

COMPUTER-GENERATED THEOREMS
ON NASH-SOLVABILITY OF BIMATRIX
GAMES BASED ON EXCLUDING CERTAIN
 2×2 SUBGAMES

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Abstract. In 1964 Shapley observed that a matrix has a saddle point whenever every 2×2 submatrix of it has one. In contrast, a bimatrix game may have no Nash equilibrium (NE) even when every 2×2 subgame of it has one. Nevertheless, Shapley's claim can be generalized for bimatrix games in many ways as follows. We partition all 2×2 bimatrix games into fifteen classes $C = \{c_1, \dots, c_{15}\}$ depending on the preference pre-orders of the two players. A subset $t \subseteq C$ is called a NE-theorem if a bimatrix game has a NE whenever it contains no subgame from t . We suggest a general method for getting all minimal (that is, strongest) NE-theorems based on the procedure of joint generation of transversal hypergraphs given by a special oracle. By this method we obtain all (six) minimal NE-theorems.

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1 Introduction, main concepts and results

1.1 Bimatrix games and Nash equilibria

Let X_1 and X_2 be finite sets of strategies of players 1 and 2. Pairs of strategies $x = (x_1, x_2) \in X_1 \times X_2 = X$ are called *situations*. A *bimatrix game* $U = (U_1, U_2)$ is a pair of real-valued matrices $U_i : X \rightarrow \mathbb{R}$, $i = 1, 2$, with common set of entries X . Value $U_i(x)$ is interpreted as utility function (also called profit or payoff) of player $i \in \{1, 2\}$ in the situation x . A situation $x = (x_1, x_2) \in X_1 \times X_2 = X$ is called a Nash equilibrium (NE) if

$$U_1(x'_1, x_2) \leq U_1(x_1, x_2) \quad \forall x'_1 \in X_1 \quad \text{and} \quad U_2(x_1, x'_2) \leq U_2(x_1, x_2) \quad \forall x'_2 \in X_2;$$

in other words, if no player can make a profit by choosing a new strategy if the opponent keeps the old one. A bimatrix game U is called a *zero sum or matrix game* if $U_1(x) + U_2(x) = 0$ for every $x \in X$. In this case the game is well-defined by one of two matrices, say, by U_1 , and a NE is called a *saddle point* (SP).

1.2 Locally minimal SP-free matrix and NE-free bimatrix games

Standardly, we define a subgame as the restriction of U to a subset $X' = X'_1 \times X'_2 \subseteq X_1 \times X_2 = X$, where $X'_1 \subseteq X_1$ and $X'_2 \subseteq X_2$. In 1964 Shapley [8] noticed that a matrix has a saddle point whenever each of its 2×2 submatrices has one. Obviously, in this case, every submatrix has a SP, too. In other words, all minimal SP-free matrices are of size 2×2 . Moreover, all locally minimal SP-free matrices are of size 2×2 , too; in other words, every SP-free matrix of larger size has a row or column whose elimination still results in an SP-free submatrix; see [1]. Other generalizations of Shapley's theorem can be found, for example, in [6, 7]. Let us also notice that a 2×2 matrix has no SP if and only if one of its diagonals is strictly larger than the other.

The "naive generalization" of Shapley's claim to bimatrix games fails: a 3×3 game might have no NE even if each its 2×2 subgame has one; moreover, for each $n \geq 3$ a $n \times n$ bimatrix game might have no NE even if every its subgame has one; see Example 1 in [6] or [1] and also examples given below. However, all locally minimal NE-free games admit the following explicit characterization [1].

For the sake of brevity, let us denote situation (x_1^i, x_2^j) by $x_{i,j}$, where $X_1 = \{x_1^1, x_1^2, \dots\}$ and $X_2 = \{x_2^1, x_2^2, \dots\}$.

Given an integer $n \geq 2$ and a bimatrix game U with $|X_1| \geq n$ and $|X_2| \geq n$, let us say that U has the *canonical strong improvement n -cycle* C_n^0 if each situation $x_{1,1}, x_{2,2}, \dots, x_{n-1,n-1}, x_{n,n}$ (respectively, $x_{1,2}, x_{2,3}, \dots, x_{n-1,n}, x_{n,1}$) is a unique largest in its row with respect to U_2 (in its column with respect to U_1) and is the second largest, not necessarily, unique, in its column with respect to U_1 (in its row with respect to U_2). Any other strong improvement n -cycle

C_n is obtained from the canonical one C_0 by arbitrary permutations of the rows of X_1 and columns of X_2 .

