it is easier to characterize the minimal rather than locally minimal true vectors.*

However, verification of local minimality of a given (true) vector is always polynomial, if only the corresponding Boolean function is given by a polynomial oracle, while it might be NP-hard to verify the minimality of a given (true) vector.

Let us also mention that, although many equivalent characterizations of the partitionable graphs are known, yet, their structure is complicated and not well understood. For example, the fact that each partitionable graph contains an odd hole or anti-hole is equivalent to SPGT.

Now let us assign Boolean variables to the edges (rather than vertices) of G.

Proposition 1 *An edge-minimal graph with $\chi > \omega$ is an odd hole plus k isolated vertices, where $k \geq 0$.*

Proof . Indeed, each graph with $\chi > \omega$ is imperfect. Hence, by SPGT, it contains an odd hole or anti-hole. In both cases it contains a simple odd cycle of length at least 5 as a (not necessarily induced) subgraph. Deleting all other edges we obtain an odd hole and some number (maybe, 0) of isolated vertices. □

In particular, odd holes are also *locally* edge-minimal graphs with $\chi > \omega$. Yet, there are many others, for example, odd wheels $W_{2\ell+1}$ with $\ell \geq 2$. Let us recall that $W_{2\ell+1}$ consists of an odd cycle $C_{2\ell+1}$ and one more vertex v_0 of degree $2\ell + 1$. It is easy to see that $4 = \chi(W_{2\ell+1}) > \omega(W_{2\ell+1}) = 3$, yet, after deleting of any edge we obtain $\chi(W') = \omega(W') = 3$ for the remaining graph W'. In general, it seems difficult to characterize or recognize the locally edge-minimal graphs with $\chi > \omega$.

Now let us consider the graphs satisfying equality $\chi = \omega$.

Proposition 2 *The only (locally) vertex-minimal graph with $\chi = \omega$ is trivial; it consists of a single vertex.*

Proof . Fix an ω-clique K in G and delete all other vertices in an arbitrary order. Thus, G is reduced to K. Then delete the vertices of K until only one remains. In is easy to see that in both phases the equation $\chi = \omega$ holds. □

Proposition 3 *The (locally) edge-minimal graphs with $\chi = \omega$ are edge-free.*

Proof . **Step 1**. Fix a maximum clique K in G and delete all other edges in an arbitrary order. Thus, G is reduced to K and k isolated vertices, where $k \geq 0$. **Step 2**. Delete an edge of K and return to Step 1. Repeat. Obviously, the equation $\chi = \omega$ holds in all steps. □

1.5 Edge- and vertex-minimal and locally minimal digraphs with and without kernels

Given a directed graph $G = (V, E)$, let us recall that a *kernel* is defined as an independent and dominating subset $K \subseteq V$. First, let us consider the kernel-free digraphs. The following simple sufficient conditions for the existence of a kernel was given by Richardson in 1953.

Theorem 7 *([22]). A digraph G has a kernel if all its (simple) directed cycles are even.* □

The original proof was simplified in 1965 by Harary, Norman, and Cartwright [17] and then in 1990 by Berge and Duchet [3]. Obviously, we can reformulate Theorem 7 as follows:

any edge-minimal kernel-free digraph is a simple directed odd cycle with k isolated vertices, where $k \geq 0$.

Of course, these digraphs are also *locally* edge minimal.

In 1980 Duchet conjectured that there are no others [9]. More precisely, if $G = (V, E)$ is a kernel-free digraph and $G_e = (V, E \setminus \{e\}$ has a kernel for every arc $e \in E$ then G is a vertex-disjoint union of simple directed odd cycles and, maybe, isolated vertices. Obviously, if true, this claim would strengthen Richardson's theorem. However, it was shown in [1] that the circulant $G_{43}(1, 7, 8)$ with 43 vertices is a counter-example.

Let us recall that a *circulant* $G_n(\ell_1, \ldots, \ell_q) = (V, E)$ is a digraph with n vertices, $V = [n] = \{1, \ldots, n\}$, and qn arcs, $E = \{(i, i + j) \mid i \in [n],\ j \in \{\ell_1, \ldots, \ell_q\}\}$.

