

CRITERIA OF SOLVABILITY OF
BIMATRIX GAMES BASED ON
EXCLUDING CERTAIN 2×2 SUBGAMES

Endre Boros¹ Khaled Elbassioni²
Vladimir Gurvich³ Kazuhisa Makino⁴
Vladimir Oudalov⁵

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Abstract.

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1 Introduction and main results

In 1964 Shapley [?] noticed that a matrix M has a saddle point whenever every its 2×2 submatrix has one. In this paper we strengthen this claim. For example, we show that M not only has a saddle point but, moreover, it is dominance-solvable and it has no strict improvement cycles, as it was recently conjectured by Kukushkin [?]. We also extend Shapley's claim to bimatrix games for which we consider the following five concepts of solution: Nash equilibrium, domination (or sophisticated) equilibrium, and acyclicity, that is, absence of weak, semi-weak, or strict improvement cycles. Although the "naive generalization" fails: a 3×3 bimatrix game might have no Nash equilibrium even if every its 2×2 subgame has one (see Example 1 in [?] or in [?]), yet, we prove that a bimatrix game with no ties is dominance-solvable and it has no strict improvement cycles whenever every its 2×2 subgame is dominance-solvable. This statement was also conjectured by Kukushkin (private communications), however, we had to add the "no tie" assumption, which is crucial.

We also obtain many similar results for games with possible ties, however, in this case more 2×2 subgames must be forbidden. Moreover, we develop a general technique for deriving all criteria of this type. Our method is based on joint generation of dual hypergraphs given by a special oracle; see [?, ?, ?, ?].

Some other generalizations of Shapley's theorem can be found, for example, in [?, ?, ?, ?, ?].

1.1 Pre-orders

Given a set Y and a mapping $\mathcal{P} : \mathcal{Y}^e \rightarrow \{<, >, =\}$ that assigns one of the above three symbols to every ordered pair $y, y' \in Y$, we say that y is *less or worse* than y' if $y < y'$, respectively, y is *more or better* than y' if $y > y'$, and finally, y and y' are *equivalent or they make a tie* if $y = y'$. Furthermore, \mathcal{P} is called a *pre-order* if the following standard properties (axioms) hold for all $y, y', y'' \in Y$:

symmetry: $y < y' \Leftrightarrow y' > y$, $y = y' \Leftrightarrow y' = y$, and $y = y$;

transitivity: $y < y' \ \& \ y' < y'' \Rightarrow y < y''$, $y < y' \ \& \ y' = y'' \Rightarrow y < y''$,
 $y = y' \ \& \ y' < y'' \Rightarrow y < y''$, $y = y' \ \& \ y' = y'' \Rightarrow y = y''$,

A pre-order without ties is called a (*linear or complete*) *order*.

We use standard notation: $y \leq y'$ if $y < y'$ or $y = y'$ and $y \geq y'$ if $y > y'$ or $y = y'$. Obviously, transitivity and symmetry still hold:

$$y \leq y' \ \& \ y' < y'' \Rightarrow y < y'', \quad y < y' \ \& \ y' \leq y'' \Rightarrow y < y'', \\ y \leq y' \ \& \ y' \leq y'' \Rightarrow y \leq y'', \quad \text{and } y \leq y' \Leftrightarrow y' \geq y.$$

In the Figures we use the following notation: arrow from y to y' if $y > y'$, two dashes if $y = y'$, and both if $y \geq y'$.

1.2 Bimatrix games and the corresponding configurations

Let X_1 and X_2 be two finite sets whose elements are called the *strategies* of the players 1 and 2, respectively. Let us consider the direct product $X = X_1 \times X_2$; its elements (the pairs of strategies) $x = (x_1, x_2) \in X_1 \times X_2 = X$ we will call *situations*. Let us assign a pre-order P_{x_i} over X_{3-i} to each $x_i \in X_i$; $i = 1, 2$, and call the obtained preference profile $P = \{P_{x_1}, P_{x_2} \mid x_1 \in X_1, x_2 \in X_2\}$ a *configuration* or bi-pre-order.

Let $U = (U_1, U_2)$ be a bimatrix game, where $U_i : X_1 \times X_2 \rightarrow R$ is the utility (or payoff) function of the player i , where $i \in \{1, 2\}$. Naturally, every bimatrix U defines a unique configuration $P = P(U) = \{P_{x_1}, P_{x_2} \mid x_1 \in X_1, x_2 \in X_2\}$, where P_{x_i} is the pre-order given by payoff U_i ; again, for $i = 1, 2$. It is clear that each configuration is realized by infinitely many bimatrix games.

Game $U = (U_1, U_2)$ is called *zero-sum* (or matrix) game if $U_1(x) + U_2(x) = 0$ for every situation $x = (x_1, x_2) \in X_1 \times X_2 = X$. In this case U is completely determined by one matrix, say U_1 . Somewhat surprisingly, the corresponding configurations are not that easy to characterize; see Section ??.

Since we do not assume that the strategies in sets X_1 and X_2 are ordered, bimatrix $U = (U_1, U_2)$ and the corresponding configuration $P = P(U)$ are defined only up to arbitrary permutations of their rows X_1 and columns X_2 . Furthermore, the permutation of players 1 and 2 (and, respectively, their strategy-sets X_1 and X_2) results in the transposition of bimatrix U (that is, both matrices U_1 and U_2 are transposed) and the corresponding configuration.

1.3 All 2×2 configurations

For brevity we will say “2-square” instead of 2×2 configuration or subconfiguration. Up to permutations and transpositions, there exist fifteen types of 2-squares, $C = \{c_1, \dots, c_{15}$. They are listed in Figure ?? together with the corresponding bimatrix games. Four 2-squares c_1, c_2, c_3, c_4 have no ties; another four, c_5, c_6, c_7, c_8 , and the next five, $c_9, c_{10}, c_{11}, c_{12}, c_{13}$, have, respectively, one and two ties each; finally, c_{14} and c_{15} have 3 and 4 ties.

2-squares $c_1, c_3, c_4, c_5, c_7, c_8, c_{11}, c_{12}$, and c_{15} can be realized by zero-sum games, which are given in Figure ??, while it is not difficult to verify that the remaining 2-squares, $c_2, c_6, c_9, c_{10}, c_{13}$ and c_{14} cannot.

2-squares $c_1 - c_6$ are frequent in the literature. For example, the non-zero-sum bimatrix games realizing c_2 and c_4 may represent classical “family dispute” and “prisoner’s dilemma”; respectively, c_5 and c_6 illustrate the concepts of the “promise” and “threat”.

Given a configuration P , let $C(P) \subseteq C$ denote the set of all types of 2-squares in P .

1.4 Dual hypergraphs

Let C be a finite set whose elements we denote by $c \in C$. A *hypergraph* H (on the ground set C) is a family of subsets $h \subseteq C$ that are called the *edges* of H . A hypergraph H is called Sperner if containment $h \subseteq h'$ holds for no distinct edges of H .

Given two hypergraphs T and E on the common ground set C , they are called *transversal* or *dual* if the following properties hold:

(i) $t \cap e \neq \emptyset$ for every $t \in T$ and $e \in E$;

(ii) for every subset $t' \subseteq C$ such that $t' \cap e \neq \emptyset$ for each $e \in E$ there exists an edge $t \in T$ such that $t \subseteq t'$;

(iii) for every subset $e' \subseteq C$ such that $e' \cap t \neq \emptyset$ for each $t \in T$ there exists an edge $e \in E$ such that $e \subseteq e'$.

Property (i) means that edges of E and T are transversal, while (ii) and (iii) mean that T contains all minimal transversals to E and E contains all minimal transversals to T , respectively. It is well-known, and not difficult to see, that (ii) and (iii) are equivalent whenever (i) holds. Although for a given hypergraph there exist infinitely many dual hypergraphs, yet, only one of them is Sperner. Thus, within the family of Sperner hypergraphs duality is an involution, that is, equations $T = E^d$ and $E = T^d$ are equivalent.

For example, the following three pairs of hypergraphs are dual:

$$E_1 = \{(c_1), (c_2)\}, T_1 = \{(c_1, c_2)\}; E_2 = \{(c_1), (c_2, c_3)\}, T_2 = \{(c_1, c_2), (c_1, c_3)\};$$

$$E_3 = \{(c_1), (c_2, c_3), (c_5, c_9), (c_3, c_5, c_6), (c_2, c_4, c_5, c_6)\},$$

$$T_3 = \{(c_1, c_2, c_5), (c_1, c_3, c_5), (c_1, c_2, c_3, c_9), (c_1, c_2, c_6, c_9), (c_1, c_3, c_4, c_9), (c_1, c_3, c_6, c_9)\}.$$

1.5 Properties of bimatrix games and concepts of solution

The main objective of this paper is to obtain criteria of solvability for a bimatrix game U based on excluding from its configuration $P(U)$ certain 2-squares. More precisely, given $U = (U_1, U_2)$, where $U_1; X_1 \times X_2 \rightarrow R$ and $U_2; X_1 \times X_2 \rightarrow R$, we will consider five cases: two types of equilibria, Nash and domination, and three types of acyclicity, strict, weak, and semi-weak.

Let us remark that, although all these concepts are introduced below for a bimatrix game U , in fact, they are uniquely defined by the corresponding configuration $P(U)$.