It is easy to see that if an $n \times n$ bimatrix game U has a strong improvement cycle then U has no NE, yet, every proper subgame obtained from U by elimination of either one row or one column has a NE. In other words, U is a *locally minimal* NE-free bimatrix game. Moreover, the inverse holds, too.

Theorem 1 ([1]). *A bimatrix game U is a locally minimal NE-free game if and only if U is of size $n \times n$ for some $n \geq 2$ and it contains a strong improvement n -cycle.* \square

Thus, locally minimal NE-free games can be arbitrary large. Several examples are given in Figures 2 - 6, where each game has the canonical strong improvement cycle. Although it seems difficult to characterize or recognize the minimal NE-free games (see [1]), yet, the above characterization of the locally-minimal ones will be sufficient for us.

1.3 Pre-orders

Given a set Y and a mapping $P : Y^2 \rightarrow \{<, >, =\}$ that assigns one of these three symbols to every ordered pair $y, y' \in Y$, we say that y is *less or worse* than y' if $y < y'$, respectively, y is *more or better* than y' if $y > y'$, and finally, y and y' are *equivalent or they make a tie* if $y = y'$. Furthermore, P is called a *pre-order* if the following standard properties (axioms) hold for all $y, y', y'' \in Y$:

symmetry: $y < y' \Leftrightarrow y' > y$, $y = y' \Leftrightarrow y' = y$, and $y = y$;

transitivity: $y < y' \ \& \ y' < y'' \Rightarrow y < y''$, $y < y' \ \& \ y' = y'' \Rightarrow y < y''$,
 $y = y' \ \& \ y' < y'' \Rightarrow y < y''$, $y = y' \ \& \ y' = y'' \Rightarrow y = y''$,

A pre-order without ties is called a (*linear or complete*) *order*.

We use standard notation: $y \leq y'$ if $y < y'$ or $y = y'$ and $y \geq y'$ if $y > y'$ or $y = y'$. Obviously, transitivity and symmetry still hold:

$y \leq y' \ \& \ y' < y'' \Rightarrow y < y''$, $y < y' \ \& \ y' \leq y'' \Rightarrow y < y''$,
 $y \leq y' \ \& \ y' \leq y'' \Rightarrow y \leq y''$, and $y \leq y' \Leftrightarrow y' \geq y$.

In Figures 1-6 we use the following notation: an arrow from y to y' for $y > y'$, a line with two dashes for $y = y'$, and an arrow with two dashes for $y \geq y'$.

1.4 Configurations; fifteen 2-squares

Let us notice that to decide whether a situation $x = (x_1, x_2) \in X_1 \times X_2 = X$ is a NE in U , it is sufficient to know only two pre-orders: in the row x_1 with respect to U_2 and in column x_2 with respect to U_1 .

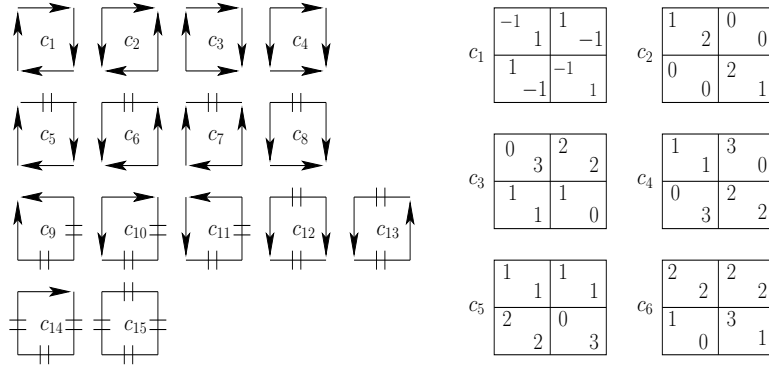


Figure 1: Fifteen 2-squares.

Given X_1 and X_2 , let us assign a pre-order P_{x_i} over X_{3-i} to each $x_i \in X_i$; $i = 1, 2$, and call the obtained preference profile $P = \{P_{x_1}, P_{x_2} \mid x_1 \in X_1, x_2 \in X_2\}$ a *configuration or bi-pre-order*.