In [1], it was shown that circulant $G_n(1, 7, 8)$ has a kernel if and only if $n \equiv 0 \mod 3$ or $n \equiv 0 \mod 29$. In particular, circulant $G_{43}(1, 7, 8)$ is kernel-free. Yet, a kernel appears whenever an arc of this circulant is deleted. Due to circular symmetry, it is sufficient to consider only three cases and delete one of the arcs $(43, 1), (43, 7)$, or $(43, 8)$. It is easy to verify that, respectively, the following three subsets become kernels:

$K_1 = \{1, 5, 10, 14, 16, 19, 25, 28, 30, 34, 39, 43\}$,
$K_7 = \{7, 9, 11, 13, 22, 24, 26, 28, 37, 39, 41, 43\}$,
$K_8 = \{3, 5, 8, 14, 17, 19, 23, 28, 32, 34, 37, 43\} \subseteq \{1, \ldots, 43\} = V$.

Thus, the set of edge-minimal kernel-free digraphs is a proper subset of the locally edge-minimal ones. It seems that the latter family, unlike the former one, is difficult to characterize. In particular, it is not known whether a circulant $G_n(\ell_1, \ell_2)$ can be a locally edge-minimal kernel-free digraph. At least, it cannot unless $n > 1,000,000$, [1].

Similarly, it seems difficult to characterize or recognize the *locally vertex*-minimal kernel-free digraphs. For example, circulant $G_{16}(1, 7, 8)$ is kernel-free, since $16 \not\equiv 0 \mod 3$ and $16 \not\equiv 0 \mod 29$. Yet, a kernel appears whenever we delete a vertex. Indeed, due to circular symmetry, without loss of generality, we can delete "the last" vertex, 16, and verify that subset $\{1, 3, 5, 7\}$ becomes a kernel.

On the other hand, the vertex-minimal kernel-free digraphs are easy to characterize.

Proposition 4 *The only vertex-minimal kernel-free digraphs are odd directed cycles.*

Proof . Indeed, by the Richardson theorem, each kernel-free digraph contains such a cycle, which is kernel-free but each its induced subgraph is acyclic and, hence, it has a kernel. □

Now let us consider digraphs with kernels. The family of (locally) edge-minimal such digraphs is trivial.

Proposition 5 *The only (locally) edge-minimal digraph with a kernel is edge-free.*

Proof . Let $G = (V, E)$ be a digraph with a kernel K. **Step 1**. Leave in it only the arcs going from K to $V \setminus K$ and delete one by one all others. **Step 2**. Then for each vertex $v \in V \setminus K$ of in-degree greater than 1, if any, delete one by one all coming arcs but one. **Step 3**. Then delete one by one the remaining arcs in an arbitrary order. It is clear that in each step, after every deleted arc the remaining graph still has a kernel. □

The family of the vertex-minimal digraphs that have kernels is trivial, too.

Proposition 6 *The only (locally) vertex-minimal digraph with a kernel is a single vertex.*

Proof . Let $G = (V, E)$ be a digraph with a kernel K. **Step 1**. Delete one by one the vertices of $V \setminus K$. **Step 2**. Delete one by one the vertices of K but one. Clearly, in each step, after every deleted vertex the remaining graph still has a kernel. □

1.6 Complementary connected d-graphs

A d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ is a complete graph whose edges are colored by d colors $I = [d] = \{1, \ldots, d\}$, or in other words, are partitioned in d subsets some of which might be empty. For example, we call $\mathcal{G}$ a 2- *or* 3-*graph* if $\mathcal{G}$ has only 2, respectively, 3, non-empty chromatic components. The following 2-graph Π and 3-graph Δ given in Figure 1 will play an important role:

$\Pi = (V; E_1, E_2)$, where
$V = \{v_1, v_2, v_3, v_4\}$; $E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$, and $E_2 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$;
$\Delta = (V; E_1, E_2, E_3)$, where
$V = \{v_1, v_2, v_3\}$, $E_1 = \{(v_1, v_2)\}$, $E_2 = \{(v_2, v_3)\}$, and $E_3 = \{(v_3, v_1)\}$.

