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WHOSE MAXIMAL CLIQUES AND  
STABLE SETS INTERSECT, EXCEPT A  
UNIQUE PAIR <sup>1</sup>

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# ON SPLIT GRAPHS AND GRAPHS WHOSE MAXIMAL CLIQUES AND STABLE SETS INTERSECT, EXCEPT A UNIQUE PAIR <sup>1</sup>

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**Abstract.** A *CIS-graph* is defined as a graph whose every maximal clique and stable set intersect. These graphs have many interesting properties, yet, it seems difficult to obtain an efficient characterization and/or polynomial-time recognition algorithm for CIS-graphs. An *almost CIS-graph* has a unique pair  $(C, S)$  of disjoint maximal clique  $C$  and stable sets  $S$ . We conjecture that almost CIS-graphs are exactly split graphs that have a unique split partition. We prove this conjecture for a large hereditary class of graphs that contains, for example, chordal graphs and  $P_5$ -free graphs, as well as their complements, etc. We also prove the conjecture in case  $|C| = |S| = 2$  and show that the vertex-set  $R = V \setminus (C \cup S)$  cannot induce a threshold graph, although we did not prove that  $R = \emptyset$ , as the conjecture suggests.

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# 1 Introduction

By definition [1], every maximal clique  $C$  and every maximal stable set  $S$  of a *CIS-graph*  $G$  intersect. In this case we also say that  $G$  has the *CIS-property*. Otherwise, clearly, there is a disjoint pair  $(C, S)$  in  $G$ , which is called a *non-CIS-pair*. The above characterization of CIS-graphs is simple but not efficient, since the numbers of maximal cliques and stable sets of a graph can be exponentially large. CIS-graphs were considered by Zang [5], Deng, Li, and Zang [3], and Andrade, Boros, and Gurvich [1]. Some necessary and some sufficient conditions were obtained for the CIS-property to hold, yet, it seems difficult to obtain an efficient characterization or polynomial recognition algorithm for CIS-graphs. A similar class of graphs, which might have much simpler structure, was also introduced in [1].

**Definition 1** *An almost CIS-graph has a unique non-CIS-pair  $(C, S)$ .*

It seems that almost CIS-graphs are closely related to the following simple and well-known class of graphs. A *split graph*  $G$  admits a partition  $A \cup B = V(G)$ , a *split partition*, such that  $A$  is a clique and  $B$  is a stable set. Split graphs are exactly  $(2K_2, C_4, C_5)$ -free graphs according to the result of Foldes and Hammer [4]. A split graph may have several split partitions. For example, Bull (called also A-graph) in Figure 1 has two split partitions, namely  $A = \{a, b, e\}$ ,  $B = \{c, d\}$  and  $A' = \{a, b\}$ ,  $B' = \{c, d, e\}$ .

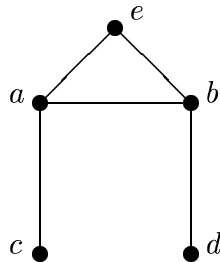


Figure 1: Bull.

If we delete vertex  $e$ , then we obtain the split graph  $P_4$ , which has a unique split partition  $A = \{a, b\}$ ,  $B = \{c, d\}$ . The sets  $A$  and  $B$  are a maximal clique and stable set, respectively, and they are disjoint. It is easy to verify that  $P_4$  is an almost CIS-graph, while Bull is a CIS-graph. The following two claims generalize above simple observations.

**Proposition 1** [1] *Every split graph has at most one non-CIS pair.*

For completeness we reproduce a simple prove from [1].

**Proof.** Let  $(A, B)$  be a split partition of a split graph  $G$ , where  $A$  is a clique and  $B$  is a stable set. Obviously, a maximal clique  $C$  distinct from  $A$  consists of a proper subset of  $A$  and one vertex  $u \in B$ ; respectively, a maximal stable set  $S$  distinct from  $B$  consists of a proper subset of  $B$  and one vertex  $v \in A$ . It is easy to see that  $C \cap S = \{u\}$  if  $u$  and  $v$  are non-adjacent, and  $C \cap S = \{v\}$  otherwise.  $\square$

In other words, every split graph is either CIS or almost CIS. The next claim shows when the first option takes place.

**Proposition 2** *A split graph  $G$  has more than one split partition if and only if  $G$  is a CIS-graph.*

**Proof.** Let  $A \cup B$  be a split partition of  $G$ . By Proposition 1,  $(A, B)$  is the only possible non-CIS-pair  $(C, S)$  in  $G$ . If, indeed,  $(A, B)$  is such a pair then  $G$  is an almost CIS-graph, by the definition. If not, then either clique  $A$  or stable set  $B$  is not maximal. In this case  $G$  is a CIS-graph.  $\square$

Thus, every split graphs with a unique split partition is an almost CIS-graph. It was conjectured in [1] that the inverse claim holds too.

**Conjecture 1** *Every almost CIS-graph is a split graph with a unique split partition.*

In other words, every non-split graph has at least two non-CIS pairs. In contrast, by Proposition 1, split graphs have at most one. Somewhat surprisingly, this simple would be characterization of split graphs is not obvious (and, perhaps, even not true). Here we obtain partial results in its support. In particular, we show that it holds for a hereditary class that contains many known extensions of cographs and split graphs, for example,  $P_5$ -free graphs, chordal graphs,  $C_4$ -free graphs, as well as their complements. We also prove the conjecture in case  $|C| = |S| = 2$  and show that the vertex-set  $R = V(G) \setminus (C \cup S)$  cannot induce a threshold graph, while the conjecture means that  $R = \emptyset$ .

## 2 Partial results

Figure 2 shows a graph  $H$  and its complement  $\overline{H}$ .

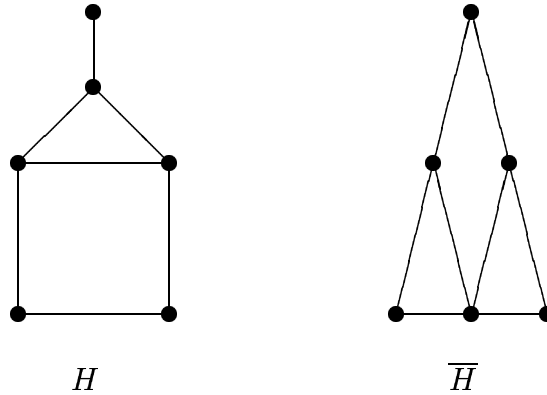


Figure 2: The graph  $H$  and its complement  $\overline{H}$ .

Using  $H$  and  $\overline{H}$ , we define a set  $\mathbf{Z}$  of graphs.