Naturally, every bimatrix game $U = (U_1, U_2)$ defines a unique configuration $P = P(U)$, where P_{x_i} is the pre-order over X_{3-i} defined by U_i ; $i = 1, 2$. Clearly, each configuration is realized by infinitely many bimatrix games. Yet, it is also clear that to get all NE in game U it is enough to know its configuration $P(U)$.

For brevity, we will refer to a 2×2 configuration as a *2-square*. Up to permutations and transpositions, there exist only fifteen different types of 2-squares. They are listed in Figure 1 together with the corresponding bimatrix games (for the first 6 squares). Four 2-squares c_1, c_2, c_3, c_4 have no ties; another four, c_5, c_6, c_7, c_8 and the next five, $c_9, c_{10}, c_{11}, c_{12}, c_{13}$, have, respectively, one and two ties each; finally, c_{14} and c_{15} have 3 and 4 ties.

Fifteen 2-squares have 0, 2, 1, 1, 1, 2, 1, 2, 3, 2, 2, 2, 2, 3, and 4 NE, respectively. Thus, only c_1 has no NE. Shapley's theorem asserts that each c_1 -free zero-sum game (or configuration) has a NE. Let us note that 2-squares $c_1 - c_6$ are frequent in the literature. For example, the non-zero-sum bimatrix games realizing c_2 and c_4 may represent classical "family dispute" and "prisoner's dilemma"; respectively, c_5 and c_6 illustrate the concepts of the "promise" and "threat".

1.5 Dual or transversal hypergraphs

Let C be a finite set whose elements we denote by $c \in C$. A *hypergraph* H (on the ground set C) is a family of subsets $h \subseteq C$ that are called the *edges* of H . A hypergraph H is called *Sperner* if containment $h \subseteq h'$ holds for no two distinct edges of H . Given two hypergraphs T and E on the common ground set C , they are called *transversal* or *dual* if the following properties hold:

- (i) $t \cap e \neq \emptyset$ for every $t \in T$ and $e \in E$;

(ii) for every subset $t' \subseteq C$ such that $t' \cap e \neq \emptyset$ for each $e \in E$ there exists an edge $t \in T$ such that $t \subseteq t'$;

(iii) for every subset $e' \subseteq C$ such that $e' \cap t \neq \emptyset$ for each $t \in T$ there exists an edge $e \in E$ such that $e \subseteq e'$.

Property (i) means that edges of E and T are transversal, while (ii) and (iii) mean that T contains all minimal transversals to E and E contains all minimal transversals to T , respectively. It is well-known, and not difficult to see, that (ii) and (iii) are equivalent whenever (i) holds. Although for a given hypergraph \mathcal{H} there exist infinitely many dual hypergraphs, yet, only one of them, which we will denote by \mathcal{H}^d , is Sperner. Thus, within the family of Sperner hypergraphs duality is well-defined; moreover, it is an involution, that is, equations $T = E^d$ and $E = T^d$ are equivalent. It is also easy to see that dual Sperner hypergraphs have the same set of elements. For example, the following two hypergraphs are dual:

$$E' = \{(c_1), (c_2, c_3), (c_5, c_9), (c_3, c_5, c_6)\}, \quad (1)$$

$$T' = \{(c_1, c_2, c_5), (c_1, c_3, c_5), (c_1, c_2, c_6, c_9), (c_1, c_3, c_9)\}; \quad (2)$$

as well as the following two:

$$E = \{(c_1), (c_2, c_3), (c_5, c_9), (c_3, c_5, c_6), (c_2, c_4, c_5, c_6)\}, \quad (3)$$

$$T = \{(c_1, c_2, c_5), (c_1, c_3, c_5), (c_1, c_2, c_3, c_9), (c_1, c_2, c_6, c_9), (c_1, c_3, c_4, c_9), (c_1, c_3, c_6, c_9)\}. \quad (4)$$

1.6 Hypergraphs of examples and theorems

Let $C = \{c_1, \dots, c_{15}\}$. We call a subset $e \subseteq C$ a *NE-example* if there is a NE-free configuration P such that e is the set of types of 2-squares in P ; respectively, a subset $t \subseteq C$ is called a *NE-theorem* if a configuration has a NE whenever it contains no 2-squares from t . Obviously, $e \cap t \neq \emptyset$ for any NE-example e and NE-theorem t , since otherwise e is a counterexample to t . Moreover, it is well-known and easy to see that the hypergraphs of all inclusion-minimal (that is, strongest) NE-examples E_{NE} and NE-theorems T_{NE} are transversal. Let us consider c_1 and four configurations in Figure 2. It is easy to verify that all five contain canonical strong cycles and hence, they are locally minimal (in fact, minimal) NE-free configurations. These five configurations are chosen because they contain few types of 2-squares; the corresponding sets are given in Figure 2; they form the hypergraph E defined by (3). Figure 2 shows that each edge of E is a NE-example.