Remark 4 *Note that both chromatic components of* Π *are isomorphic to* P_4 *and that* Δ *is a three-colored triangle.*

Let us also notice that, formally, d-graphs $\Pi(d)$ *(respectively,* $\Delta(d)$*) are defined for every fixed* $d \geq 2$ *(respectively,* $d \geq 3$*) and* $d-2$ *(respectively,* $d-3$*) of their chromatic components are empty. Yet, we will omit argument d assuming that it is a fixed parameter.*

Both d-graphs Π *and* Δ *were first considered in 1967 by Gallai in his fundumental paper [10]. In particular,* Δ*-free d-graphs are called Gallai's graphs. Furthermore,* Π*- and* Δ*-free d-graphs have important applications in theory of positional games [13, 14, 15].*

Figure 1: 2-graph Π and 3-graph Δ.

We say that a d-graph $\mathcal{G}$ is *complementary connected* (CC) if the complement to each chromatic component of $\mathcal{G}$ is connected on V, or in other words, if for each two vertices $u, w \in V$ and color $i \in I = \{1, \dots, d\}$ there is a path between u and w without edges of E_i.

A CC d-graph $\mathcal{G}$ with $d \geq 1$ will be called a *CC-graph.*

By definition, the one-vertex d-graph is not a CC-graph, since $d = 0$. Clearly, there is no CC-graph with two vertices. It is easy to verify that Δ (respectively, Π) is a unique CC-graph with three (respectively, four) vertices. It is also easy to see that Π and Δ are minimal CC-graphs, that is, they do not contain non-trivial induced CC-subgraphs. The next statement shows that, except Π and Δ, no other d-graph has this property.

Theorem 8 *([13, 15, 16]). Every CC-graph contains an induced* Π *or* Δ*.*

Furthermore, we shall strengthen this result and prove that Π and Δ are the only locally vertex-minimal CC-graphs.

Theorem 9 *Every CC-graph* $\mathcal{G}$*, except* Π *and* Δ*, has a vertex* $v \in V$ *such that the induced subgraph* $\mathcal{G}[V \setminus \{v\}]$ *is still CC.*

In other words, not only every CC-graph $\mathcal{G}$ contains a Π or Δ but also $\mathcal{G}$ can be reduced to it by successive elimination of vertices in such a way that all intermidiate d-graphs are CC. This claim was announced in [15]; in Section 3 we will proof both thorems.

2 Proof of Theorem 4

Part 1. It is easy to notice that an odd hole, C_5 or C_{2k+1}, appears whenever we delete an edge of G, because, as we already mentioned, each edge belongs to a triangle and a chordless even cycle: C_4 or C_{2k}.

Part 2. Let us show that G_k is perfect. By the Strong Perfect Graph Theorem, [8], it is sufficient to show that G_k has no odd holes and anti-holes. It is more or less obvious that G_k contains no odd anti-hole of length greater than 6 and we leave the proof to the reader. Let us show that G_k contains no odd holes. Let C be a cycle of G_k. It may outline a facet of P_k: one of $4k$ triangles, $2k$ squares, or two $2k$-gons. In all these cases C is not an odd hole. Let C be a chordless "non-facet" cycle. Then it can be partitioned in $4m$ simple paths: $p_1, \ldots, p_{2m-1}$ in the upper $2k$-gon of P_k and $p_2, \ldots, p_{2m}$ in the lower one, and $2m$ connecting paths $p_{1,2}, p_{2,3}, \ldots, p_{2m-1,2m}, p_{2m,1}$. For each $i = 1, \ldots, 2m$ the path p_i contains between 0 (in this case p_i consists of a single vertex) and $2k - 2$ edges (in this case $m = 1$). Furthermore, each connecting path consists of two edges that form a right or obtuse angle in the polytope P_k. It is easy to see that if at least one of these angles in C is right then $m = 1$ and the other angle must be right too (otherwise C has a chord). Yet, in this case C is even. It is also not difficult to see that C is even whenever all angles are obtuse. Several examples are given in Figures 2 and 3.