1.5.1 Improvement cycles and acyclicity

Here and in what follows let $x_{i,j}$ denote the situation $(x_1^i, x_2^j) \in X_1 \times X_2 = X$. Given an integral $n \geq 2$ and a bimatrix game U (or only its configuration $P(U)$) such that $|X_1| \geq n$ and $|X_2| \geq n$, the following set of $2n$ situations:

$$C = \{x_{1,1}, x_{1,2}, x_{2,2}, x_{2,3}, \dots, x_{n-1,n}, x_{n,n}, x_{1,n}\} \subseteq X_1 \times X_2 = X.$$

will be called a *cycle* C (or n -cycle C_n) *in canonical form*. Furthermore, it will be called a *weak improvement cycle* if every situation in it is not worse than the next one for the corresponding player; more precisely, if

$$U_2(x_{1,1}) \geq U_2(x_{1,2}), U_1(x_{1,2}) \geq U_1(x_{2,2}), U_2(x_{2,2}) \geq U_2(x_{2,3}), \dots, U_1(x_{n-1,n}) \geq U_1(x_{n,n}), U_2(x_{n,n}) \geq U_2(x_{n,1}), U_1(x_{n,1}) \geq U_2(x_{1,1}).$$

Finally, n -cycle C will be called *strict* (respectively, *semi-weak*) if each (respectively, at least one) of the above inequalities is strict.

The (improvement) cycles are defined above in canonical form. A general (improvement) cycle is obtained from a canonical one by a permutation of strategies in X_1 and in X_2 .

A bimatrix game U and the corresponding configuration $P(U)$ is called (weakly, semi-weakly, or strictly) *acyclic* if it has no (weak, semi-weak, or strict) improvement cycle, respectively.

Let us notice that weak acyclicity is a stronger property than semi-weak one and this in its turn is stronger than strict acyclicity. This is obvious, since each strict cycle is semi-weak and each semi-weak one is weak.

Remark 1 *Unlike strict acyclicity, which is standard for non-cooperative game theory, the concepts of weak and semi-weak acyclicity belong rather to order or potential theory. Indeed, given a configuration P , that is, the direct product $X = X_1 \times X_2$ and a pre-order in every row $x_1 \in X_1$ and column $x_2 \in X_2$, the (semi-) weak acyclicity of P guarantees that given one-dimensional pre-orders can be extended to the whole two-dimensional X .*

Let us also notice that weak-acyclicity appears crucial for characterizing the configurations of zero-sum (matrix) games; see Section ??

Among fifteen 2-squares $c_1, c_5, c_{11}, c_{13}, c_{14}$, and c_{15} , are weak cycles, five of them, except c_{15} , are semi-weak, and only one, c_1 , is strict.

Remark 2 *To give an indirect proof of acyclicity, in Section XXX, typically, we will assume that C_n is the shortest cycle in the considered configuration $P = P(U)$. In other words, P might contain a larger cycle $C_{n'}$, where $n' > n$, or another n -cycle C'_n , but no shorter cycle $C_{n''}$ with $n'' < n$ can exist in P .*

For this reason, we ignore “pseudo-cycles”, which contain more than two situations in one line, row or column. Indeed, it is easy to see that for each such improvement pseudo-cycle there exists a shorter improvement cycle.

1.5.2 Nash equilibria

A situation $x = (x_1, x_2) \in X_1 \times X_2 = X$ is called a *Nash equilibrium* (NE) if $U_1(x_1, x_2) \geq U_1(x'_1, x_2)$ for each $x'_1 \in X_1$ and $U_2(x_1, x_2) \geq U_2(x_1, x'_2)$ for each $x'_2 \in X_2$; in other words, a player cannot make a profit if the opponent keeps the same strategy.

A bimatrix game U and the corresponding configuration $P(U)$ is called Nash-solvable (NS) if it has at least one Nash equilibrium. It is obvious and well-known that a bimatrix game is Nash-solvable whenever it has no strict improvement cycle.

Fifteen 2-squares have 0, 2, 1, 1, 1, 2, 1, 2, 3, 2, 2, 2, 2, 3, and 4 Nash equilibria, respectively. Thus, only c_1 has no NE. Shapley's theorem asserts that each c_1 -free zero-sum game (or configuration) has a NE.

1.5.3 Nash-solvability and strong acyclicity

As we already mentioned, NS is a weaker property than even strict acyclicity, which, in its turn, is weaker than (semi-) weak acyclicity. Indeed, any game without strict improvement

cycle, obviously, has a NE. Now, we will introduce *strong* improvement cycles and show that NS is equivalent to strong acyclicity.

Given an integral $n \geq 2$ and a bimatrix game $U = (U_1, U_2)$ (or only its configuration $P = P(U)$), the canonical improvement n -cycle C_n in U is called *strong* if each situation from the set $x_{1,1}, x_{2,2}, \dots, x_{n-1,n-1}, x_{n,n}$ (from $x_{1,2}, x_{2,3}, \dots, x_{n-1,n}, x_{n,1}$) is a unique largest in its row with respect to U_2 (in its column with respect to U_1) and it is the second largest (not necessarily, unique) in its column with respect to U_1 (in its row with respect to U_2). All other strong improvement n -cycles are obtained from the canonical one by arbitrary permutations of the rows of X_1 and columns of X_2 . By definition, a strong improvement cycle is strict.

It is not difficult to verify that if an $n \times n$ bimatrix game U has a strong improvement cycle then U has no NE, yet, every subgame obtained from U by elimination of one row or column has a NE. In other words, U is a *locally minimal* NE-free bimatrix game [?]. Moreover, the inverse holds, too.

Theorem 1 ([?]). *A bimatrix game U is a locally minimal NE-free game if and only if U is of size $n \times n$ for some $n \geq 2$ and it contains a strong improvement n -cycle.*

Although, it seems difficult to characterize the minimal NE-free games, yet, the above characterization of the locally-minimal ones will be sufficient, for us for the following reason. A class \mathcal{U} of bimatrix games (or configurations) defined by a family of forbidden subconfigurations \mathcal{F} is **hereditary**, that is, $U' \in \mathcal{U}$ whenever $U \in \mathcal{U}$ and U' is a subgame of U . Furthermore, for a hereditary class, Nash solvability and strong acyclicity are equivalent properties. Indeed, a game $U \in \mathcal{U}$ has a strong n -cycle if and only if the corresponding $n \times n$ subgame U' has no NE, yet, $U' \in \mathcal{U}$, too, since \mathcal{U} is hereditary.

1.5.4 Dominance-solvability and dominance equilibria

We say that strategy $x'_i \in X_i$ is dominated by $x_i \in X_i$ and use the notation $x_i \geq x'_i$ if $(x_i, x_{3-i}) \geq (x'_i, x_{3-i})$ for every $x_{3-i} \in X_{3-i}$; in other words, for player i strategy x_i is not worse than x'_i for any strategy of the opponent $3 - i$, where i equals 1 or 2. A bimatrix game U is called *domination-free* if $x_i \geq x'_i$ for no two distinct strategies of a player. Among fifteen 2-squares only c_1 and c_2 are domination-free.

If $x_i \geq x'_i$ then let us delete x'_i , consider the obtained subgame, and repeat the procedure if possible. Let us notice that the resulting subgame may depend on the chosen sequence of domination. There are simple conditions, see, e.g., [?], implying that the result is well-defined. We will consider these conditions in Section ???. Yet, already now we may call U *dominance-solvable* (DS) if domination results in a single situation $x = (x_1, x_2) \in X_1 \times X_2 = X$ for *at least one* possible sequence of eliminating dominated strategies. The obtained situation x is called a *domination equilibrium* (DE) or, sometimes, sophisticated equilibrium; see, for example, [?].

It is well-known and easy to see that every DE is a NE, too.

[Indeed, let $x = (x_1, x_2)$ be a DE. Every strategy $x'_1 \neq x_1$ and $x_2 \neq x_2$ was eliminated at certain step. Hence, by transitivity, we get $U_1(x_1, x_2) \geq U_1(x'_1, x_2)$ and $U_2(x_1, x_2) \geq U_2(x_1, x'_2)$. These inequalities exactly mean that x is a NE.]

In particular, we conclude that for bimatrix games DS implies NS.

Let us notice, however, that NE may exist even in a domination-free game. The simplest such example is given by the following zero-sum (matrix) game:

$$\begin{array}{r} -101 \\ 000 \\ 10 - 1 \end{array}$$

Thus, for any game (or configuration) NS follows from DS, as well as from strict acyclicity. Yet, it is not difficult to show that DS and strict acyclicity do not imply one another.

Since elimination of dominated strategies is a recursive procedure (which is not uniquely defined), it is easier to study domination-free games than DS games. Yet, again these two properties are equivalent for any hereditary class \mathcal{U} . Indeed, a game $U \in \mathcal{U}$ is not DS whenever a subgame U' of U is domination free. Yet, $U' \in \mathcal{U}$, too, since class \mathcal{U} is hereditary.

Remark 3 *Unfortunately, we did not succeed in reducing DS (like NS) to a type of acyclicity. In fact, locally minimal domination free (cf. NE-free) bimatrix games and configurations can be characterized, too, but this characterization is less efficient and we do not make use of it.*

1.6 Solvability of bimatrix games with no ties

Naturally, in this section, we restrict ourselves by the strict acyclicity only, since we assume that there are no ties.

Only four 2-squares have no ties: c_1, c_2, c_3 , and c_4 . Among them, c_1 has an improvement cycle, in fact c_1 itself is a strict 2-cycle. Moreover, c_1 has no NE. In contrast, c_2, c_3 , and c_4 are acyclic and, hence, they have NE. Furthermore, c_3 and c_4 are DS, while c_1 and c_2 are domination free. Thus, in order to guarantee dominance-solvability **in terms of excluding some 2-squares**, we must forbid c_1 and c_2 . The following claim shows that it will suffice.

Theorem 2 *A tie-free bimatrix game U (and the corresponding configuration $P = P(U)$) is DS whenever P contains no 2-squares c_1 and c_2 , or in other words, if every 2×2 subgame of U (subconfiguration of P) is DS.*

This theorem can be reformulated in a “more game theoretic way” as follows: A bimatrix game U with no ties is DS whenever each its 2×2 subgame is DS.

Let us underline that this condition is only sufficient, but not necessary, for dominance-solvability. Yet, every set of 2-squares whose exclusion implies the dominance-solvability must contain c_1 and c_2 .