Let us consider the dual hypergraph T given by (4). We will prove that every edge $t \in T$ is a NE-theorem, thus, showing that the “research is complete”, that is, $E = E_{NE}$ and $T = T_{NE}$ are the hypergraphs of all strongest NE-examples and theorems.

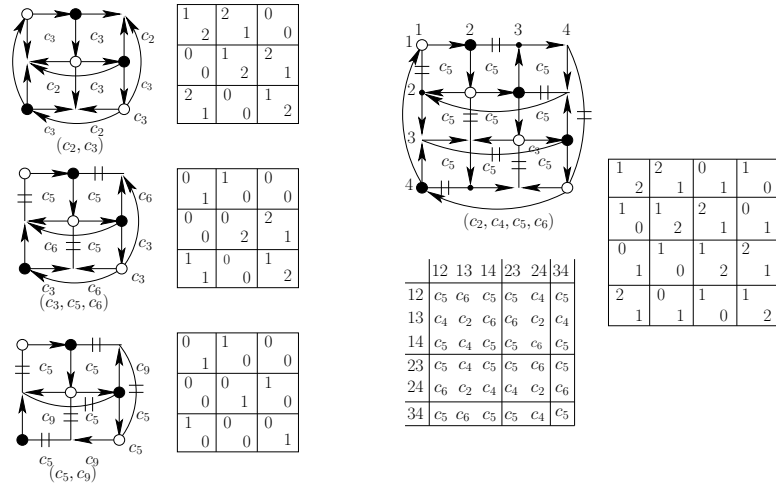


Figure 2: NE-examples.

Remark 1 Given a family of NE-examples E' , the dual hypergraph T' should be viewed as a hypergraph of conjectures rather than theorems. Indeed, some inclusion-minimal examples might be missing in E' ; moreover, some examples of E' might be reducible. In this case some conjectures from the dual hypergraph $T' = E'^d$ will fail, being too strong. For instance, let us consider E' given by (1) in which the NE-example (c_2, c_4, c_5, c_6) is missing. (In fact, it is not that easy to obtain a minimal 4×4 example without computer.) Respectively, conjecture (c_1, c_3, c_9) appears in $T' = E'^d$. This conjecture is too strong, so it fails. In $T = T_{NE}$ we substitute for it three weaker (but correct) NE-theorems (c_1, c_3, c_9, c_2) , (c_1, c_3, c_9, c_4) , and (c_1, c_3, c_9, c_6) . Thus, if it seems too difficult to prove a conjecture, one should look for new examples.

1.7 Joint generation of examples and theorems

Of course, this approach can be applied not only to NE-free bimatrix games.

In general, given a set of objects Q (in our case, configurations), list C of subsets (properties) $Q_c \subseteq Q$, $c \in C$ (in our case, c -free configurations), the target subset $Q_0 \subseteq Q$ (configurations that have a NE), we introduce a pair of hypergraphs $E = E(Q, Q_0, C)$ and $T = T(Q, Q_0, C)$ (examples and theorems) defined on the ground set C as follows:

- (i) every set of properties assigned to an edge $t \in T$ (a theorem) implies Q_0 , that is, $q \in Q_0$ whenever q satisfies all properties of t , or in other words, $\bigcap_{c \in t} Q_c \subseteq Q_0$; in contrast,
- (ii) each set of properties corresponding to the complement $C \setminus e$ of an edge $e \in E$ (an example) does not imply Q_0 , i.e., there is an object $q \in Q \setminus Q_0$ satisfying all properties of $C \setminus e$, or in other words, $\bigcap_{c \notin e} Q_c \not\subseteq Q_0$.

If hypergraphs E and T are dual then we can say that “our understanding of Q_0 in terms of C is perfect”, that is, every new example $e' \subset C$ (theorem $t' \subseteq C$) is a superset of some

old example $e \in E$ (theorem $t \in T$).

Without loss of generality we can assume that examples of $e \in E$ and theorems $t \in T$ are inclusion-wise minimal in C ; or in other words both hypergraphs E and T are Sperner.