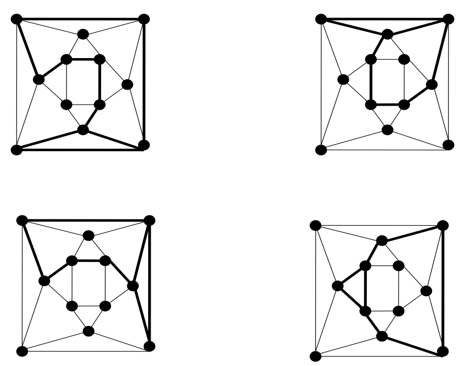

Figure 2: Rotterdam graph G_2; even chordless cycles and odd cycles with chords, m=1.

Part 3. A simple case analysis shows that an odd hole appears whenever we add an edge to G_k. Let us recall the Rotterdam polytope P_k. Its $6k$ vertices lie in three parallel layers (planes), $2k$ in each. Denote these three sets of vertices by L_1, L_2, and L_3. Due to symmetry of L_1 and L_3, it is sufficient to consider the following four types of chords: (L_1, L_1), (L_1, L_2), (L_1, L_3), and (L_2, L_2) It is easy to show that in each case an odd hole does appear; examples are given in Figure 4. We leave complete case analysis to the reader. □

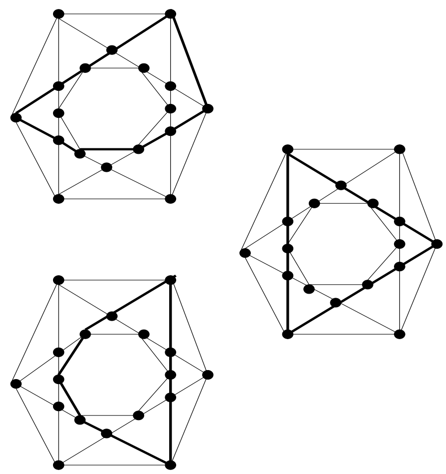

Figure 3: Rotterdam graph G_3; even holes, m=2 and m=3.

3 Proof of Theorems 8 and 9

Given a Π- and Δ-free d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$, we will show that it is not CC, that is, the graph $G_i = (V, \overline{E_i}) = (V, \cup_{j \neq i} E_j)$ is not connected for some $i \in I$. (In fact, there is exactly one such $i \in I$; see [16]).

Let us assume indirectly that $\mathcal{G}$ is CC and also Π- and Δ-free. Then $\mathcal{G}$ has the following property.

Lemma 1 *For each edge $(v', v'') \in E_i$ there exist a vertex $v \in V$ such that $(v, v'), (v, v'') \in E_j$ for some $j \neq i$.*

Proof . Since v', v'', and v cannot form a Δ, it would suffice to show that $(v, v'), (v, v'') \notin E_i$. Since $\mathcal{G}$ is complementary connected, there exists a path between v' and v'' that contains no edge from E_i. Let p be a shortest such path. Then each chord of p is of color i. Let ℓ be the length (that is, the number of edges) of p. Clearly, $\ell \neq 1$, because $(v', v'') \in E_i$. If $\ell = 2$ then $p = \{(v', v), (v, v'')\}$ and we are done. Let us show that if $\ell \geq 3$ then $\mathcal{G}$ contains a Π or Δ. Indeed, if p is monochromatic then a Π exists. Otherwise, p contains two successive edges of distinct colors, say, $(v_1, v_2) \in E_{i_1}$ and $(v_2, v_3) \in E_{i_2}$, where $i_1 \neq i_2$. Obviously, $i_1 \neq i$ and $i_2 \neq i$, since p contains no edges of color i. Thus, v_1, v_2, v_3 form a Δ. □

Now we proceed with the proof of Theorem 8 as follows.