Two hypergraphs, $E_{DE} = \{(c_1), (c_2)\}$ and $T_{DE} = \{(c_1, c_2)\}$, will be called, respectively, the family of *examples* and *theorems* for dominance-solvability. Let us notice that E_{DE} and T_{DE} are dual hypergraphs.

Now, let us consider NS and acyclicity of bimatrix games with no ties. Recall that c_1 has no NE and it has an improvement cycle. Let us consider also the second configuration P in Figure ??; it also has an improvement cycle (the canonical 3-cycle) and P has no NE. Furthermore, it is not difficult to verify that P contains only two types of 2-squares, c_2 and c_3 . Thus, we obtain the same hypergraph of examples $E = E_{NE} = E_{St} = \{(c_1), (c_2, c_3)\}$ for both, NS and acyclicity. The dual hypergraph $T = T_{NE} = T_{St} = \{(c_1, c_2), (c_1, c_3)\}$ represents the corresponding theorems.

Let us notice that at this stage it would be more accurate to call them “conjectures” rather than “theorems”. Indeed, we are not yet sure that E consists of **all** minimal examples: some might be not minimal and some minimal might be missing. In both cases, $T = E^d$ will contain wrong conjectures. Only if we prove all conjectures of $T = E^d$, we can conclude that both E and T consist of all minimal examples and theorems, respectively. This is the case with tie-free NS and acyclicity.

Theorem 3 *A tie-free bimatrix game U (and the corresponding configuration $P = P(U)$) has no improvement cycle (and hence, it is Nash-solvable) whenever P is (c_1, c_2) - or (c_1, c_3) -free.*

We will prove Theorems ?? and ?? in Section ??.

Remark 4 *Let us note that we got the same examples and theorems for NS and strict acyclicity, although the latter property is stronger than the former one. To explain this we have to recall again that excluding (c_1, c_2) or (c_1, c_3) provide only sufficient (but not necessary) conditions for both NS and acyclicity.*

Let us also remark that, unlike c_1 , configurations c_2 and c_3 are acyclic and have NE; however, in Theorem ?? we exclude one of them together with c_1 .

1.7 Joint generation of examples and theorems

In general, given a set of objects \mathcal{P} , list \mathcal{C} of subsets (properties) $\mathcal{P}_j \subseteq \mathcal{P}$, $j \in \mathcal{C}$, the target subset $\mathcal{P}_i \subseteq \mathcal{P}$, we introduce a pair of hypergraphs $E = E(\mathcal{P}, \mathcal{P}_i, \mathcal{C})$ and $T = T(\mathcal{P}, \mathcal{P}_i, \mathcal{C})$ (examples and theorems) defined on the ground set \mathcal{C} as follows:

(i) every set of properties assigned to an edge $t \in T$ (a theorem) implies \mathcal{P}_i , that is, $p \in \mathcal{P}_i$ whenever p satisfies all properties of t , or in other words, $\bigcap_{c \in t} \mathcal{P}_j \subseteq \mathcal{P}_i$; in contrast,

(ii) each set of properties corresponding to the complement $C \setminus e$ of an edge $e \in E$ (an example) does not imply \mathcal{P}_i , i.e., there is an object $p \in \mathcal{P} \setminus \mathcal{P}_i$ satisfying all properties of $C \setminus e$, or in other words, $\bigcap_{c \notin e} \mathcal{P}_j \not\subseteq \mathcal{P}_i$

Once more, let us notice that $\mathcal{P}_i \subseteq \bigcap_j \mathcal{P}_j$ but this containment might be strict.

If hypergraphs E and T are dual then we can say that “our understanding of \mathcal{P}_i in terms of C is complete”, that is, every new example $e' \subset C$ (respectively, theorem $t' \subseteq C$) is a superset of some old example $e \in E$ (respectively, theorem $t \in T$).

Obviously, without loss of generality we can assume that examples of $e \in E$ and theorems $t \in T$ are inclusion-minimal in C ; or in other words both hypergraphs E and T are Sperner.

Given \mathcal{P} , \mathcal{P}_l and C , we try to generate hypergraphs E and T jointly. Of course, the oracle may be a problem. Given a subset $C' \subseteq C$, it may be difficult to decide whether C' is a theorem (that is, $p \in \mathcal{P}_l$ whenever p satisfies all properties of C') or an example (i.e. there is a $p \in \mathcal{P} \setminus \mathcal{P}_l$ satisfying all properties of $C \setminus C'$). However, the stopping criterion, $E^d = T$, is well-defined and, moreover, it can be verified in quasi-polynomial time [?].

Remark 5 In [?], the above approach was illustrated by a simple model problem in which \mathcal{P} is the set of 4-gons, \mathcal{P}_l is the set of squares, C is a set of eight properties of a 4-gon. Two dual hypergraphs of all minimal theorems T and examples E were constructed. In [?], the same approach was applied to a more serious problem related to families of Berge graphs.

Now we shall apply the same approach to games with ties and consider NS, DS, and three types of acyclicity. In all cases, first we obtain the hypergraph E of examples. Then we dualize it and get $T = E^d$. As we already mentioned, the edges t are conjectures rather than theorems, since hypergraph E may be “not perfect”: some examples of E might be not really minimal, while some other minimal examples might be missing.

1.8 Examples

Let us consider list \mathcal{L}_ϵ of fifteen 2-squares in Figure ?? and recall that c_1, c_5, c_{11}, c_{13} , and c_{15} are weak improvement cycles; among them c_1 is strict, all, except c_{15} , are semi-weak; furthermore, c_1 and c_2 are DE-free and c_1 is also NE-free. Then, let us consider list \mathcal{L} of configurations given in Figure ?. It is easy to see that each configuration $C \in \mathcal{L}$ is uniquely determined by the set L_C of its 2-squares; see Figure ?.

As usual, we consider the following five types of configurations: NE- and DE-free, weakly, semi-weakly, and strictly acyclic. It is not difficult to verify that in the joint list $\mathcal{L} \cup \mathcal{L}_\epsilon$ the configurations

$(c_1), (c_2, c_3), (c_3, c_5, c_6), (c_2, c_4, c_5, c_6)$, and (c_5, c_9)

have strong improvement cycles, that is, they are locally minimal NE-free. Furthermore,

$(c_1), (c_2, c_3), (c_3, c_5, c_6), (c_2, c_4, c_5, c_6), (c_5, c_9), (c_5, c_{10}), (c_3, c_4, c_5, c_9), (c_3, c_4, c_5, c_{10}), (c_3, c_4, c_6, c_{13}, c_7, c_8), (c_3, c_4, c_6, c_{13}, c_7, c_{11}), (c_3, c_4, c_6, c_{13}, c_8, c_{11}), (c_4, c_5, c_{15}, c_9)$, and $(c_4, c_5, c_{15}, c_{10})$

have strict improvement cycles. They and also the following configurations:

$(c_1), (c_5), (c_{11}), (c_{13}), (c_2, c_6, c_7), (c_2, c_6, c_8), (c_2, c_4, c_6, c_9), (c_2, c_4, c_6, c_{10}), (c_2, c_4, c_7, c_8, c_9)$, and $(c_2, c_4, c_7, c_8, c_{10})$

have semi-weak improvement cycles. Three configurations, $(c_2, c_9), (c_2, c_{10})$, and (c_{15}) , have weak improvement cycles. We should take into account that families of configurations with weak, semi-weak, strict, and strong improvement cycles are embedded, by definition.

Finally, the following configurations are domination free:

$(c_1), (c_2), (c_2, c_3), (c_2, c_4, c_5, c_6), (c_3, c_5, c_6), (c_4, c_5, c_6), (c_5, c_9)$ and (c_6, c_{11}) .

By definition, a hypergraph E of examples is Sperner, in other words, only minimal examples are of interest. So, for each of the considered five concepts, let us select all inclusion-minimal examples from $L_2 \cup \mathcal{L}$ to obtain five Sperner hypergraphs :

$$\begin{aligned}
E_{NE} &= \{(c_1), (c_2, c_3), (c_2, c_4, c_5, c_6), (c_3, c_5, c_6), (c_5, c_9)\}; \\
E_{DE} &= \{(c_1), (c_2), (c_3, c_5, c_6), (c_4, c_5, c_6), (c_5, c_9), (c_5, c_{10}), (c_6, c_{11})\}; \\
E_{St} &= \{(c_1), (c_2, c_3), (c_2, c_4, c_5, c_6), (c_3, c_5, c_6), (c_5, c_9), (c_5, c_{10}), \\
&(c_3, c_4, c_6, c_{13}, c_7, c_8), (c_3, c_4, c_6, c_{13}, c_7, c_{11}), (c_3, c_4, c_6, c_{13}, c_8, c_{11}), \\
&(c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_7, c_8), (c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_7, c_{12}), (c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_8, c_{12})\}; \\
E_{SW} &= \{(c_1), (c_5), (c_{11}), (c_{13}), (c_{14}), (c_2, c_3), (c_2, c_4, c_6, c_9), (c_2, c_4, c_6, c_{10}), (c_2, c_4, c_7, c_8, c_9), \\
&(c_2, c_4, c_7, c_8, c_{10}), (c_2, c_6, c_7), (c_2, c_6, c_8)\}; \\
E_{We} &= \{(c_1), (c_5), (c_{11}), (c_{13}), (c_{14}), (c_{15}), (c_2, c_3), (c_2, c_9), (c_2, c_{10}), (c_2, c_6, c_7), (c_2, c_6, c_8)\}.
\end{aligned}$$

Computer analysis shows that these five lists contain all minimal examples given by configurations of size up to 4×4 . However, there might exist other minimal examples produced by larger configurations. Moreover, these new examples might reduce some already obtained “minimal” examples (although, this cannot happen to E_{NE} and E_{DE}). Anyway, we conjecture that our five lists contain all minimal examples and prove this conjecture for two cases: E_{NE} and E_{We} .