Given Q, Q_0 and C , we try to generate hypergraphs E and T jointly [5]. Of course, the oracle may be a problem: Given a subset $C' \subseteq C$, it may be difficult to decide whether C' is a theorem (i.e., if $q \in Q_0$ whenever q satisfies all properties of C') or an example (i.e., if there is a $q \in Q \setminus Q_0$ satisfying all properties of $C \setminus C'$). However, the stopping criterion, $E^d = T$, is well-defined and, moreover, it can be verified in quasi-polynomial time [3].

Remark 2 *Let us notice that containment $\bigcap_{c \in t} Q_c \subseteq Q_0$ might be strict. In other words, theorem t gives sufficient but not always necessary conditions for $q \in Q_0$. We can also say that theorems $t \in T$ give all optimal “inscribed approximations” of $Q_0 \subseteq Q$ in terms of C .*

Remark 3 *In [4], this approach was illustrated by a simple model problem in which Q is the set of 4-gons, Q_0 is the set of squares, C is a set of six properties of a 4-gon. Two dual hypergraphs of all minimal theorems T and examples E were constructed. In [2], the same approach was applied to a more serious problem related to families of Berge graphs.*

1.8 Strengthening NE-theorems; main results

We will prove all six NE-theorems $t \in T_{NE}$. Formally, they cannot be strengthened, since t' is not a NE-theorem whenever $t' \subset t \in T_{NE}$ and the containment $t' \subset t$ is strict. Still, we can get stronger claims in slightly different terms.

Let us notice that for any t the class of t -free configurations (games) is hereditary. Indeed, if a configuration (game) is t -free then every subconfiguration (subgame) of it is t -free, too. Hence, we can restrict ourselves by the locally minimal NE-free examples, which are characterized by Theorem 1.

Now, let us consider NE-theorems (c_1, c_2, c_5) , (c_1, c_3, c_5) , and (c_1, c_2, c_6, c_9) . Formally, since 2-square c_1 has no NE, it must be eliminated. Yet, in a sense, it is the only exception. More precisely, we can strengthen the above three NE-theorems as follows.

Theorem 2 *The 2-square c_1 is a unique locally minimal NE-free configuration that is also (c_2, c_5) - or (c_3, c_5) -, or (c_2, c_6, c_9) -free.*

Furthermore, theorems (c_1, c_3, c_9, c_2) , (c_1, c_3, c_9, c_4) , (c_1, c_3, c_9, c_6) can be strengthened, too. In fact, we will characterize explicitly the configurations that are locally minimal NE-free and also (c_3, c_9) -free. This family is sparse but still infinite. In particular, we obtain the following result. Let $C(P)$ denote the set of all types of 2-squares of configuration P ; furthermore, let $C' = \{c_2, c_4, c_5, c_6, c_7, c_8, c_{13}, c_1\}$ and $C'' = C' \cup \{c_{12}\}$.

Theorem 3 *Let P be a locally minimal NE-free $n \times n$ configuration that is also (c_3, c_9) -free. Then*

- (i) n is even unless $n = 1$; (ii) if $n = 2$ then P is c_1 ;
- (iii) if $n = 4$ then P is a unique (c_2, c_4, c_5, c_6) -configuration in Figure 2;
- (iv) if $n = 6$ then $C(P) = C'$;
- (v) if $n = 8$ then $C' \subseteq C(P) \subseteq C''$ and there exist P with $C(P) = C'$;
- (vi) finally, if $n \geq 10$ is even then $C(P) = C''$.

It is clear that this statement implies the remaining three NE-theorems: (c_1, c_3, c_9, c_2) , (c_1, c_3, c_9, c_4) , and (c_1, c_3, c_9, c_6) .

2 Proof of Theorems 2 and 3

As we already mentioned, we can restrict ourselves to the locally minimal NE-free configurations. By Theorem 1, each such configuration P is of size $n \times n$ for some $n \geq 2$ and P contains a strong improvement cycle C_n . Without loss of generality we can assume that $C_n = C_n^0$ is canonical. In particular,

$$x_{i,i+1} \geq x_{i,j}, \quad x_{i,i+1} > x_{j,i+1}, \quad \text{for } j \neq i, \quad x_{j,j} \geq x_{i,j}, \quad x_{j,j} > x_{j,i+1}, \quad \text{for } j \neq i+1. \quad (5)$$

Furthermore, if $n = 2$ then 2-square c_1 is a unique NE-free configuration (in fact, c_1 is a strong 2-cycle). Hence, we will assume that $n \geq 3$. Additionally, we assume that P is t -free and consider successively the following subsets t : (c_2, c_5) , (c_3, c_5) , (c_2, c_6, c_9) , and (c_3, c_9) . Theorem 2 will follow, since in the first three cases we get a contradiction. For $t = (c_3, c_9)$ we will characterize the corresponding configurations explicitly, thus proving Theorem 3.