Let $(v_{j_0}, v_{j_1}) \in E_{i_1}$. By Lemma 1, there exists $v_{j_2} \in V$ and $i_2 \in I$ such that $i_2 \neq i_1$ and $(v_{j_0}, v_{j_2}), (v_{j_1}, v_{j_2}) \in E_{i_2}$. Furthermore, since $(v_{j_1}, v_{j_2}) \in E_{i_2}$, by Lemma 1, there exists $v_{j_3} \in V$ and $i_3 \in I$ such that $(v_{j_1}, v_{j_3}), (v_{j_2}, v_{j_3}) \in E_{i_3}$ and $i_3 \neq i_2$, though $i_3 = i_1$ may hold.

Obviously, $v_{j_3} \neq v_{j_2}$ and $v_{j_3} \neq v_{j_1}$, by construction. It is also clear that $v_{j_3} \neq v_{j_0}$, because $(v_{j_0}, v_{j_2}) \in E_{i_2}$, while $(v_{j_3}, v_{j_2}) \in E_{i_3}$ and $i_3 \neq i_2$.

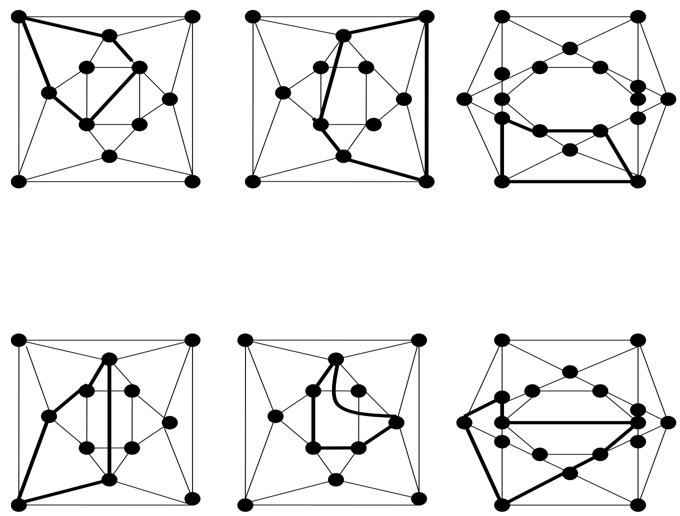

Figure 4: Rotterdam graphs with one extra edge; odd holes appear in all cases.

We will show that $(v_{j_3}, v_{j_0}) \in E_{i_3}$ too. Let us consider two cases: $i_3 = i_1$ and i_3 is distinct from both i_1 and i_2. If $i_3 = i_1$ then (v_{j_0}, v_{j_3}) must be of color i_1 too. Indeed, if $(v_{j_0}, v_{j_3}) \in E_{i_2}$ then all four vertices form a Π; if $(v_{j_0}, v_{j_3}) \in E_{i_4}$, where $i_4 \neq i_1$ and $i_4 \neq i_2$, then $(v_{j_0}, v_{j_2}, v_{j_3}$ form a Δ. If $i_3 \neq i_1$ and $i_3 \neq i_2$ then (v_{j_0}, v_{j_3}) must be in E_{i_3} too. Indeed, if $(v_{j_0}, v_{j_3}) \in E_{i_1}$ then $v_{j_0}, v_{j_2}, v_{j_3}$ form a Δ; if $v_{j_0}, v_{j_3}) \in E_{i_2}$ then $(v_{j_0}, v_{j_1}, v_{j_3}$ form a Δ; finally, if $(v_{j_0}, v_{j_3}) \in E_{i_4}$, where where $i_4 \neq i_1$ and $i_4 \neq i_2$, then both above triangles form Δs.

Furthermore, since $(v_{j_2}, v_{j_3}) \in E_{i_3}$, by Lemma 1, there exists $v_{j_4} \in V$ such that and $i_4 \in I$ such that $(v_{j_2}, v_{j_4}), (v_{j_3}, v_{j_4}) \in E_{i_4}$ and $i_4 \neq i_3$, although i_4 may coincide with i_1 or i_2; etc. We will show that V cannot be finite.