**Definition 2** *A graph  $G$  belongs to  $\mathbf{Z}$  if and only if  $G$  contains both  $H$  and  $\overline{H}$  as induced subgraphs, and  $G$  is minimal with this property.*

Here minimality of  $G$  means that every proper induced subgraph of  $G$  does not contain either  $H$  or  $\overline{H}$ . Every graph in  $\mathbf{Z}$  has at least 7 vertices and at most 12 vertices. A vertex-minimal member of  $\mathbf{Z}$  is shown in Figure 3.

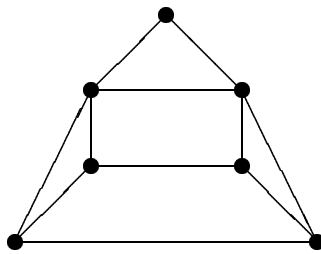


Figure 3: A minimal member of the set  $\mathbf{Z}$ .

We shall use  $\mathbf{Z}$  as the set of minimal forbidden induced subgraphs for a hereditary class.

We use notation  $X \sim Y$  (respectively,  $X \not\sim Y$ ) for disjoint subsets  $X, Y \subseteq V(G)$  in a graph  $G$  to indicate that every vertex of  $Y$  is adjacent (respectively, non-adjacent) to every vertex of  $X$ .

**Theorem 1** Conjecture 1 holds for the class  $\mathcal{P}$  of all  $\mathbf{Z}$ -free graphs.

**Proof.** Let  $G \in \mathcal{P}$  be a non-split almost CIS-graph, and let  $(C, S)$  be the unique pair consisting of a maximal clique  $C$  disjoint from a maximal stable set  $S$ . The set  $C \cup S$  induces a split graph, therefore there is a vertex  $u \in V(G) \setminus (C \cup S)$ . We denote:

$$C^- = C \setminus N(u),$$

$$C^+ = C \cap N(u),$$

$$S^- = S \setminus N(u),$$

$$S^+ = S \cap N(u),$$

where  $N(u)$  is the neighborhood of  $u$  in  $G$ . Let  $L$  be the set of all vertices  $x \notin C \cup S$  that are non-adjacent to the largest number of vertices in  $S$ . Assumption (1):  $u \in L$  and  $u$  is adjacent to the maximum number of vertices in  $C$  among all vertices  $x \in L$ .

The set  $C^+ \cup \{u\}$  induces a clique which is disjoint from the maximal stable set  $S$ . It implies that there exists a vertex  $s_1 \in S$  with  $s_1 \sim C^+ \cup \{u\}$  [ $s_1$  is adjacent to all vertices in  $C^+ \cup \{u\}$ ]. Clearly,  $s_1 \in S^+$ . We subdivide  $C^-$ :  $C_1^- = \{x \in C^- : x \sim s_1\}$  and  $C_2^- = \{x \in C^- : x \not\sim s_1\}$ .

Similarly, the set  $S^- \cup \{u\}$  induces a stable set which is disjoint from the maximal clique  $C$ . It implies that there exists a vertex  $c_1 \in C$  with  $c_1 \not\sim S^- \cup \{u\}$  [ $c_1$  is non-adjacent to all vertices in  $S^- \cup \{u\}$ ].

**Claim 1**  $c_1 \sim S^+$ .

**Proof.** Suppose that  $c_1$  is non-adjacent to some vertex  $x \in S^+$ . Since  $S$  a maximal stable set, the vertex  $c_1$  is adjacent to some  $y \in S$ . Clearly,  $y \in S_+$ , and therefore  $y$  is adjacent to  $u$ . Consider the stable set  $S'$  consisting of  $c_1$  and all its non-neighbors in  $S$ . We have  $S^- \subseteq S'$  and  $x \in S'$ . The clique  $C' = \{u, y\}$  is disjoint from  $S'$ , therefore there exists a vertex  $z$  such that  $z \sim C'$  and  $z \not\sim S'$ . We have  $z \notin C \cup S$ . Indeed,  $z$  is non-adjacent to  $c_1 \in C$  and  $z$  is adjacent to  $y \in S$ . We obtain a contradiction to Assumption (1):  $z$  is non-adjacent to a larger number of vertices  $S$  than  $u$ . Indeed,  $z \not\sim S^- \cup \{x\}$  while  $u \not\sim S^-$  and  $u \sim S^+$ .  $\square$

Claim 1 shows that  $c_1 \sim s_1$  and therefore  $c_1 \in C_1^-$ , see an illustration in Figure 4.

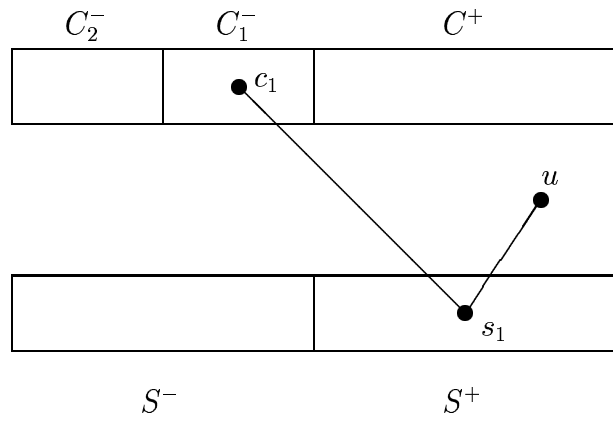


Figure 4: An illustration.

By maximality of  $C$ , the vertex  $s_1$  is non-adjacent to some  $c_2 \in C$ . Clearly,  $c_2 \in C_2^-$ . Now we subdivide  $S^-$ :  $S_1^- = \{x \in S^- : x \not\sim c_2\}$  and  $S_2^- = \{x \in S^- : x \sim c_2\}$ .

The set  $C_1^- \cup C^+ \cup \{s_1\}$  induces a clique which is disjoint from the stable set  $S_1^- \cup \{u, c_2\}$ . Hence there exists a vertex  $v$  such that  $v \sim C_1^- \cup C^+ \cup \{s_1\}$  and  $v \not\sim S_1^- \cup \{u, c_2\}$ , see Figure 5.

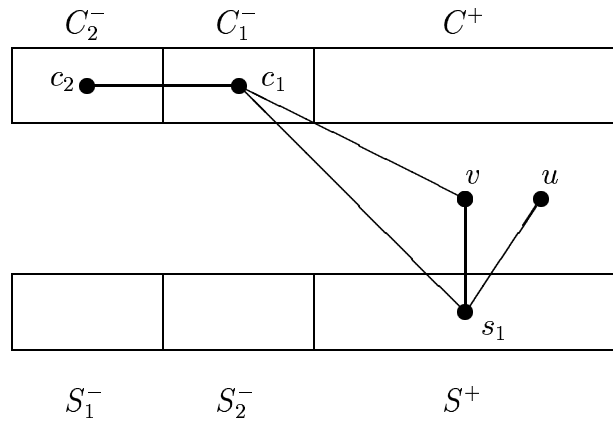


Figure 5: Another illustration.