2.1 Locally minimal NE-free and (c_2, c_5) -free configurations

Let us consider C_n^0 in Figure 3 (where $n = 7$). By (5), $x_{i,i} > x_{i,j}$ (with respect to U_2) whenever $j \neq i$; in particular, $x_{i,i} > x_{i,i-1}$ for $i \in [n] = \{1, \dots, n\}$, where standardly, $0 \equiv n$. Similarly, $x_{i,i} \geq x_{j,i}$ whenever $j \neq i-1$ (with respect to U_1); in particular, $x_{i,i} \geq x_{i+1,i}$ for $i \in [n] = \{1, \dots, n\}$, where standardly, $n+1 \equiv 1$. Moreover, the latter n inequalities are also strict, since otherwise c_5 would appear.

By similar arguments we show that $x_{i,i+1} > x_{i,i+2}$ and $x_{i,i+1} > x_{i-1,i+1}$ for $i = 1, \dots, n-1$; see Figure 3.

Next, let us notice that $x_{i,i} = x_{i-2,i}$ for $i = 2, \dots, n$. Indeed, $x_{i,i} \geq x_{i-2,i}$, since C_n is a strong cycle, and c_2 would appear in case $x_{i,i} > x_{i-2,i}$.

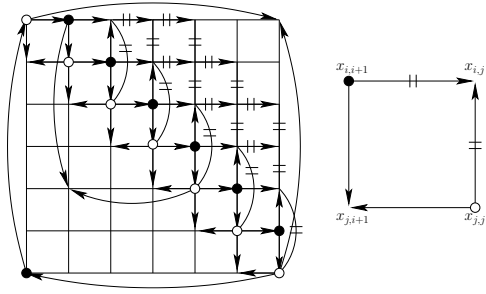


Figure 3: Locally minimal NE-free and (c_2, c_5) -free configurations do not exist, except c_1 .

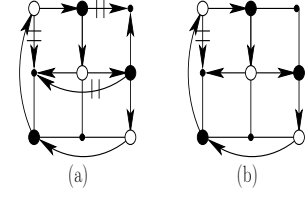


Figure 4: Locally minimal NE-free and (c_2, c_6, c_9) - or (c_3, c_5) -free configurations do not exist, except c_1 .

Furthermore, $x_{i,i+2} \geq x_{i,i+3}$ for $i = 1, \dots, n-3$, since otherwise $x_{i,i+2}, x_{i,i+3}, x_{i+2,i+2}, x_{i+2,i+3}$ would form a c_5 .

Next, let us notice that $x_{i,i+3} = x_{i+1,i+3}$ for $i = 1, \dots, n-3$. Indeed, $x_{i,i+3} \leq x_{i+3,i+3} = x_{i+1,i+3}$, and if $x_{i,i+3} < x_{i+1,i+3}$ then $x_{i,i+1}, x_{i,i+3}, x_{i+3,i+1}, x_{i+3,i+3}$ would form a c_2 , by (5).

Similarly, by induction on j , we show that $x_{i,i+j} \geq x_{i,i+j+1}$ and $x_{i,i+j} = x_{i+1,i+j}$ for $1 \leq i \leq n-3$ and $2 \leq i+j \leq n-1$.

In particular, $x_{n,n} = x_{n-2,n} = x_{n-3,n} = \dots = x_{2,n} = x_{1,n}$ in contradiction with the strict inequality $x_{n,n} > x_{1,n}$ obtained before. \square

2.2 Locally minimal NE-free and (c_2, c_6, c_9) - or (c_3, c_5) -free configurations

These two cases are easy. Let us consider C_n^0 in Figures 4 (a) and (b) (where $n = 3$), corresponding respectively to the two cases. By definition, in both cases $x_{2,2} > x_{2,1}$ $x_{1,1} \geq x_{2,1}$. In case (b) we already got a contradiction, since four above situations form c_3 or c_5 .