More precisely, we will prove by induction that for each k there is a sequence of vertices $v_{j_0}, v_{j_1}, \ldots, v_{j_{k-1}}, v_{j_k}$ and colors $i_1, i_2, \ldots, i_{k-1}, i_k$ such that:

(i) all vertices are pairwise distinct;

(ii) every two successive colors are distinct, that is, $i_m \neq i_{m+1}$ for every $m = 1, 2, \ldots, k-1$;

(iii) every vertex is connected by the same color to all preceding vertices, that is, $(v_{j_k}, v_{j_m}) \in E_{i_k}$ whenever $k > m$.

Suppose that we already got such vertices $\{v_{j_0}, v_{j_1}, \ldots, v_{j_{k-1}}\}$ and colors $\{i_1, i_2, \ldots, i_{k-1}\}$ for $k-1$. Since $(v_{j_{k-2}}, v_{j_{k-1}}) \in E_{i_{k-1}}$, by Lemma 1, there is a vertex $v_{j_k} \in V$ such that

$(v_{j_{k-2}}, v_{j_k})$, $(v_{j_{k-1}}, v_{j_k}) \in E_{i_k}$, where $i_k \neq i_{k-1}$. First, let us show that v_{j_k} is distinct from all preceding vertices, that is, $v_{j_k} = v_{j_m}$ for no $m < k$. Indeed, by the induction hypothesis, $(v_{j_{k-1}}, v_{j_m}) \in E_{i_{k-1}}$, while, by construction, $(v_{j_{k-1}}, v_{j_k}) \in E_{i_k}$ and $i_k \neq i_{k-1}$. Hence, $v_{j_k} \neq v_{j_m}$.

Now, let us prove that $(v_{j_k}, v_{j_m}) \in E_{i_k}$ for all $m < k$. Indeed, for $m = k-1$ and $m = k-2$ this holds by construction. Given $m < k-2$, let us consider four vertices $v_{j_{k-2}}, v_{j_{k-1}}, v_{j_k}$ and v_{j_m}. They are connected by six edges five of which are colored as follows: $(v_{j_{k-2}}, v_{j_k}), (v_{j_{k-1}}, v_{j_k}) \in E_{i_k}$, by construction; $(v_{j_{k-2}}, v_{j_{k-1}})$, $(v_{j_m}, v_{j_{k-1}}) \in E_{i_{k-1}}$, and $(v_{j_m}, v_{j_{k-2}}) \in E_{i_{k-2}}$, by the induction hypothesis.

Let us show that $(v_{j_m}, v_{j_k}) \in E_{i_k}$. We know that $i_k \neq i_{k-1} \neq i_{k-2}$, though i_k and i_{k-2} may coincide. If they do then $(v_{j_m}, v_{j_k}) \in E_{i_k}$. Indeed, if $(v_{j_m}, v_{j_k}) \in E_{i_{k-1}}$ then all four vertices, $v_{j_{k-2}}, v_{j_{k-1}}, v_{j_k}$, and v_{j_m}, form a Π; if $(v_{j_m}, v_{j_k}) \in E_{i_\ell}$ where $i_\ell \neq i_k$ and $i_\ell \neq i_{k-1}$ then $v_{j_{k-1}}, v_{j_k}$, and v_{j_m} form a Δ.

Now, let us suppose that $i_k \neq i_{k-2}$ and show that again $(v_{j_m}, v_{j_k}) \in E_{i_k}$. Indeed, if $(v_{j_m}, v_{j_k}) \in E_{i_{k-2}}$ then $v_{j_{k-1}}, v_{j_k}$, and v_{j_m} form a Δ; if $(a_{j_m}, a_{j_k}) \in E_{i_\ell}$ where $i_\ell \neq i_k$ and $i_\ell \neq i_{k-2}$, then $a_{j_{k-2}}, a_{j_k}$, and a_{j_m} form a Δ.