Let us also recall that c_1 and all NE-free configurations of \mathcal{L} are (locally) minimal and, hence, each of them has a strong improvement cycle. Furthermore, weakly, semi-weakly, strictly, and strongly acyclic configurations form four strictly nested sets. In particular, every NE-free configuration is strictly acyclic (and also DE-free). These observations show, in particular, that hypergraphs \mathcal{L} and $\mathcal{L} \cup \mathcal{L}_\epsilon$ are not Sperner.

1.9 Main results

1.9.1 Dualizing examples we obtain conjectures

Let us dualize the above five hypergraphs to get the following conjectures:

$$\begin{aligned}
T_{NE} &= \{(c_1, c_2, c_5), (c_1, c_2, c_3, c_9), (c_1, c_2, c_6, c_9), (c_1, c_3, c_4, c_9), (c_1, c_3, c_5), (c_1, c_3, c_6, c_9)\}; \\
T_{DE} &= \{(c_1, c_2, c_3, c_4, c_9, c_{10}, c_{11}), (c_1, c_2, c_5, c_6), (c_1, c_2, c_5, c_{11}), (c_1, c_2, c_6, c_9, c_{10})\}; \\
T_{St} &= \{(c_1, c_2, c_3, c_9, c_{10}), (c_1, c_2, c_4, c_5), (c_1, c_2, c_5, c_6), (c_1, c_2, c_5, c_7, c_8), (c_1, c_2, c_5, c_7, c_{11}), \\
&(c_1, c_2, c_5, c_8, c_{11}), (c_1, c_2, c_5, c_{13}, c_7, c_{12}), (c_1, c_2, c_5, c_{13}, c_8, c_{12}), (c_1, c_2, c_5, c_{13}, c_9), (c_1, c_2, c_5, c_{13}, c_{10}), \\
&(c_1, c_2, c_5, c_{13}, c_{11}), (c_1, c_2, c_6, c_9, c_{10}), (c_1, c_3, c_4, c_9, c_{10}), (c_1, c_3, c_5), (c_1, c_3, c_6, c_9, c_{10})\}; \\
T_{SW} &= \{(C', c_2), (C', c_3, c_4, c_6), (C', c_3, c_4, c_7, c_8), (C', c_3, c_6, c_7), (C', c_3, c_6, c_8), (C', c_3, c_6, c_9, c_{10}), \\
&(C', c_3, c_7, c_8, c_9, c_{10})\};
\end{aligned}$$

where $C' = \{c_1, c_5, c_{11}, c_{13}, c_{14}\} \subset C$ is the set of all semi-weak 2-square cycles.

$$T_{We} = \{(C'', c_2), (C'', c_3, c_6, c_9, c_{10}), (C'', c_3, c_7, c_8, c_9, c_{10})\};$$

where $C'' = C' \cup \{c_{15}\} = \{c_1, c_5, c_{11}, c_{13}, c_{14}, c_{15}\} \subset C$ is the set of all weak 2-square cycles.

1.9.2 Theorems and conjectures

We will prove all NE-, DE-, We-, and SW-conjectures in Sections ??, ??, and ??, respectively.

Somewhat surprisingly, the concept of strict acyclicity appears more complicated than weak or semi-weak acyclicity. In particular, the hypergraphs E_{St} and T_{St} have more edges (examples and theorems) than E_{We} and T_{We} , E_{SW} and T_{SW} . We will prove six St-conjectures:

(c_1, c_2, c_4, c_5) , (c_1, c_2, c_5, c_6) , $(c_1, c_2, c_6, c_9, c_{10})$, $(c_1, c_3, c_4, c_9, c_{10})$, (c_1, c_3, c_5) , $(c_1, c_3, c_6, c_9, c_{10})$
leaving open the remaining nine:

$(c_1, c_2, c_3, c_9, c_{10})$, $(c_1, c_2, c_5, c_7, c_8)$, $(c_1, c_2, c_5, c_7, c_{11})$, $(c_1, c_2, c_5, c_8, c_{11})$, $(c_1, c_2, c_5, c_{13}, c_7, c_{12})$,
 $(c_1, c_2, c_5, c_{13}, c_8, c_{12})$, $(c_1, c_2, c_5, c_{13}, c_9)$, $(c_1, c_2, c_5, c_{13}, c_{10})$, $(c_1, c_2, c_5, c_{13}, c_{11})$.

It seems that there may exist more St-examples. Let us remark, however, that these examples (if any) must be of size at least 6×6 , since all St-examples of size at most 5×5 were found out by a computer code.

1.9.3 Comments

Let us notice that the results stated above are much stronger than Theorems ?? and ?? for the tie-free games. Indeed, only the first four of fifteen 2-squares are tie-free. Moreover, weak and semi-weak acyclicity make no sense for the tie-free games and for the remaining three concepts we obtain the following hypergraphs of examples and theorems: $E_{NE}^0 = E_{St}^0 = \{(c_1), (c_2, c_3)\}$, $T_{NE}^0 = T_{St}^0 = \{(c_1, c_2), (c_1, c_3)\}$ (Theorems ?? and ??) $E_{DE}^0 = \{(c_1), (c_2)\}$, $T_{DE}^0 = \{(c_1, c_2)\}$ (Theorem ??). Let us remark that every conjecture of T_{DE} (respectively, of T_{NE} or T_{St}) contains (c_1, c_2) (respectively, (c_1, c_2) or (c_1, c_3)) as a subset. Hence, every such claim is a refinement of Theorem ?? or ??.

To summarize, first we construct examples; then we dualize the obtained hypergraph and get conjectures. If we prove all these conjectures then “the research is finished”. We can be sure that we already have all minimal examples and all minimal (strongest) theorems. Yet, if some conjectures remain not proven, several explanations are possible: (i) there are too many of them to prove all; (ii) some of them are too difficult; (iii) our list of examples is not perfect: some minimal examples might be missing, in particular, some of the obtained examples might be reduced. Thus, if a conjecture seems untractable we can try to get a counterexample and add it to the list of examples.

For instance, instead of E_{NE} let us consider $E' = \{(c_1), (c_2, c_3), (c_2, c_4, c_5, c_6), (c_3, c_5, c_6), (c_5, c_9)\}$, in which the NE-example (c_2, c_4, c_5, c_6) is missing. (Without computer it is easy to miss a 4×4 example.) Respectively, in $T' = E'^d = \{(c_1, c_2, c_5), (c_1, c_3, c_5), (c_1, c_2, c_6, c_9), (c_1, c_3, c_9)\}$ the new conjecture (c_1, c_3, c_9) substitutes for three conjectures (c_1, c_3, c_9, c_2) , (c_1, c_3, c_9, c_5) , and (c_1, c_3, c_9, c_6) of T_{NE} . Yet, this new conjecture is too strong, so it fails. Thus, if it seems too difficult to prove a conjecture, one should look for more examples.

1.9.4 Strengthening NE-theorems

We will prove all six NE-theorem $t \in T_{NE}$. Formally, they cannot be strengthened, since t' is not a NE-theorem whenever $t' \subset t \in T_{NE}$ and the containment is strict. Still, we can get stronger claims in slightly different terms.

Let us notice that for any t the class of t -free configurations (games) is hereditary. Indeed, if a configuration (game) is t -free then every its subconfiguration (subgame) is t -free, too. Hence, we can restrict ourselves by the locally minimal NE-free examples, which are characterized by Theorem ??.

Now, let us consider NE-theorems (c_1, c_2, c_5) , (c_1, c_3, c_5) , and (c_1, c_2, c_6, c_9) . Formally, since 2-square c_1 has no NE, it must be eliminated. Yet, in a sence, it is the only exception. More precisely, we can strengthen the above three NE-theorems as follows.

Theorem 4 *The 2-square c_1 is a unique locally minimal NE-free configuration that is also (c_2, c_5) - or (c_3, c_5) -, or (c_2, c_6, c_9) -free.*

Furthermore, theorems (c_1, c_3, c_9, c_2) , (c_1, c_3, c_9, c_4) , (c_1, c_3, c_9, c_6) can be strengthened, too. In fact, we will characterize explicitly the configurations that are locally minimal NE-free and also (c_3, c_9) -free. This family is sparse but still infinite. In particular, we obtain the following result. Recall that $C(P)$ denote the set of all types of 2-squares of configuration P . Furthermore, let $C' = \{c_2, c_4, c_5, c_6, c_7, c_8, c_{13}, c_1\}$ and $C'' = C' \cup \{c_{12}\}$.

Theorem 5 *Let P be a locally minimal NE-free $n \times n$ configuration that is also (c_3, c_9) -free. Then*

- (i) n is even unless $n = 1$; (ii) if $n = 2$ then P is c_1 ;
- (iii) if $n = 4$ then P is a unique (c_2, c_4, c_5, c_6) -configuration in Figure ??;
- (iv) if $n = 6$ then $C(P) = C'$;
- (v) if $n = 8$ then $C' \subseteq C(P) \subseteq C''$ and there exist P with $C(P) = C'$;
- (vi) finally, if $n \geq 10$ is even then $C(P) = C''$.

Obviously, this statement implies the remaining three NE-theorems: (c_1, c_3, c_9, c_2) , (c_1, c_3, c_9, c_4) , and (c_1, c_3, c_9, c_6) .