In case (a) we have to proceed a little further. Clearly, $x_{2,3} \geq x_{2,1}$, $x_{1,2} \geq x_{1,3}$, $x_{2,3} > x_{1,3}$, and again we get a contradiction, since situations $x_{1,1}, x_{1,3}, x_{2,1}, x_{2,3}$ form c_9 if two equalities hold, c_6 if exactly one, and c_2 if none. \square

2.3 Locally minimal NE-free and (c_3, c_9) -free configurations

Let us consider C_n^0 in Figure 5 (where $n = 8$). By (5), for all i we have:

$$\begin{aligned} x_{i,i} &> x_{i,i+1}, x_{i,i} > x_{i,i-1}, x_{i,i} \geq x_{i+1,i}, x_{i,i} \geq x_{i-2,i}; \\ x_{i,i+1} &> x_{i+1,i+1}, x_{i,i+1} > x_{i-1,i+1}, x_{i,i+1} \geq x_{i,i+2}, x_{i,i+1} \geq x_{i,i-1}. \end{aligned}$$

Furthermore, it is not difficult to show that

$$x_{i,i} = x_{i+1,i} \text{ and } x_{i,i+1} = x_{i,i+2}, \quad (6)$$

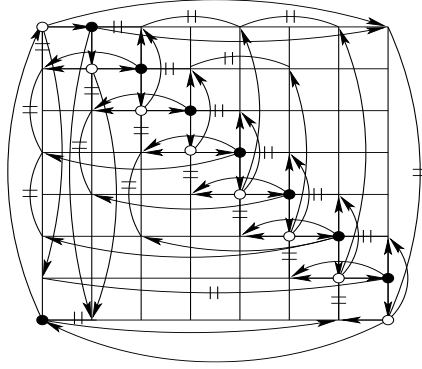


Figure 5: Locally minimal NE-free and (c_3, c_9) -free configurations.

since otherwise c_3 appears, while

$$x_{i,i} > x_{i-2,i} \text{ and } x_{i,i+1} > x_{i,i-1}, \quad (7)$$

since otherwise c_9 appears; see Figure 5.

Standardly, we prove all four claims in (6) and (7) by induction introducing situations in the following (alternating diagonal) order:

$$x_{2,1}, x_{1,3}, \dots, x_{i,i-1}, x_{i-1,i+1}, \dots, x_{n,n-1}, x_{n-1,1}, x_{1,n}, x_{n,2}.$$

Furthermore, $x_{1,1} = x_{2,1} \geq x_{4,1}$ unless $n < 5$; moreover, $x_{2,1} = x_{4,1}$, since otherwise situations $x_{2,1}, x_{4,1}, x_{2,4}$, and $x_{4,4}$ form c_3 .

Similarly, we prove that $x_{1,3} = x_{1,5}$ unless $n < 5$.

Then let us recall that $x_{4,5} \geq x_{4,1}$ and conclude that $x_{4,5} > x_{4,1}$, since otherwise situations $x_{1,1}, x_{4,1}, x_{1,5}$, and $x_{4,5}$ form c_9 .

In general, it is not difficult to prove by induction that

$$x_{i,i} = x_{i+1,i} = x_{i+3,i} = \dots = x_{i+2j-1,i}, \text{ while } x_{i-1,i} > x_{i,i} > x_{i+2j,i}; \quad (8)$$

$$x_{i,i+1} = x_{i,i+2} = x_{i,i+4} = \dots = x_{i,i+2j}, \text{ while } x_{i,i} > x_{i,i+1} > x_{i,i+2j+1}. \quad (9)$$

In both cases each sum is taken \pmod{n} (in particular, $n = 0$) and $1 \leq j < n/2$ (in particular, j takes values 1, 2, and 3 for $n = 7$ and $n = 8$).

If $n > 1$ is odd we immediately get a contradiction, since in this case, by (8), $x_{1,1} = x_{n-1,1}$, while, by (7), $x_{1,1} > x_{n-1,1}$ for all $n > 1$. Yet, for each even n , the family F_n of all locally minimal NE-free and (c_3, c_9) -free configurations is not empty.

Up to an isomorphism, F_2 (respectively, F_4) consists of a unique configuration: c_1 in Figure 1 (respectively, (c_2, c_4, c_5, c_6) in Figure 2). Two larger examples, from F_6 and F_8 , are given in Figures 6 (a) and (b), respectively.