Finally, let us note that for any fixed k the d-graph induced by $V_k = \{v_{j_0}, v_{j_1}, \ldots, v_{j_{k-1}}, v_{j_k}\}$ is not complementary connected, just because v_{j_k} is an isolated vertex in $G_k = (V_k, \overline{E_{i_k}})$. Thus, V cannot be finite and we get a contradiction. □

Remark 5 *In fact, we proved a little more than Theorem 8 claims. Let us denote by $\mathcal{G}_\infty$ the family of infinite d-graphs that, for some numbering of their vertices, satisfy all properties (i), (ii), and (iii) mentioned above It is easy to see that each $\mathcal{G} \in \mathcal{C}_\infty$ is complementary connected, though each finite subgraph of $\mathcal{G}$ is not. Let us mention that if $d = 2$ then $\mathcal{G}_\infty$ contains only two graphs, since in this case the two involved colors must alternate.*

Our arguments show that each CC-graph (finite or infinite) must contain a Π, or Δ, or an infinite subgraph from the family $\mathcal{G}_\infty$.

Remark 6 *The proof of Theorem 8 was given in [13]. The statement appears without proof in [15]. The case $d = 2$ is a little simpler than the general one, since Δ cannot exist when $d \leq 2$. This case was considered earlier, in [24, 25, 23, 12, 13, 15]. It was also suggested as a problem for Moscow Mathematical Olympiad in 1971 (Problem 72 in [11]) and was successfully solved by five high school students.*

Now let us prove Theorem 9. Given a CC-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$, we will show that either $\mathcal{G}$ is Π or Δ, or there is a vertex $v \in V$ such that the induced subgraph $\mathcal{G}[V \setminus \{v\}]$ is still CC.

As we already mentioned, such a vertex does not exist for Π and Δ, that is, they are minimal CC-graphs. Moreover, by Theorem 8 there are no others. Theorem 9 strengthen these claims further stating that Π and Δ are the only *locally* minimal CC-graphs.

Remark 7 *Since, by definition, d-graphs are complete, the concept of edge-minimality makes no sense for them and we restrict ourselves to vertex-minimality.*

Our proof is based on counting cut-vertices. Let us recall that $v \in V$ is a *cut-vertex* of a given connected graph $G = (V, E)$ if the induced subgraph $G[V \setminus \{v\}]$ is not connected.

Lemma 2 *Let $G = (V, E)$ be a connected graph with n vertices, m edges ($|V| = n$, $|E| = m$), and k cut-vertices. Then $0 \le k \le n-2$ and $m \le \binom{n-k}{2} + k$.*

Proof . Clearly, reducing a graph to its spanning tree one does not reduce the set of its cut-vertices. It is also clear that between all trees with n vertices, the maximum number of the cut-vertices, $k = n-2$, has the simple path.

Furthermore, given an integral k such that $0 \le k \le n-2$, let us introduce graph G_k with n vertices that consists of a clique on $n-k$ vertices and a simple path with $k+1$ vertices (and k edges) one of whose terminal vertices is in the clique, while all others are not. Clearly, each vertex of this path, except for one terminal vertex (which is not in the clique) is a cut-vertex of G_k. Hence, G_k has k cut-vertices and $\binom{n-k}{2} + k$ edges.

Let us prove that no graph with n vertices and k cut-vertices can have more edges.

Indeed, let G be a connected graph with n vertices, m edges, k cut-vertices, and ℓ 2-connected components. Furthermore, let $n_1, \ldots, n_\ell$ denote the numbers of vertices in these components. Then, clearly,

$$\ell \ge k+1, \text{ and } \sum_{j=1}^{\ell} n_j = n + \ell - 1, \text{ and } a_j \ge 2 \text{ for } j = 1, \ldots, \ell.$$

The following chain of inequalities proves the lemma.

$$m \le \sum_{j=1}^{\ell} \binom{n_j}{2} \le \binom{n-\ell+1}{2} + (\ell-1)\binom{2}{2} \le \binom{n-k}{2} + k.$$

The first one is obvious, while the last two result from the following simple implication:

$$a \ge b \ge 2 \;\Rightarrow\; \binom{a}{2} + \binom{b}{2} < \binom{a+1}{2} + \binom{b-1}{2}$$

□

This lemma shows that the more cut-vertices the less edges there are in G. Now let us assume that $\mathcal{G}$ is a locally vertex-minimal CC-graph and prove that $\mathcal{G}$ is either Π or Δ.