Remark 6 *Saying “a unique or two exceptions” we mean that there is only one or two exceptional locally minimal NE-free configurations, while there may be infinitely many not locally minimal ones without forbidden 2-squares. Thus, although theorems ?? and ?? strengthen theorems (c_1, c_2, c_5) , (c_1, c_3, c_5) , (c_1, c_2, c_6, c_9) and (c_1, c_3, c_9, c_2) , (c_1, c_3, c_9, c_4) , (c_1, c_3, c_9, c_6) , yet, to guarantee NS we still have to forbid c_1 and, c_2 , or c_4 , or c_6 , respectively.*

1.10 Zero-sum or matrix games

1.10.1 Semi-weakly acyclic and matrix configurations

A bimatrix game $U = (U_1, U_2)$ is zero-sum if $U_1(x) + U_2(x) = 0$ for every $x = (x_1, x_2) \in X_1 \times X_2 = X$. In this case the bimatrix zero-sum game U is also called a *matrix* game, since it is determined by only one of two matrices, say, U_1 . Then, player 2 becomes minimizer, while 1 remains maximizer. Respectively, in the corresponding configuration $P = (P_1, P_2)$, preference pre-order P_2 must be inversed. Let us denote the obtained configuration by

$P'(U_1) = P' = (P_1, P_2^{-1})$. Obviously, P' has no semi-weak cycles, since it is the pre-order over the entries of U_1 , which are real numbers. (Let us notice that P' may have a weak cycle whose all entries are equal; for example, c_{15} is such a 2-cycle.) It is also clear that, conversely, every semi-weakly acyclic configuration is realized by infinitely many matrices. Let us call a configuration P *matrix* if $P = P(U)$ for a zero-sum (matrix) game U . We obtain the following characterization.

Proposition 1 *Configuration P is matrix if and only if the corresponding configuration P' is semi-weakly acyclic.*

Thus, there is a one-to-one correspondence between semi-weakly acyclic and matrix configurations.

1.10.2 Corollaries for the zero-sum case

Since P' is semi-weakly acyclic it contains no 2-squares that are semi-weak cycles, i.e., $c_1, c_5, c_{11}, c_{13}, c_{14}$ (the weak cycle c_{15} is allowed.) It is easy to verify that the corresponding 2-squares in P are $F = \{c_2, c_6, c_9, c_{10}, c_{13}, c_{14}\}$. Notice that both, c_9 and c_{10} , correspond to the same semi-weak cycle c_{11} . Let us also notice that $\{c_1, c_2, c_6, c_9, c_{10} \subset F \cup \{c_1\}$ and recall that $\{c_1, c_2, c_6, c_9\}$ imply NS, while $\{c_1, c_2, c_6, c_9, c_{10}$ imply DS and strict acyclicity for the original configuration P . Thus, as a corollary, we obtain the following statement.

Theorem 6 *A zero-sum game has a saddle point and, moreover, it is dominance-solvable and has no strict improvement cycles whenever every its 2×2 subgame has a saddle point.*

The first claim is exactly Sapley's (1964) theorem, while the last two are generalizations conjectured by Kukushkin in 2007; private communications.

There is a "much simpler natural plan" to prove Theorem ?? but unfortunately it fails. Let us notice that strict (and even semi-weak) acyclicity, as well as DS, clearly, hold for a matrix M satisfying the following monotonicity condition:

$$i_1 \geq i'_1, i_2 \geq i'_2, \Rightarrow U_1(x_{i_1, i_2}) \geq U_1(x_{i'_1, i'_2}) \quad (1)$$

However, it is not true that (??) can be enforced by permutations of rows and columns of an absolutely solvable matrix M , that is, whenever each 2×2 submatrix of M has a saddle point. Indeed, the following matrix M' is a counterexample.

Furthermore, a 2×2 matrix might have a saddle point and a semi-weak improvement cycle; see, for example, M'' , whose configuration is c_5 .

12 11
03 20

1.10.3 Characterizing matrix configurations

Instead of matrix, we will characterize semi-weakly acyclic configurations. As we already know, these two problems are equivalent.

Let F_2 denote the family of all semi-weak 2-cycles, $F_2 = \{c_1, c_5, c_{11}, c_{13}, c_{14}\}$.

Obviously, semi-weakly acyclic configurations contain no semi-weak 2-cycles. Yet, there are more forbidden subconfigurations, unfortunately, an infinite family of them.

Let $X = X_1 \times X_2$, where $|X_1| = |X_2| = n \geq 2$ and let C_n be a semi-weak n -cycle in X . Given an arbitrary partition $X = C_n \cup O \cup I$, we extend C_n to a $n \times n$ configuration P as follows. For any two situations $x, x' \in X$ that belong to one line, a row or column, let $x > x'$ whenever $x \in O$, $x' \in C_n \cup I$ or $x' \in I$, $x \in C_n \cup O$. Furthermore, between the situations of O (or I) we define the pre-order P arbitrarily, yet, it cannot contain a semi-weak n' -cycle with $n' < n$. (Although, P can contain other n -cycles.) It is easy to check that P contains no semi-weak 2-cycles but, by construction, it contains the semi-weak cycle C_n . Let us denote by F_n the family of all such configurations. (For example, $O = I = \emptyset$ in case of F_2 .)

Proposition 2 *A configuration is semi-weakly acyclic if and only if it contains no subconfiguration from F_n for $n \geq 2$.*

It is not difficult to obtain similar criteria for weak and strict acyclicity. Thus, exact characterization of acyclicity in terms of a family F of all forbidden configurations is possible but not efficient, since F is infinite. It seems easier to verify acyclicity directly following the definitions. However, an “approximation of acyclicity” in terms of forbidden 2-squares looks more successful.

1.11 Tie-transitive configurations and games

1.11.1 Game forms and formal configurations

A *game form* is defined as a mapping $g : X_1 \times X_2 \rightarrow A$, where A is the set of *outcomes*. It is convenient to specify a game form by a matrix whose entries are the elements of A . For example, let us consider three game forms $g', g'',$ and g''' given by the following matrices:

$$\begin{array}{ccc} a_1a_2 & a_1a_2 & a_1a_1a_2a_2 \\ a_3a_3 & a_2a_1 & a_3a_4a_3a_4 \end{array}$$

Furthermore, let $Q = (Q_1, Q_2)$ be a pair of complete orders over A specifying the preferences of players 1 and 2, respectively. Naturally, pair (g, Q) defines a configuration $P = P(g, Q)$.

A configuration P is called *formal* if $P = P(g, Q)$ for a game form g and pair of complete orders $Q = (Q_1, Q_2)$ over A .

For example, 2-squares $c_5, c_6, c_7,$ and c_8 (respectively, c_1 and c_2) can be realized by game form g_1 (respectively, by g_2). Hence, these six configurations are formal. Yet, it is easy to see that c_{13} and c_{14} are not formal.

1.11.2 Tight and totally tight game forms and their solvability

A game form g is called *Nash-solvable* (NS) if configuration (g, Q) has a NE for every preference profile $Q = (Q_1, Q_2)$; furthermore, g is called *zero-sum-solvable* if a NE exists in (g, Q) whenever $Q = (Q_1, Q_2)$ is a pair of opposite complete orders over A . In this case a NE is called a saddle point.

Similarly, g is called *dominance-solvable* (DS) and, respectively, weakly, semi-weakly, or strictly acyclic if configuration (g, Q) has the corresponding property for every preference profile $Q = (Q_1, Q_2)$ over A .

Given a game form $g : X_1 \times X_2 \rightarrow A$, let us introduce two hypergraphs $H_1 = H_1(g)$ and $H_2 = H_2(g)$ on the ground set A whose edges are the sets of outcomes in the rows and columns of g ; more accurately, $H_i = \{e_{x_i}^i \mid x_i \in X_i\}$, where $e_{x_i}^i = \{g(x_i, x_{3-i}) \mid x_{3-i} \in X_{3-i}\}$ for $i = 1, 2$.

A game form g is called *tight* if its two hypergraphs, $H_1(g)$ and $H_2(g)$, are dual.

It was shown in [?] (see also [?]) that a game form is zero-sum-solvable if and only if it is tight. Moreover, in [?] (see also [?]) it was shown that a game form is Nash-solvable if and only if it is tight. Summarizing, we get the following statement.

Theorem 7 ([?, ?, ?, ?]). *The following three propertipes of a game form g are equivalent: (i) tightness, (ii) zero-sum-solvability, and (iii) NS.*

It is easy to verify that a 2×2 game form g is tight if and only if there is a line, row $x_1 \in X_1$ or column $x_2 \in X_2$, in which g is a constant.

For example, g' is tight, while g'' is not. Indeed, $A' = \{a_1, a_2, a_3\}$ and $g'(x_{1,1}) = a_1, g'(x_{1,2}) = a_2$, while $g'(x_{2,1}) = g'(x_{2,2}) = a_3$; thus, $H_1' = \{(a_1, a_2), (a_3)\}$, $H_2' = \{(a_1, a_3), (a_2, a_3)\}$, and g' is tight, since $H_1'^d = H_2'$. In contrast, $A'' = \{a_1, a_2\}$ and $g''(x_{1,1}) = g''(x_{2,2}) = a_1$, while $g''(x_{1,2}) = g''(x_{2,1}) = a_2$; then $H_1'' = H_2'' = \{(a_1, a_2)\}$ and g is not tight, since $H_1''^d = \{(a_1), (a_2)\} \neq H_2''$. Furthermore, g''' is tight again. Indeed, $A''' = \{a_1, a_2, a_3, a_4\}$ and $g'(x_{1,1}) = g'(x_{1,2}) = a_1, g'(x_{1,3}) = g'(x_{1,4}) = a_2, g'(x_{2,1}) = g'(x_{2,3}) = a_3$, and $g'(x_{2,2}) = g'(x_{2,4}) = a_4$; thus, $H_1''' = \{(a_1, a_2), (a_3, a_4)\}$, $H_2''' = \{(a_1, a_3), (a_1, a_4), (a_2, a_3), (a_2, a_4)\}$, and g''' is tight, since $H_1'''^d = H_2'''$. Let us recall that $x_{i,j}$ denotes the situation $(x_1^i, x_2^j) \in X_1 \times X_2 = X$.