We already know that each configuration $P \in F_{2k}$ must satisfy (5) - (9). Yet, P has one more important property:

$$x_{i,i+2j+1} \neq x_{i,i+2j'+1}, \quad x_{i+2j,i} \neq x_{i+2j',i} \quad (10)$$

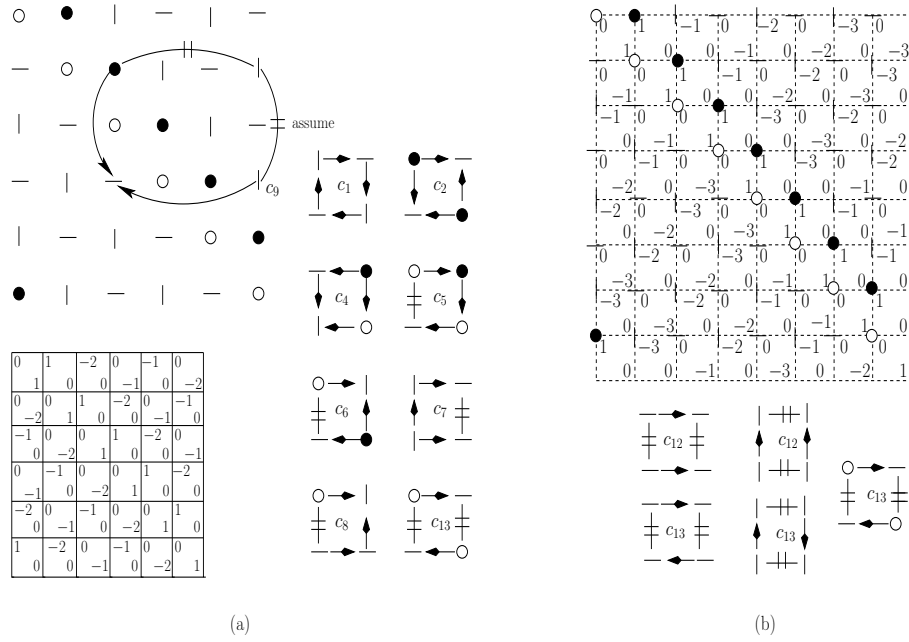


Figure 6: Two examples from F_6 and F_8 : horizontal (respectively, vertical) bars indicate second largest elements with respect to U_1 (respectively U_2).

for all $i \in [n]$ and for all positive distinct $j, j' < n/2$. Indeed, it is easy to see that otherwise c_9 appears; see Figure 6(a).

Let us denote by G_n the family of all configurations satisfying (5) - (10). We already know that $F_n \subseteq G_n$ and $F_n = G_n = \emptyset$ if $n > 1$ is odd. Let us show that $F_n = G_n$ for even n . Obviously, G_4 consists of a unique configuration (c_2, c_4, c_5, c_6) in Figure 2 and $G_2 = \{c_1\}$. Examples of configurations from G_6 and G_8 are given in Figures 6 (a) and (b). It is easy to verify that each configuration of G_n contains eight 2-squares $C' = \{c_2, c_4, c_5, c_6, c_1, c_7, c_8, c_{13}\}$ whenever $n \geq 6$; see Figure 6 (a). Moreover, c_{12} appears, too, when $n \geq 10$.

On the other hand, no configuration $P \in G_n$ contains $c_9, c_{10}, c_{11}, c_{14}$, or c_{15} , since no 2-square in P can have two adjacent equalities. It is also easy to verify that P cannot contain c_3 . Thus, P can contain only nine 2-squares of $C'' = C' \cup \{c_{12}\}$. In particular, each $P \in G_n$ is (c_3, c_9) -free; in other words, $G_n \subseteq F_n$ and, hence, $G_n = F_n$ for each n . This implies Theorem 3 and provides an explicit characterization for family F_n of locally minimal NE-free and (c_3, c_9) -free configurations. \square

Remark 4 *Interestingly, for even n each configuration $P \in F_n = G_n$ contains the same set of nine 2-squares C'' if $n \geq 10$; for $P \in G_8$ there are two options: C'' or C' (see example in Figure 6 (b), where c_{12} does not appear); for $P \in G_6$ only C' ; furthermore, G_4 consists of a unique configuration (c_2, c_4, c_5, c_6) in Figure 2 and G_2 only of c_1 ; finally, $F_n = G_n$ is empty if $n > 1$ is odd.*

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