As before, let $G_i = (V, \overline{E}_i) = (V, \cup_{j \ne i} E_j)$ be the complement of the ith chromatic components of $\mathcal{G}$ and let k_i be the number of cut-vertices in G_i. By definition,

$$\sum_{i=1}^{d} k_i \ge n, \tag{3.1}$$

since $\mathcal{G}$ is a locally minimal CC-graph. By Lemma 2,

$$1 \le k_i \le n-2 \;\; \forall i \in I = [d] = \{1, \ldots, d\}. \tag{3.2}$$

On the other hand, equality

$$\sum_{i=1}^{d} m_i = (d-1)\binom{n}{2}, \tag{3.3}$$

obviously, holds for every d-graph $\mathcal{G}$. This and Lemma 2 imply the inequality

$$\sum_{i=1}^{d}[\binom{n-k_i}{2} + k_i] \geq (d-1)\binom{n}{2}. \tag{3.4}$$

We will prove that (3.1) and (3.4) can hold only for Π and Δ and, hence, they are the only locally minimal CC-graphs.

First, let us notice that for both, Π and Δ, the equality holds in (3.1) and (3.4).

Indeed, for Π we have: $n = 4$, $k_1 = k_2 = 2$ and $k_3 = \ldots = k_d = 0$ whenever $d > 2$; furthermore, $m_1 = m_2 = 3$ and $m_3 = \ldots = m_d = \binom{4}{2} = 6$ whenever $d > 2$.

For Δ we have: $n = 3$, $k_1 = k_2 = k_3 = 1$ and $k_4 = \ldots = k_d = 0$ whenever $d > 3$; furthermore, $m_1 = m_2 = m_3 = 2$ and $m_4 = \ldots = m_d = \binom{3}{2} = 3$ whenever $d > 3$.

It is easy to verify that (3.1) and (3.4) hold with equality in both cases.

Furthermore, without loss of generality, we can make the following assumptions.

The d sets of cutting vertices form a *minimal* set-cover of V. Indeed, they must form a set-cover, since otherwise d-graph $\mathcal{G}$ is not complementary connected. Moreover, if this set-cover is not minimal then we can assume that the corresponding superfluous chromatic components are empty (respectively, their complements are the complete graphs on V) and reduce d to d' deleting all these components. Obviously, such a reduction respects (3.1) - (3.4). It is also clear that now $n \geq d'$. Then, we can assume that d' sets of cutting vertices form a partition (not just minimal set-cover) of V, or in other words, that (3.1) holds with equality. Indeed, by Lemma 2, the more is k_i the less is the upper bound for m_i in (3.4).

By simple computations, it is easy to verify that (3.1) - (3.4) imply

$$\sum_{i=1}^{d} k_i^2 \geq n(n-2). \tag{3.5}$$

If $d' = 2$ then $k_1^2 + k_2^2 \geq n(n-2)$, where $1 \leq k_1 \leq n-2$, $1 \leq k_2 \leq n-2$, by Lemma 2, and $k_1 + k_2 = n$. From this we derive successively that $k_1 \geq 2$, $k_2 \geq 2$, $n \geq k_1 k_2$, $n \leq 4$, and, finally, that Π is a unique solution.

If $d' = 3$ then $k_1^2 + k_2^2 + k_3^2 \geq n(n-2)$, where $1 \leq k_i \leq n-2$, by Lemma 2, and $k_1 + k_2 + k_3 = n$. Since $1 + 1 + (n-2)^2 \geq n(n-2)$ implies that $n \leq 3$, we conclude that Δ is a unique solution.

Finally, if $d' > 3$ then $(d-1) + (n-d+1)^2 \geq n(n-2)$ and, hence, $n \leq \frac{d'(d'-1)}{2((d'-2)} < d'$ in contradiction to $n \geq d'$. Thus, Π and Δ are the only two solutions of (3.1) - (3.4). $\square$

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