**Claim 2** *The vertex  $v$  is adjacent to some vertex in  $S_2^-$ .*

**Proof.** If  $v \not\sim S_2^-$  then  $v \not\sim S^-$ , and therefore  $v \in L$ . However, the vertex  $v$  is adjacent to more vertices in  $C$  than  $u$  does:  $v \sim C^+ \cup C_1^+$  while  $u \sim C^+$  and  $u \not\sim C^-$ . This is a contradiction to Assumption (1).  $\square$

According to Claim 2, there exists a vertex  $s_2 \in S_2^-$  which is adjacent to  $v$ . By the definition of  $S_2^-$ ,  $c_2 \sim s_2$ . Also,  $s_2 \not\sim \{c_1, u, s_1\}$ . Thus, the set  $\{u, v, c_1, c_2, s_1, s_2\}$  induces a subgraph isomorphic to  $H$ , see Figure 6.

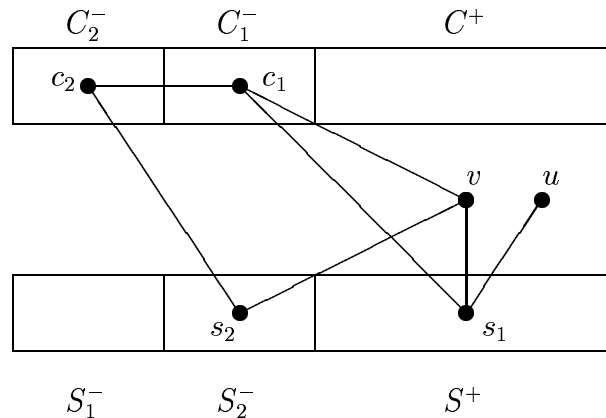


Figure 6: The set  $\{u, v, c_1, c_2, s_1, s_2\}$  induces a subgraph isomorphic to  $H$ .

The complementary arguments show that  $G$  also contains an induced subgraph isomorphic to  $\overline{H}$ . It implies that  $G$  contains an induced subgraph belonging to  $\mathbf{Z}$ , a contradiction to the assumption  $G \in \mathcal{P}$ .  $\square$

Every graph in  $\mathbf{Z}$  contains the 5-path  $P_5$  as an induced subgraph.

**Corollary 1** Conjecture 1 holds for all  $P_5$ -free graphs.

Also, every graph in  $\mathbf{Z}$  contains the 4-cycle  $C_4$  as an induced subgraph.

**Corollary 2** Conjecture 1 holds for all  $C_4$ -free graphs, and therefore for all chordal graphs.

The set  $\mathbf{Z}$  is self-complementary, that is  $G \in \mathbf{Z}$  implies  $\overline{G} \in \mathbf{Z}$ . Hence the statements complementary to Corollary 1 and to Corollary 2 hold: Conjecture 1 is true for all  $\overline{P}_5$ -free graphs and for all  $\overline{C}_4$ -free graphs. Finally, the set  $\mathbf{Z}$  is finite, it contains graphs with at most 12 vertices, therefore the class  $\mathcal{P}$  is polynomial-time recognizable.



### 3 Some useful properties

The proof of Theorem 1 contains some interesting properties of non-split almost CIS-graphs. Let  $G$  be a non-split graph with a non-CIS pair  $(C, S)$ , and let  $U = V(G) \setminus (C \cup S)$ . We define a partial order  $\leq_C$  on  $U$ :  $u \leq_C u'$  if and only if  $N(u) \cap C \subseteq N(u') \cap C$ . Maximal elements of  $(U, \leq_C)$  will be called  $C$ -maximal vertices. Similarly, a partial order  $\leq_S$  can be defined on  $U$ :  $u \leq_S u'$  if and only if  $N(u') \cap S \subseteq N(u) \cap S$ , that is the set of all non-neighbors of  $u'$  in  $S$  contains the set of all non-neighbors of  $u$  in  $S$ . Maximal elements of  $(U, \leq_S)$  are called  $S$ -maximal vertices.

**Property 1** *A non-split almost CIS graph  $G$  does not have a vertex which is both  $C$ -maximal and  $S$ -maximal.*

For example, suppose that  $|C| = |S| = 2$  for the unique non-CIS pair  $(C, S)$  of  $G$ . It is easy to see that  $C \cup S$  induces  $P_4$ , say  $P_4 = (s_1, c_1, c_2, s_2)$ , where  $C = \{c_1, c_2\}$  and  $S = \{s_1, s_2\}$ . We apply Property 1 to an arbitrary vertex  $u \in U = V(G) \setminus (C \cup S)$ . If  $u$  is  $S$ -maximal then it is non-adjacent to exactly one vertex of  $s$ . Note that  $u$  cannot be non-adjacent to both vertices of  $S$  by maximality of  $S$ . According to Property 1,  $u$  is not  $C$ -maximal, that is  $u$  must be non-adjacent to both vertices of  $C$ . Thus, the set  $C \cup S \cup \{u\}$  induces the graph  $P_5$  shown in Figure 7 [or the symmetric graph, where  $u$  is adjacent to  $s_2$  instead of  $s_1$ ].

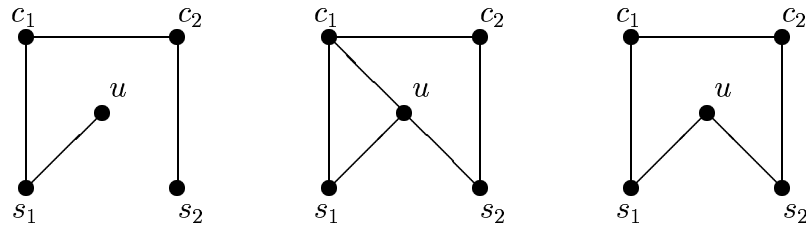


Figure 7: The graphs  $P_5$ ,  $\overline{P}_5$  and  $C_5$ .

If  $u$  is  $C$ -maximal then it is adjacent to exactly one vertex of  $C$ . Note that  $u$  cannot be adjacent to both vertices of  $C$  by maximality of  $C$ . According to Property 1,  $u$  is not  $S$ -maximal, that is  $u$  must be adjacent to both vertices of  $S$ . Thus, the set  $C \cup S \cup \{u\}$  induces the graph  $\overline{P}_5$  shown in Figure ?? [or the symmetric graph].