Let us also remark that for game forms DS implies NS, since every DE is a NE, but not vice versa. For example the following game form g is tight (and hence, NS and zero-sum-solvable) but it is not DS:

$$\begin{array}{l} a_1 a_0 a_2 \\ a_0 a_0 a_0 \\ a_2 a_0 a_1 \end{array}$$

Indeed, it is easy to check that the corresponding hypergraphs $H_1 = H_1(g)$ and $H_2 = H_2(g)$ are dual; $H_1 = H_2 = \{(a_0), (a_0, a_1, a_2)\} \approx \{(a_0)\}$. However, let us set $U_1(a_0) = 0$, $U_1(a_1) = -1$, $U_1(a_2) = 1$ and $U_2 \equiv -U_1$. Obviously, Obviously, the obtained zero-sum game (g, U) is not DS. Hence, game form g is not DS, either.

A subform g' of g is defined as the restriction of g to a product $X'_1 \times X'_2$ defined by subsets $X'_1 \subseteq X_1$ and $X'_2 \subseteq X_2$. A game form $g : X_1 \times X_2 \rightarrow A$ is called *totally tight* if every its subform (including g itself) is tight.

Proposition 3 *The following three properties of a game form g are equivalent:*

- (i) g is totally tight;
- (ii) every 2×2 subform of g is tight;
- (iii) no formal configuration $P = P(g, Q)$ contains c_1, c_2, c_3 , or c_4 .

Proof Implication (i) \Rightarrow (ii) is straightforward. The inverse one follows immediately from Shapley's (1964) theorem [?] and equivalence between tightness and zero-sum-solvability [?, ?]. Finally, as we already mentioned, a 2×2 game form is tight if and only if it has a line, row or column, with two identical outcomes, in other words, (ii) \Leftrightarrow (iii).

Let us notice that total tightness implies tightness but not vice versa. For example, game form g''' is tight but not totally tight, since it contains a 2×2 subform with four distinct outcomes.

In particular, total tightness implies NS. This claim was recently strengthened by Kukushkin [?] who proved that total tightness implies strict acyclicity. Moreover, after this, a characterization of totally tight game forms implying both their strict acyclicity and also dominance-solvability was obtained in [?].

Theorem 8 ([?, ?]). *A totally tight game form is strictly acyclic and dominance-solvable.*

Let us remark, however, that even total tightness does not imply semi-weak (and hence, weak) acyclicity. For example, c_5 is a weak cycle, yet, obviously, it is formal and realized by the totally tight game form g' .

1.11.3 Why (c_1, c_2, c_3, c_4) is not a theorem ?

Indeed, c_1, c_2, c_3 and c_4 are the only four 2×2 configurations with no ties. Hence, in a configuration P that does not contain these four every 2-square has a tie. Hence, the corresponding game form is totally tight. Thus, by Theorem ??, (c_1, c_2, c_3, c_4) should be a theorem in three cases: strict acyclicity, Nash- and dominance-solvability. However, it is not the case; see Section ??.

This "paradox" can be easily resolved: (c_1, c_2, c_3, c_4) would be a theorem if we restrict ourselves by formal configurations only. (This is exactly what we plan to do in this section.) Yet, as we saw, there exist not formal configurations, for example, c_{13} and c_{14} .

Indeed, let us assume indirectly that $c_{13} = P(g, Q)$, i.e., it is realized by a game form g and preference profile $Q = (Q_1, Q_2)$. Then $g(x_{1,1} = g(x_{1,2} = a_1, g(x_{1,2} = g(x_{2,2} = a_2, and $A = \{a_1, a_2\}$. Yet, $a_1 > a_2$ and $a_2 > a_1$ with respect to Q_2 , in contradiction to transitivity. Similarly, for c_{14} we derive that A consists of a unique outcome $A = \{a\}$, that is, $g(x_{1,1} = g(x_{1,2} = g(x_{2,1} = g(x_{2,2} = a$. Yet, $a > a$ with respect to Q_2 , and again we get a contradiction. It is easy to verify that the remaining thirteen 2-squares are formal.$

1.11.4 Tie-transitivity assumption

The following assumption is standard in non-cooperative game theory: $U_1(x) = U_1(x')$ if and only if $U_2(x) = U_2(x')$ for every two situations $x = (x_1, x_2), x' = (x'_1, x'_2) \in X_1 \times X_2 = X$. In other words, if two situations are equivalent for one player then they must be also equivalent for the other one. A bimatrix game U satisfying this assumption and the corresponding configuration $P(U)$ will be called *tie-transitive*.

Remark 7 *For example, it is known that under tie-transitivity assumption the concept of dominance-solvability becomes well-defined, that is, if a sequence of eliminating dominated strategies results in a unique situation (domination equilibrium) then the same holds for every such sequence; moreover, $U(x) = U(x')$ for any two situations $x, x' \in X$ obtained by two sequences; see, e.g., [?].*

It is easy to verify that among fifteen 2-squares all, except c_{13} and c_{14} , are tie-transitive. Moreover, in general

Proposition 4 *A configuration is tie-transitive if and only if it is formal.*

Proof Indeed, given a configuration $P = P(U)$ generated by a bimatrix game $U = (U_1, U_2)$, let us call two situations $x = (x_1, x_2), x' = (x'_1, x'_2) \in X = X_1 \times X_2$ *elementary equivalent* if they make a tie, that is, if $x_1 = x'_1$ and $U_2(x_1, x_2) = U_2(x_1, x_2)$ or $x_2 = x'_2$ and $U_1(x_1, x_2) = U_1(x_1, x_2)$. Furthermore, let us extend this relation by transitivity and partition $X = X_1 \times X_2$ into equivalence classes. Let us assign an outcome to each class and consider the obtained game form $g = g(P)$. It is easy to see that configuration $P = P(U)$ is tie-transitive if and only if functions $U_1; X_1 \times X_2 \rightarrow R$ and $U_2; X_1 \times X_2 \rightarrow R$ define complete orders over the equivalence classes, that is, each of them is a constant within each class and between classes transitivity holds. Indeed, in this and only in this case P is well-defined by game form g .

Let us also notice that tie-transitivity is a hereditary property, that is, if a configuration (or game) is tie-transitive then every its subconfiguration (respectively, subgame) is tie-transitive, too.

1.11.5 Tie-transitive examples

We will apply the above arguments to verify tie-transitivity.

First, let us recall again that among fifteen 2-squares, all are tie-transitive, except c_{13} and c_{14} . Since, tie-transitivity is a hereditary property, any configuration that contains c_{13} or c_{14} is not tie-transitive.

Now, let us consider configurations in Figure ???. It is not difficult (although time consuming) to verify that the following configurations are tie-transitive:

$(c_2, c_3), (c_2, c_6, c_9), (c_2, c_7, c_{10}), (c_2, c_4, c_6, c_9), (c_2, c_4, c_6, c_{10}), (c_2, c_4, c_7, c_8, c_9), (c_2, c_4, c_7, c_8, c_{10}),$
 $(c_2, c_6, c_7, c_{10}), (c_2, c_6, c_8, c_9), (c_2, c_9), (c_2, c_{10}), (c_3, c_4, c_5, c_9), (c_3, c_4, c_5, c_{10}), (c_3, c_5, c_6), (c_4, c_5, c_6),$
 $(c_4, c_5, c_9, c_{15}), (c_4, c_5, c_{10}, c_{15}),$

while the following configurations in Figure ?? are not tie-transitive:

(c_2, c_4, c_5, c_6) , (c_2, c_6, c_7) , (c_2, c_6, c_8) , $(c_3, c_4, c_6, c_7, c_8, c_{13})$, $(c_3, c_4, c_6, c_7, c_{11}, c_{13})$, $(c_3, c_4, c_6, c_8, c_{11}, c_{13})$, (c_5, c_9) , (c_5, c_{10}) .

Let us notice that in Figure ?? for some non-tie-transitive examples we provide weaker (that is, “larger”) examples that are tie-transitive; e.g.,

(c_2, c_6, c_7, c_{10}) , (c_2, c_6, c_8, c_9) , (c_4, c_5, c_9, c_{15}) , $(c_4, c_5, c_{10}, c_{15})$

are tie-transitive, while the following four are not.

(c_2, c_6, c_7) , (c_2, c_6, c_8) , (c_5, c_9) , (c_5, c_{10}) .

We should verify tie-transitivity for all examples in Figure ?. Instead, we will check just a few leaving complete case analysis to the reader.

Let us show that (c_5, c_9) is not tie-transitive. Indeed, nine situations are partitioned into three classes as follows: $g(x_{1,1}) = g(x_{2,1}) = g(x_{2,3}) = a_1$, $g(x_{1,2}) = g(x_{1,3}) = g(x_{3,3}) = a_2$, $g(x_{2,2}) = g(x_{3,1}) = g(x_{3,2}) = a_3$, and $A = \{a_1, a_2, a_3\}$. Then, $a_1 > a_2 > a_3 > a_1$ with respect to P_1 , in contradiction to transitivity. Indeed, from columns 3, 2, and 1 we can see that $a_1 > a_2$, $a_2 > a_3$, and $a_3 > a_1$, with respect to P_1 . Similarly, we obtain that $a_1 > a_2 > a_3 > a_1$, for P_2 , by considering rows 1, 3, and 2, respectively. Thus, configuration (c_5, c_9) is not tie-transitive.

Similarly, we show that (c_5, c_{10}) is not tie-transitive. Indeed, $g(x_{1,1}) = g(x_{1,3}) = g(x_{2,3}) = a_1$, $g(x_{1,2}) = g(x_{3,2}) = g(x_{3,3}) = a_2$, $g(x_{2,1}) = g(x_{2,2}) = g(x_{3,1}) = a_3$, and $A = \{a_1, a_2, a_3\}$. Then, $a_1 > a_2 > a_3 > a_1$ with respect to both, P_1 and P_2 , in contradiction to transitivity.