Finally, it is possible that  $u$  neither  $C$ -maximal nor  $S$ -maximal, in which case the set  $C \cup S \cup \{u\}$  induces  $C_5$ , see the rightmost graph in Figure 7.

A  $C$ -mirror of a vertex  $u \in U$  is a vertex  $s \in S$  adjacent to  $u$  and such that  $N(u) \cap C = N(s) \cap C$ . An  $S$ -mirror of a vertex  $u \in U$  is a vertex  $c \in C$  non-adjacent to  $u$  and such that such that  $N(u) \cap C = N(s) \cap C$ .

**Property 2** *Every  $C$ -maximal vertex has a  $C$ -mirror, and every  $S$ -maximal vertex has an  $S$ -mirror.*

A  $P_4$ -based graph has a non-CIS pair  $(C, S)$  such that  $|C| = |S| = 2$ .

**Theorem 2** *Conjecture 1 holds for the class of all  $P_4$ -based graphs.*

**Proof.** Consider an arbitrary almost CIS  $P_4$ -based graph  $G$  with the unique non-CIS pair  $(C, S)$ . We may assume that  $G$  is not a split graph. Let  $C \cup S$  induces  $P_4 = (s_1, c_1, c_2, s_2)$ , where  $\{c_1, c_2\} = C$  and  $\{s_1, s_2\} = S$ . We denote

$$X_1 = \{u : u \sim \{c_1, s_1, s_2\} \text{ and } u \not\sim c_2\},$$

$$X_2 = \{u : u \sim \{c_2, s_1, s_2\} \text{ and } u \not\sim c_1\},$$

$$Y_1 = \{u : u \not\sim \{s_2, c_1, c_2\} \text{ and } u \sim s_1\}, \text{ and}$$

$$Y_2 = \{u : u \not\sim \{s_1, c_1, c_2\} \text{ and } u \sim s_2\}.$$

**Fact 1** *Every vertex of  $X_1$  is adjacent to every vertex of  $X_2$ .*

**Proof.** Suppose that a vertex  $x_1 \in X_1$  is non-adjacent to some vertex  $x_2 \in X_2$ . An arbitrary maximal stable set that contains both  $x_1$  and  $x_2$  is disjoint from the maximal clique  $C = \{c_1, c_2\}$ , a contradiction.  $\square$

Here is the complementary statement.

**Fact 2** *Every vertex of  $Y_1$  is non-adjacent to every vertex of  $Y_2$ .*

Now consider a maximal clique  $C_i$  in the subgraph induced by  $X_i$ ,  $i = 1, 2$ , and a maximal stable set  $S_i$  in the subgraph induced by  $Y_i$ ,  $i = 1, 2$ . Fact 1 and Fact 2 show that  $C_1 \cup C_2$  is a clique in  $G$ , and  $S_1 \cup S_2$  is a stable set. The set  $A = C_1 \cup \{c_1, s_1\}$  is a clique, and the set  $B = S_1 \cup S_2 \cup \{c_2\}$  is a stable set.

**Fact 3** *There is a vertex  $u \in C_1$  which is non-adjacent to all vertices of  $B$ .*

**Proof.** We extend  $A$  and  $B$  to a maximal clique  $A'$  and a maximal stable set  $B'$ , respectively. Since  $(A', B') \neq (C, S)$ , there must be a vertex  $u \in A' \cap B'$ . The proof of Theorem 1 implies that  $S_1$  is non-empty, therefore  $u \neq s_1$ . The vertex  $u$  is adjacent to both  $c_1$  and  $s_1$ , and  $u$  is non-adjacent to  $c_2$ . It follows that  $u \in X_1$ . Since  $u \sim C_1$ ,  $u \in C_1$  by maximality of  $C_1$ . Finally,  $u \in B'$  implies that  $u$  is non-adjacent to all vertices of  $B \subseteq B'$ .  $\square$

Now, the set  $Q = C_1 \cup C_2 \cup \{s_2\}$  is a clique, and the set  $R = S_2 \cup \{c_2, s_1\}$  is a stable set.

**Fact 4** *There is a vertex  $v \in S_2$  which is adjacent to all vertices of  $Q$ .*

**Proof.** We extend  $Q$  and  $R$  to a maximal clique  $Q'$  and a maximal stable set  $R'$ , respectively. Since  $(Q', R') \neq (C, S)$ , there must be a vertex  $v \in Q' \cap R'$ . By Fact 3, the set  $C_1$  is non-empty, therefore  $v \neq c_2$ . The vertex  $v$  is non-adjacent to both  $c_2$  and  $s_1$ , and  $v$  is adjacent to  $s_2$ , hence  $v \in Y_2$ . Since  $v \not\sim S_2$ , maximality of  $S_2$  implied that  $v \in S_2$ . Finally,  $v \in Q'$  shows that  $v$  is adjacent to all vertices of  $Q \subseteq Q'$ .  $\square$

Fact 3 and Fact 4 produce a contradiction: the vertices  $u$  and  $v$  are non-adjacent and adjacent simultaneously.  $\square$

## 4 An extension

We can further extend the graph  $H$  of Figure 6 using the maximal clique  $C'$  containing  $v$  and  $s_2$  and the maximal stable set  $S'$  containing  $s_1$  and  $c_2$ . There must be a vertex  $w \in C' \cap S'$ . We have  $w \notin C \cup S$ , since  $w \not\sim c_2 \in C$  and  $w \sim s_2 \in S$ , see Figure 8, where  $w$  can be adjacent to the vertices  $c_1$  and/or  $u$ .

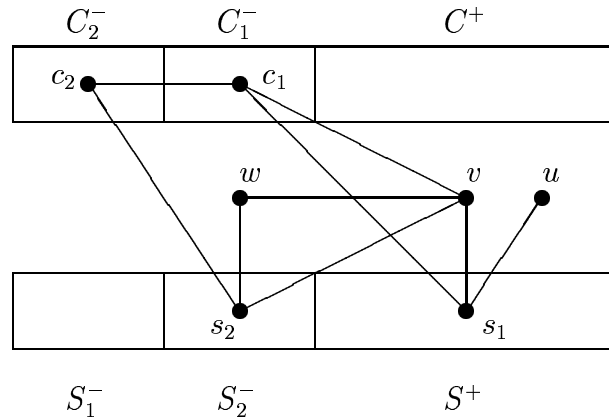


Figure 8: An extension.

The configuration of Figure 8 contains four graphs  $H_1$ ,  $H_2$ ,  $H_3$  and  $H_4$  shown in Figure 9:

- $H_1$ : the vertex  $w$  is non-adjacent to both  $u$  and  $c_1$ ,
- $H_2$ : the vertex  $w$  is non-adjacent to  $u$ , and  $w$  is adjacent to  $c_1$ ,
- $H_3$ : the vertex  $w$  is adjacent to  $u$ , and  $w$  is non-adjacent to  $c_1$ , and
- $H_4$ : the vertex  $w$  is adjacent to both  $u$  and  $c_1$ .

Now we specify a maximal clique  $C'$  and a maximal stable set  $S'$  in each  $H_i$ ,  $i = 1, 2, 3, 4$ .

- $H_1$ :  $C' = \{c_1, s_1, v\}$  and  $S' = \{u, w, c_2\}$ ,
- $H_2$ :  $C' = \{c_1, s_1, v\}$  and  $S' = \{u, w, c_2\}$ ,
- $H_3$ :  $C' = \{u, s_1\}$  and  $S' = \{w, c_1\}$ , and
- $H_4$ :  $C' = \{v, w, c_1\}$  and  $S' = \{s_1, s_2\}$ .

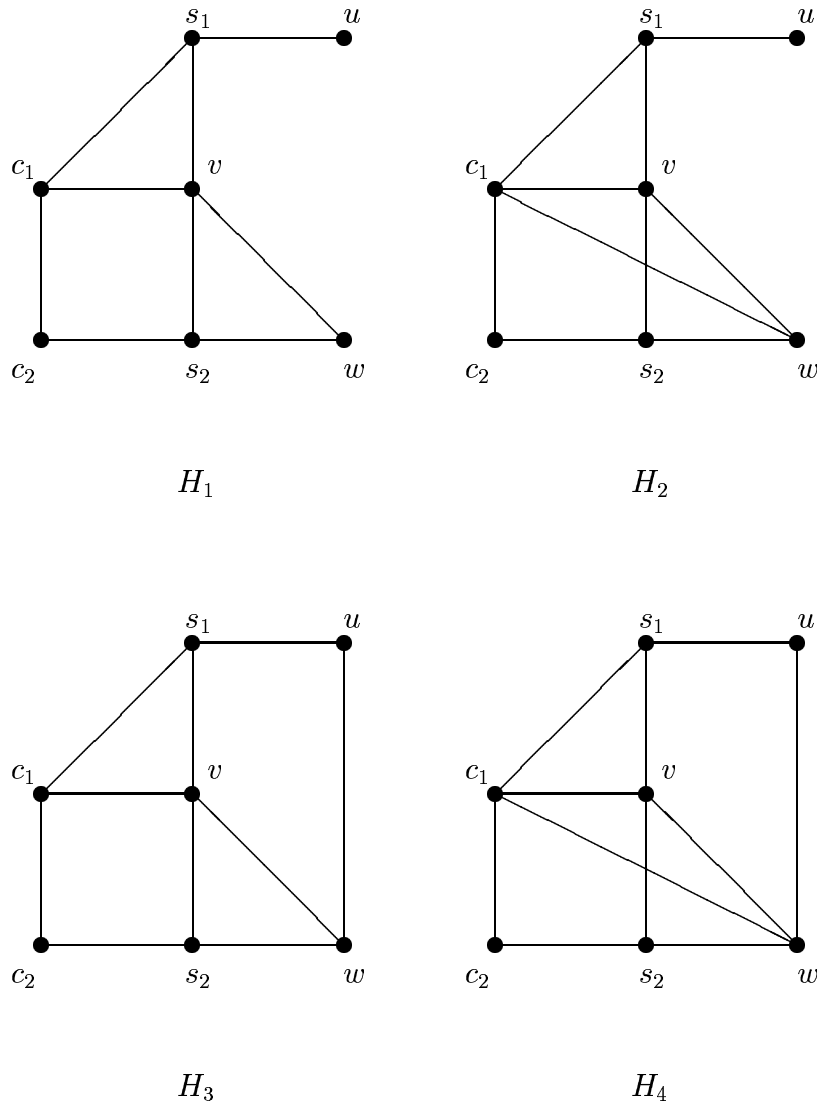


Figure 9: The four variants.

We introduce a new vertex  $x$  such that  $x \sim C'$  and  $x \not\sim C'$ . For the graph  $H_1$ ,  $x \notin C \cup S$ , since  $x \not\sim c_2 \in C$  and  $x \sim s_1 \in S$ . Depending on adjacency of  $x$  to the vertex  $s_2$ , we obtain two extensions  $F_1$  and  $F_2$  of  $H_1$ , see Figure 10.

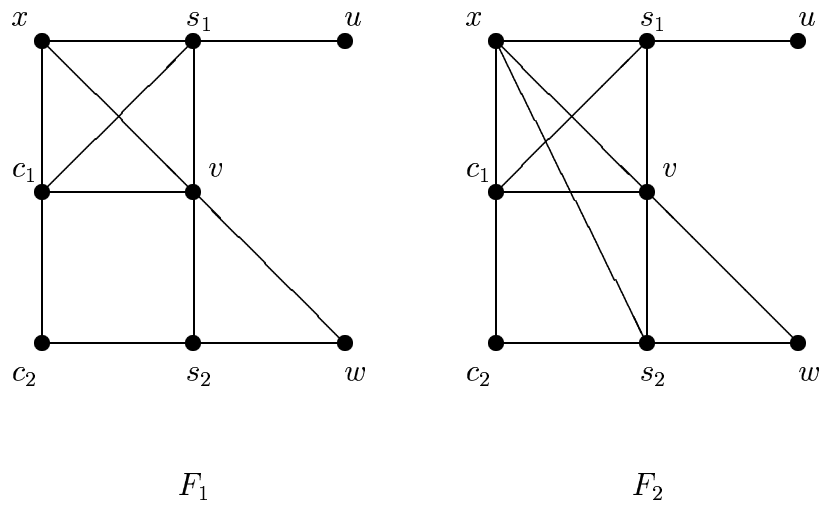


Figure 10: The two extensions  $F_1$  and  $F_2$  of  $H_1$ .

The graph  $H_2$  is similar to  $H_1$ , and we have two extensions  $F_3$  and  $F_4$  of  $H_2$  shown in Figure 11.