Similarly, (c_6, c_{11}) is not tie-transitive. Indeed, $g(x_{1,1}) = g(x_{1,2}) = g(x_{3,2}) = a_1$, $g(x_{1,3}) = g(x_{2,3}) = g(x_{2,2}) = a_2$, $g(x_{2,1}) = g(x_{3,1}) = g(x_{3,3}) = a_3$, and $A = \{a_1, a_2, a_3\}$. Then, $a_1 > a_2 > a_3 > a_1$ with respect P_1 and $a_1 > a_3 > a_2 > a_1$ with respect to P_2 , in contradiction to transitivity.

Similarly, (c_2, c_6, c_7) and (c_2, c_6, c_8) are not tie-transitive. Indeed, in both cases $g(x_{1,1}) = g(x_{1,2}) = g(x_{3,2}) = a_1$, $g(x_{2,2}) = g(x_{2,3}) = a_2$, $g(x_{1,3}) = g(x_{3,3}) = a_3$, and $a_1 > a_2 > a_3 > a_1$ with respect P_2 , in contradiction to transitivity.

It is a little more difficult to come to the same conclusion for (c_2, c_4, c_5, c_6) . In this case $g(x_{1,1}) = g(x_{2,1}) = a_1$, $g(x_{1,2}) = g(x_{1,3}) = a_2$, $g(x_{1,4}) = g(x_{4,4}) = a_3$, $g(x_{2,3}) = g(x_{2,4}) = a_4$, $g(x_{2,2}) = g(x_{3,2}) = a_5$, $g(x_{3,1}) = g(x_{3,4}) = a_6$, $g(x_{4,1}) = g(x_{4,2}) = a_7$, $g(x_{3,3}) = g(x_{4,3}) = a_8$, $A = \{a_1, \dots, a_8\}$.

Then, $a_1 > a_2 > a_3 > a_7 > a_8 > a_6 > a_5 > a_4 > a_1$, with respect to P_2 , a contradiction.

Indeed, rows 1, 4, 3, and 2 result in $a_1 > a_2 > a_3$, $a_3 > a_7 > a_8$, $a_8 > a_6 > a_5$, and $a_5 > a_4 > a_1$, for P_2 , respectively.

Similarly, considering the columns, we obtain that $a_6 > a_3 > a_4 > a_8 > a_2 > a_5 > a_7 > a_1 > a_6$, with respect to P_1 .

Thus, configuration (c_2, c_4, c_5, c_6) is not tie-transitive, either.

Let us also remark that configurations $(c_3, c_4, c_6, c_{13}, c_7, c_8)$, $(c_3, c_4, c_6, c_{13}, c_7, c_{11})$, and $(c_3, c_4, c_6, c_{13}, c_8, c_{11})$ are not tie-transitive. Indeed, although each of them has only four ties, yet, they all contain a 2-square c_13 , which is already not tie-transitive.

Now, let us consider several tie-transitive examples. Clearly, all 2-squares are tie-tran

in Figures ?? and ?. Obviously, every configuration with no ties is tie-transitive; see for example (c_1) , (c_2) , (c_3) , (c_4) , and (c_2, c_3) .

Configuration (c_3, c_5, c_6) in Figure ?? is tie-transitive, too. Indeed, it is easy to see that $g(x_{1,1} = g(x_{2,1} = a_1, g(x_{1,2} = g(x_{1,3} = a_2, g(x_{2,2} = g(x_{3,2} = a_3, g(x_{2,3} = a_4, g(x_{3,1} = a_5, g(x_{3,3} = a_6, and $A = \{a_1, \dots, a_6\}$. Choosing P_2 such that $a_1 > a_2 > a_6 > a_5 > a_3 > a_4$ and P_1 such that $a_2 > a_3 > a_4 > a_6$ and $a_5 > a_1$ we define the configuration. Clearly, it is tie-transitive, since the constraints for P_1 and P_2 do not contradict transitivity.$

Also, configuration (c_4, c_5, c_6) in Figure ?? is tie-transitive. Indeed, it is easy to see that $g(x_{1,1} = g(x_{1,3} = a_1, g(x_{1,2} = g(x_{3,2} = a_2, g(x_{2,1} = g(x_{2,2} = a_3, g(x_{2,3} = g(x_{3,3} = a_4, g(x_{3,1} = a_5, and $A = \{a_1, \dots, a_5\}$. Choosing P_2 such that $a_3 > a_4 > a_5 > a_2 > a_1$ and P_1 such that $a_2 > a_3 > a_5 > a_1 > a_4$ we define the configuration. Clearly, it is tie-transitive, since the constraints for P_1 and P_2 do not contradict transitivity.$

Similarly, we can check that the following configurations in Figure ?? are tie-transitive:

(c_2, c_9) , (c_2, c_{10}) , (c_2, c_4, c_6, c_9) , (c_2, c_4, c_6, c_{10}) , $(c_2, c_4, c_7, c_8, c_9)$, $(c_2, c_4, c_7, c_8, c_{10})$, (c_2, c_6, c_8, c_9) , (c_2, c_6, c_7, c_{10}) , (c_3, c_4, c_5, c_9) , (c_3, c_4, c_5, c_{10}) , (c_4, c_5, c_{15}, c_9) , $(c_4, c_5, c_{15}, c_{10})$.

In summary, for five considered concepts we obtain the following five hypergraphs of tie-transitive examples:

$$\begin{aligned} E'_{NE} &= \{(c_1), (c_2, c_3), (c_3, c_5, c_6)\}; \\ E'_{DE} &= \{(c_1), (c_2), (c_3, c_5, c_6), (c_4, c_5, c_6)\}; \\ E'_{St} &= \{(c_1), (c_2, c_3), (c_3, c_5, c_6), (c_3, c_4, c_5, c_9), (c_3, c_4, c_5, c_{10}), (c_4, c_5, c_9, c_{15}), (c_4, c_5, c_{10}, c_{15}); \\ & (c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_7c_8), (c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_7c_{12}), \\ & (c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_8c_{12})\}; \\ E'_{SW} &= \{(c_1), (c_5), (c_{11}), (c_2, c_3), (c_2, c_4, c_6, c_9), (c_2, c_4, c_6, c_{10}), (c_2, c_4, c_7, c_8, c_9), (c_2, c_4, c_7, c_8, c_{10}), \\ & (c_2, c_6, c_8, c_9), (c_2, c_6, c_7, c_{10})\}; \\ E'_{We} &= \{(c_1), (c_5), (c_{11}), (c_{15}), (c_2, c_3), (c_2, c_9), (c_2, c_{10})\}. \end{aligned}$$

1.11.6 Tie-transitive conjectures and theorems. Main results

Dualizing these five hypergraphs we obtain five hypergraphs of conjectures:

$$\begin{aligned} T'_{NE} &= \{(c_1, c_3), (c_1, c_2, c_5), (c_1, c_2, c_6)\}; \\ T'_{DE} &= \{(c_1, c_2, c_3, c_4), (c_1, c_2, c_5), (c_1, c_2, c_6)\}; \\ T'_{St} &= \{(c_1, c_3, c_4), (c_1, c_3, c_5), (c_1, c_3, c_{15}), (c_1, c_3, c_9, c_{10}); (c_1, c_2, c_6, c_4), \\ & (c_1, c_2, c_6, c_9, c_{10}); (c_1, c_2, c_5, c_4), (c_1, c_2, c_5, c_6), (c_1, c_2, c_5, c_9), (c_1, c_2, c_5, c_{10}), \\ & (c_1, c_2, c_5, c_{11}), (c_1, c_2, c_5, c_7, c_8), (c_1, c_2, c_5, c_7, c_{12}), (c_1, c_2, c_5, c_8, c_{12})\}; \\ T'_{SW} &= \{(c_1, c_5, c_{11}, c_2), (c_1, c_5, c_{11}, c_3, c_4, c_6), (c_1, c_5, c_{11}, c_3, c_6, c_7), \\ & (c_1, c_5, c_{11}, c_3, c_6, c_8), (c_1, c_5, c_{11}, c_3, c_4, c_7, c_8), (c_1, c_5, c_{11}, c_3, c_4, c_7, c_9), \\ & (c_1, c_5, c_{11}, c_3, c_4, c_8, c_{10}), (c_1, c_5, c_{11}, c_3, c_9, c_{10})\}; \\ T'_{We} &= \{(c_1, c_5, c_{11}, c_{15}, c_2), (c_1, c_5, c_{11}, c_{15}, c_3, c_9, c_{10})\}. \end{aligned}$$

All tie transitive NE-, We-, and SW-conjectures are proved in Sections [?], [?], and [?], respectively.

Yet, only one tie-transitive DE-conjecture, (c_1, c_2, c_3, c_4) , is proved, while (c_1, c_2, c_5) and (c_1, c_2, c_6) remain open. However, as it will be shown in Section ?? these two open claims are equivalent.