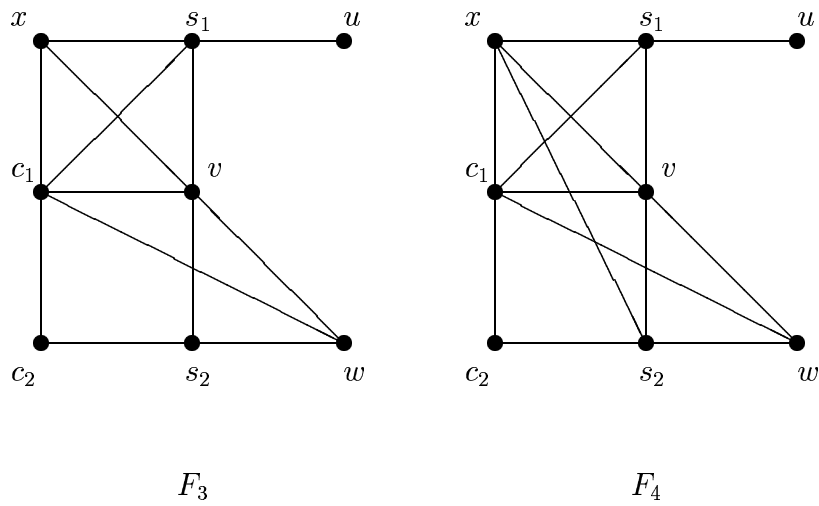
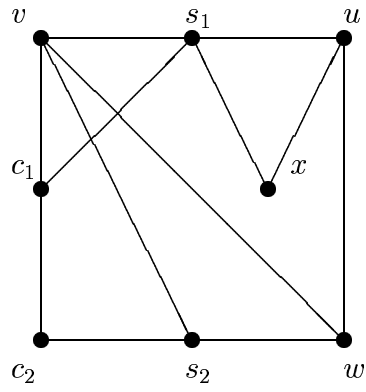
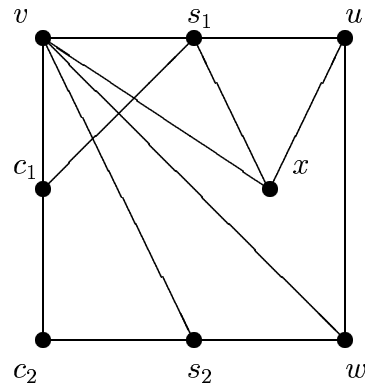
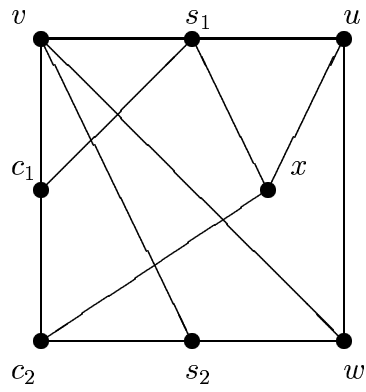
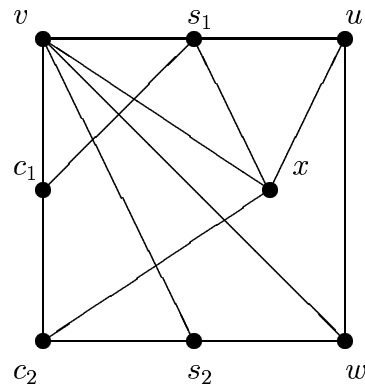


Figure 11: The two extensions  $F_3$  and  $F_4$  of  $H_2$ .

In case of  $H_3$ , the new vertex  $x$  does not belong to  $C \cup S$ , since  $x \not\sim c_1 \in C$  and  $x \sim s_1 \in S$ . The vertex  $x$  may or may not be adjacent to  $v, c_2, s_2$  independently of each other. Thus, we have 8 variants – the graphs  $F_5, F_6, F_7, F_8$  of Figure 12 and the graphs  $F_9, F_{10}, F_{11}, F_{12}$  of Figure 13.

 $F_5$  $F_6$  $F_7$  $F_8$ Figure 12: The graphs  $F_5, F_6, F_7, F_8$  (extensions of  $H_3$ ).



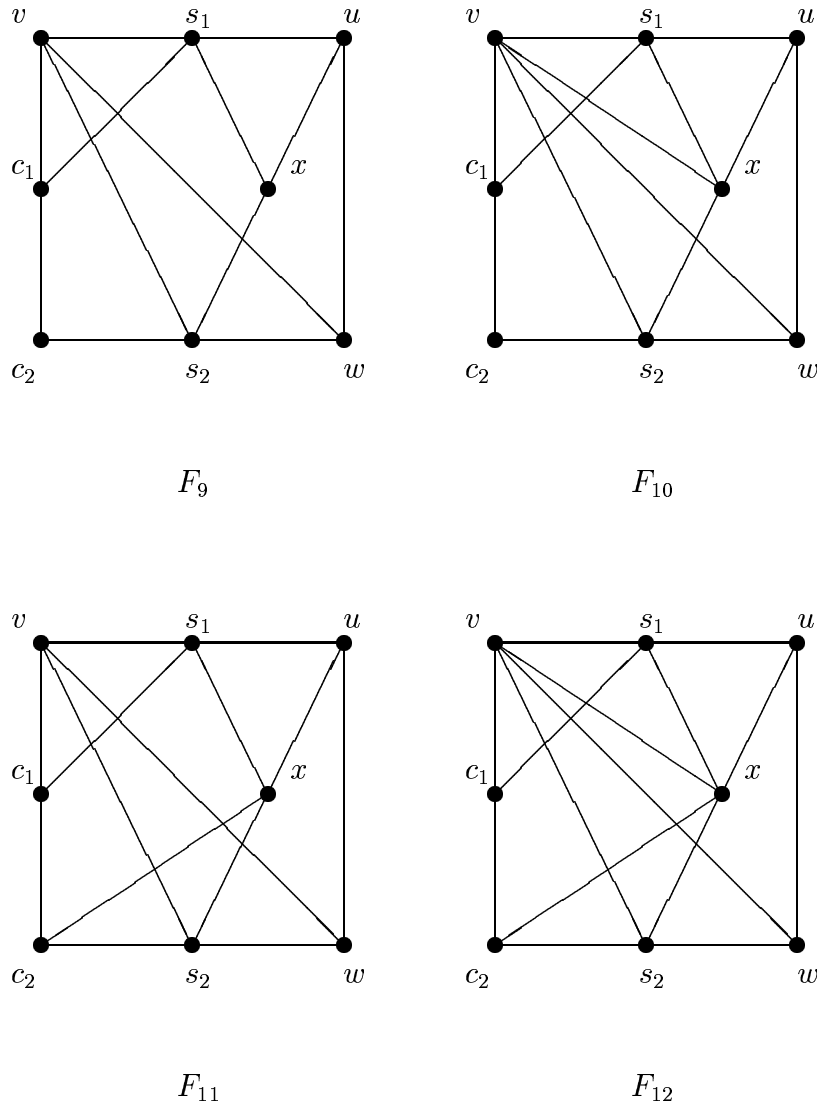
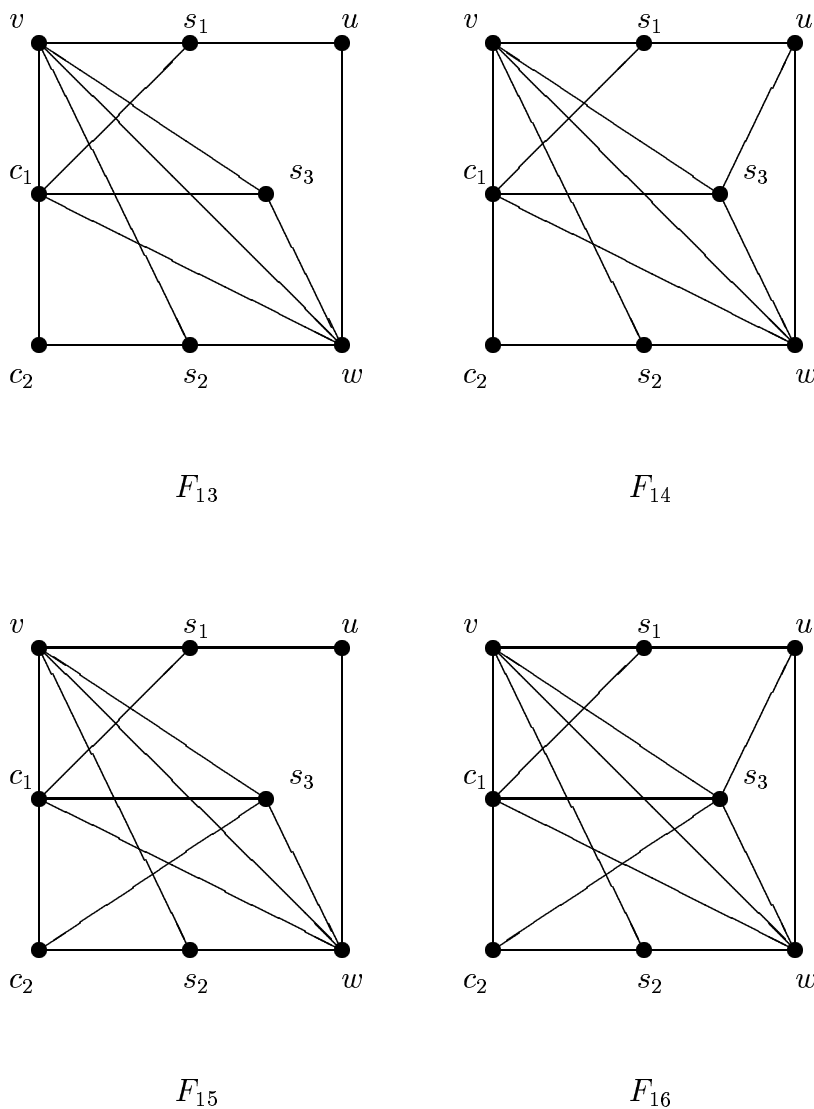


Figure 13: The graphs  $F_9, F_{10}, F_{11}, F_{12}$  (extensions of  $H_3$ ).

In case of  $H_4$ , we can extend  $S'$  to the original maximal stable set  $S$ . As a result, the vertex  $x$  will be in  $S$ , and therefore we rename it as  $s_3$ . Specifying potential edges  $s_3u$  and  $s_3c_2$ , we obtain the four graphs  $F_{13}, F_{14}, F_{15}, F_{16}$  of Figure 14.

Figure 14: The extensions of  $H_4$ .

Now we define a finite set  $\mathbf{Z}'$  of graphs. A graph  $G$  belongs to  $\mathbf{Z}'$  if and only if

- $G$  contains at least one of  $F_1, F_2, \dots, F_{16}$  as induced subgraph, and
  - $G$  contains at least one of  $\overline{F}_1, \overline{F}_2, \dots, \overline{F}_{16}$  as induced subgraph,
- and  $G$  is minimal with this property.

**Theorem 3** Conjecture 1 holds for the class  $\mathcal{P}$  of all  $\mathbf{Z}'$ -free graphs.

## 5 Alternating sequences

Let  $G$  be an almost CIS-graph with a unique non-CIS pair  $(C, S)$ . Here we derive some properties of the subgraph induced by the set  $R = V(G) \setminus (C \cup S)$ . In particular, we show that  $R$  cannot induce a threshold graph.

**Property 3** *For an arbitrary  $C$ -maximal vertex  $u \in R$ , there exists a vertex  $v \in R$  such that*

- (i)  $v$  is an  $S$ -maximal vertex,
- (ii)  $v$  is non-adjacent to all non-neighbors of  $u$  in  $S$ ,
- (iii)  $v$  is adjacent to  $u$ ,
- (iv)  $v$  is non-adjacent to an arbitrary  $C$ -mirror  $s \in S$  of  $u$ , and
- (v)  $u$  is non-adjacent to an arbitrary  $S$ -mirror  $c \in C$  of  $v$ .

**Proof.** As in the proof of Theorem 1, we denote

$$C^- = C \setminus N(u),$$

$$C^+ = C \cap N(u),$$

$$S^- = S \setminus N(u),$$

$$S^+ = S \cap N(u).$$

Let  $s \in S$  be an arbitrary  $C$ -mirror of  $u$ , see Property 2. Also, there is a vertex  $c' \in C^-$  such that  $c' \not\sim \{u\} \cup S^-$ . The proof of Theorem 1 implies that  $s$  and  $c'$  are non-adjacent. The set  $S \cup \{c'\}$  is not stable, therefore  $c'$  is adjacent to some  $s' \in S$ . Clearly,  $s' \in S^+$  and  $s' \sim u$ . Consider the clique  $\{u, s'\}$  and the stable set  $\{c', s\} \cup S^-$ . Their maximal extensions must have a common vertex, say  $v'$ . Since  $v' \sim s'$  and  $v' \not\sim c'$ , we have  $v' \in R$ . The vertex  $v'$  is non-adjacent to all non-neighbors of  $u$  in  $S$  and to the vertex  $s$ , therefore  $R$  contains a maximal vertex with this property. Thus, we can choose an  $S$ -maximal vertex  $v'' \not\sim S^- \cup \{s\}$ , and both (i) and (ii) hold for  $v''$ .

Suppose that  $v''$  is non-adjacent to  $u$ . According to Property 2,  $v''$  has an  $S$ -mirror  $c \in C$ . Since  $v'' \not\sim s$ , we have  $c \not\sim s$ , and therefore  $c \not\sim u$ . The set  $S \cup \{c\}$  is not stable, therefore  $c$  is adjacent to some vertex  $s'' \in S$ . Clearly,  $s'' \in S^+$  and  $s'' \sim \{u, v''\}$ . Consider the clique  $\{u, s''\}$  and the stable set  $\{c\} \cup X$ , where  $X$  consists of all non-neighbors of  $v''$  in  $S$ , including  $s$ . Their maximal extensions must have a common vertex  $v$ . The vertex  $v$  has the same set of non-neighbors in  $S$  as  $v''$  by  $S$ -maximality of  $v''$ . In particular,  $c$  is an  $S$ -mirror for  $v$ . Now  $v$  satisfies all the properties (i), (ii), (iii), (iv) and (v).  $\square$

Here is the complementary result.

**Property 4** For an arbitrary  $S$ -maximal vertex  $v \in R$ , there exists a vertex  $u \in R$  such that

- (i)  $u$  is a  $C$ -maximal vertex,
- (ii)  $u$  is adjacent to all neighbors of  $v$  in  $C$ ,
- (iii)  $u$  is non-adjacent to  $v$ ,
- (iv)  $u$  is adjacent to an arbitrary  $S$ -mirror  $c \in C$  of  $v$ , and
- (v)  $v$  is non-adjacent to an arbitrary  $C$ -mirror  $s \in S$  of  $u$ .

Now we consider the shortest alternating cyclic sequence  $P$  of  $C$ -maximal vertices  $u_i \in R$  and  $S$ -maximal vertices  $v_j \in R$ :

$$P = (u_1, v_1, u_2, v_2, \dots, u_k, v_k, u_{k+1} = u_1), \quad (1)$$

such that each  $v_i$  is constructed for  $u_i$  according to Property 3, and each  $u_{i+1}$  is constructed for  $v_i$  according to Property 4. The sequence  $P$  has at least four vertices. Indeed,

$$\{u_1, u_2, \dots, u_k\} \cap \{v_1, v_2, \dots, v_k\} = \emptyset,$$

since no vertex is  $C$ -maximal and  $S$ -maximal simultaneously. In particular,  $v_1 \neq u_1$ . The vertex  $u_2$  is non-adjacent to  $v_1$ , while the vertices  $u_1$  and  $v_1$  are adjacent, therefore  $u_2 \neq u_1$ . Similarly,  $v_2 \neq v_1$ . However, it is possible that  $u_3 = u_1$ , in which case  $P$  has length four [ $k = 2$ ].

**Theorem 4** Let  $G$  be an almost CIS non-split graph with a non-SIC pair  $(C, S)$ , and let  $R = V(G) \setminus (C \cup S)$ . Then  $G$  contains a sequence (1) of pairwise distinct vertices  $u_i, v_i \in R$ ,  $k \geq 2$ , and two sequences

$$(c_1, c_2, \dots, c_k), (s_1, s_2, \dots, s_k) \quad (2)$$

of vertices  $c_i \in C$  and  $s_j \in S$  such that

- (i) each  $u_i$  is a  $C$ -maximal vertex,
- (ii) each  $v_i$  is an  $S$ -maximal vertex,
- (iii) each  $v_i$  is non-adjacent to all non-neighbors of  $u_i$  in  $S$ ,
- (iv) each  $u_i$  is adjacent to all neighbors of  $v_{i-1}$  in  $C$ ,
- (v) each  $u_i$  is non-adjacent to  $v_{i-1}$ ,
- (vi) each  $v_i$  is adjacent to  $u_i$ ,
- (vii) each  $s_i$  is a  $C$ -mirror of  $u_i$ ,
- (viii) each  $c_i$  is an  $S$ -mirror of  $v_i$ ,
- (ix) each  $v_i$  is non-adjacent to  $s_i$ , and
- (x) each  $u_{i+1}$  is adjacent to  $c_i$ .

Note that the vertices in (2) are not necessarily pairwise distinct. Figure 15 illustrates the minimal case  $k = 2$ , where  $u_1 \not\sim u_2$  and  $v_1 \not\sim v_2$ . However, the vertices  $u_1$  and  $u_2$  may be adjacent as well as  $v_1$  and  $v_2$ .

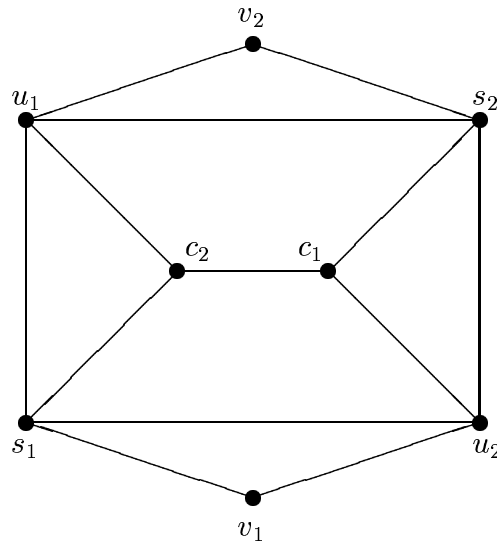


Figure 15: An illustration for  $k = 2$ .

A graph is *threshold* if it does not contain  $2K_2$ ,  $P_4$  and  $C_4$  as induced subgraphs, see Figure 16.

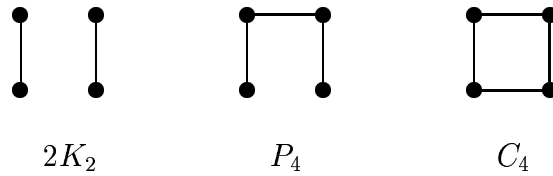


Figure 16: Minimal forbidden induced subgraph for threshold graphs.

**Corollary 3** *If  $G$  is an almost CIS non-split graph with a non-SIC pair  $(C, S)$ , then the set  $R = V(G) \setminus (C \cup S)$  does not induce a threshold graph.*

**Proof.** For  $k = 2$ , the sequence (1) contains four vertices that induce a configuration  $C$  shown in see Figure 17, where  $u_1$  and  $u_2$  may or may not be adjacent, as well as  $v_1$  and  $v_2$ . Clearly,  $C$  contains exactly the graphs  $2K_2$ ,  $P_4$  and  $C_4$ .

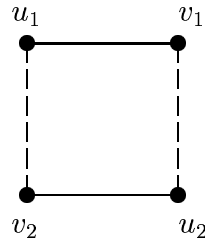


Figure 17: The configuration  $C$ .

If  $k \geq 3$  then the set  $\{u_i, v_i : i = 1, 2, \dots, k\}$  induce a configuration that contains  $C$ .  $\square$

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