Furthermore, in Sections ?? and ?? we prove tie-transitive St-conjectures:

(c_1, c_3, c_4) , (c_1, c_3, c_5) , $(c_1, c_2, c_6, c_9, c_{10})$. (c_1, c_2, c_5, c_4) , (c_1, c_2, c_5, c_6) .

and we leave open the remaining St-conjectures:

(c_1, c_3, c_{15}) , (c_1, c_3, c_9, c_{10}) ; (c_1, c_2, c_6, c_4) , (c_1, c_2, c_5, c_9) , (c_1, c_2, c_5, c_{10}) , (c_1, c_2, c_5, c_{11}) , $(c_1, c_2, c_5, c_7, c_8)$, $(c_1, c_2, c_5, c_7, c_{12})$, $(c_1, c_2, c_5, c_8, c_{12})$

Yet, in Section ?? it will be shown that two pairs of statements:

(c_1, c_2, c_5, c_9) and (c_1, c_2, c_5, c_{10}) , $(c_1, c_2, c_5, c_7, c_{12})$ and $(c_1, c_2, c_5, c_8, c_{12})$

are equivalent. Let us also remark that general St-conjectures:

(c_1, c_3, c_5) , $(c_1, c_2, c_6, c_9, c_{10})$, (c_1, c_2, c_5, c_4) , (c_1, c_2, c_5, c_6) , $(c_1, c_2, c_5, c_7, c_8)$, $(c_1, c_2, c_5, c_9, c_{13})$, $(c_1, c_2, c_5, c_{10}, c_{13})$, $(c_1, c_2, c_5, c_{11}, c_{13})$, $(c_1, c_2, c_5, c_7, c_{12}, c_{13})$, $(c_1, c_2, c_5, c_8, c_{12}, c_{13})$

remain the tie-transitive St-conjectures; only c_{13} must be deleted, since this 2-square is not tie-transitive. The first four of these conjectures,

(c_1, c_3, c_5) , $(c_1, c_2, c_6, c_9, c_{10})$ (c_1, c_2, c_5, c_4) , (c_1, c_2, c_5, c_6) ,

were proved in general and hence, they hold for the considered tie-transitive case too. The next five remain open in both cases:

(c_1, c_2, c_5, c_9) , (c_1, c_2, c_5, c_{10}) , (c_1, c_2, c_5, c_{11}) , $(c_1, c_2, c_5, c_7, c_{12})$, $(c_1, c_2, c_5, c_8, c_{12})$.

1.11.7 Strengthening (c_1, c_2, c_3, c_4) -theorems

As we already mentioned, $\{c_1, c_2, c_3, c_4\}$ is the set of all tie-free 2-squares. In other words, each 2-square of a tie-transitive (c_1, c_2, c_3, c_4) -free configuration P contains a tie. Hence, the corresponding game form is totally tight; see Section [?]. In [?], Kukushkin proved that each totally tight game form g is St-solvable (that is, for any utility functions (U_1, U_2) the obtained bimatrix game (g, U_1, U_2) has no no strict cycles) and also conjectured that g is dominance-solvable. This conjecture was proved in [?], where a characterization of the totally tight game forms was given. Let us also recall that strict acyclicity, as well as dominance solvability, imply Nash-solvability. Hence, in the tie-transitive case, (c_1, c_2, c_3, c_4) is a NE-, St-, and DE-theorem simultaneously.

Hypergraphs T'_{NE} and T'_{St} given above show that the first two theorems can be strengthened. Indeed, (c_1, c_3) is NE-theorem and (c_1, c_3, c_4) is St-theorem.

1.11.8 TT-cycles, -examples, and -theorems

Similar to Nash-solvability, tie-transitivity also can be characterized as a certain type acyclicity. A TT - n -cycle C_n^0 in canonical form is defined by the following chain of alternating equations and inequalities:

$$x_{1,1} = x_{1,2} \geq x_{2,2} = x_{2,3} \geq x_{3,3} = \dots = x_{n-2,n-1} \geq x_{n-1,n-1} = x_{n-1,n} \geq x_{n,n} = x_{n,1} > x_{1,1}.$$

Let us notice that in the rows we have equalities, while inequalities are in the columns; notice also that the last inequality is strict. The canonical TT-4-cycle is shown in Figure ??.

Standardly, general TT- n -cycles are obtained from the canonical one by prmutations of rows and columns and, perhaps, by transposition. A configuration is called *TT-acyclic* if it has no TT-cycles.

Proposition 5 *A configuration is tie-transitive if and only if it is TT-acyclic.*

Proof Obviously, a TT-cycle is not tie-transitive; see Figure ?. Conversely, let configuration P be TT-acyclic. Partition all entries of P into equivalent classes defined by all equalities (in both rows and columns). Let A be the set of these classes. The preferences of player 1 within the rows and 2 within the columns define two acyclic relations over A . Since P is TT-acyclic both these relations are acyclic, too. Hence, they can be extended to complete orders over A . Thus, we associated with P a game form $g = g(P)X_1 \times X_2 \rightarrow A$. Hence, by Proposition ??, P is tie-transitive.

It is easy to see that, by definition, TT-cycles are semi-weak. Yet, the sets of TT-cycles and strict cycles are not nested. For example, c_1 is a tie-transitive strict cycle, while c_{13} and c_{14} are TT-cycles but not strict. Let us also recall that there are three tie-transitive 5×5 configurations that have strict cycles; see St-examples $(c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_7, c_8)$, $(c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_7, c_{12})$, $(c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_8, c_{12})$ in Figure ?. Thus we get two chains of containments: (i) weak, semi-weak, strict, and strong cycles; (ii) weak, semi-weak, and TT-cycles.

Remark 8 *More generally, we could introduce the concept of alternating cycles, in which every second edge is an equality This can be done for weak, semi-weak, and strict cycles. It is easy to see that semi-weak alternating cycles are exactly TT-cycles. As another option for alternating cycles, we could also assume that every second inequality is strict.*

Standardly, we can get TT-examples and TT-thorems, that is, the minimal (strongest) examples and theorems for the concept of TT-acyclicity.

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1.12 Hereditary and non-hereditary classes of configurations, games, and game forms

Given an object (a configuration, bimatrix game, or game form) defined on the set of situations $X = X_1 \times X_2$, standardly, we define a subobject (respectively, a subconfiguration, subgame, or subform) as a restriction of the corresponding mapping to a subset $X'_1 \times X'_2 \subseteq X_1 \times X_2 = X$, where $X'_1 \subseteq X_1$ and $X'_2 \subseteq X_2$. Also standardly, a class F of

objects is called hereditary if every subobject is in F whenever the corresponding object is in F .

It is not difficult to verify that the following classes are hereditary: (C1-C9) Weakly, semi-weakly, or strictly acyclic objects: bimatrix games, game forms, or configurations; (C10, C11) Zero-sum (matrix) games and configurations; (C12, C13) Tie-transitive (matrix) games and configurations; (C14) Totally tight game forms.

It is also easy to check that, in contrast, the following classes are not hereditary: (C15, C16) Tight (equivalently, NS or zero-sum-solvable), as well as DS game forms; (C17, C18) NS (equivalently, strongly acyclic), as well as DS bimatrix games.

For example, game form g''' is tight. Moreover, it is easy to verify that it is DS, too. In other words, for every payoff $U = (U_1, U_2)$ the obtained (tie-transitive) bimatrix game (g, U) has a DE, which, as we know, is a NE, too. Yet, g''' contains a 2×2 subform with four distinct outcomes, which is clearly not NS (hence, not tight) and not DS, either. It is easy to define a zero-sum payoff such that the corresponding 2×2 subgame has no NE (and hence, no DE, either).

It is clear that any class of object defined by a family of forbidden subobject is hereditary. In particular, configurations that do not contain c_1, c_2, c_3 , and c_4 form a hereditary class. (They correspond to the totally tight game forms.)

Conversely, every hereditary class F of objects is characterized by a family F' of forbidden subobjects. Yet, some of them might be larger than 2×2 . Moreover, F' might be infinite; see for example, the characterization of zero-sum configurations given by Propositions ?? and ??.

Thus, all our theorems formulated in terms of forbidden 2×2 subconfigurations give only sufficient conditions. In other words, they provide an approximation of a hereditary (zero-sum or acyclic configurations) and non-hereditary (NS or DS configurations) class by some special hereditary classes. Yet, we obtain all optimal such approximations.

1.13 Semi-inverse configurations; DE-conjectures (c_1, c_2, c_5, c_{11}) and $(c_1, c_2, c_6, c_9, c_{10})$ are equivalent

Given a configuration $P = \{P_{x_1}, P_{x_2} \mid x_1 \in X_1, x_2 \in X_2\}$, let us inverse pre-orders P_{x_1} for all rows $x_1 \in X_1$ (respectively, P_{x_2} for all columns $x_2 \in X_2$) and keep unchanged pre-orders in all columns (respectively, rows). We will denote the obtained two configurations P^1 and P^2 , respectively, and call them semi-inverse transformations (or semi-inversions) of P .

Clearly, both transformations are involutive, that is, $(P^1)^1 = (P^2)^2 = P$.

It is also clear that all three configurations P^1 , P^2 , and P are domination-free (or not) simultaneously, that is, all three are domination-free whenever one of them is domination-free.

Let us remark that although similar claim does not hold for dominance-solvability, in general, yet, all configurations of a hereditary class are dominance-solvable if and only if none of them is domination-free. Hence, semi-inversions respect dominance-solvability within the considered classes.

Let us also remark that semi-inversions do not respect acyclicity of any kind: strong (NS), strict, weak, or semi-weak. Indeed, it is easy to see that P may have cycles, while P^1 and/or P^2 have not, or vice versa.

Let us compute semi-inversions for fifteen 2-squares:

$$\begin{aligned} c_i^1 &= c_i^2 = c_i \text{ for } i \in \{3, 4, 12, 13, 14, 15\}; \\ c_1^1 &= c_1^2 = c_2 \text{ and } c_2^1 = c_2^2 = c_1; \quad c_5^1 = c_5^2 = c_6 \text{ and } c_6^1 = c_6^2 = c_5; \\ c_7^1 &= c_7, \quad c_7^2 = c_8, \text{ or vice versa, } c_8^1 = c_8, \quad c_8^2 = c_7, \text{ or vice versa.} \\ c_9^1 &= c_9^2 = c_{10}^1 = c_{10}^2 = c_{11}, \quad c_{11}^1 = c_9, \quad c_{11}^2 = c_{10}, \text{ or vice versa.} \end{aligned}$$

Let us recall that 2-squares are defined up to permutations of their rows and columns and also up to transposition.

Making use of semi-inversions, we can easily prove that DE-theorems (c_1, c_2, c_5, c_{11}) and $(c_1, c_2, c_6, c_9, c_{10})$ are equivalent.

Indeed, let P be a counter-example to (c_1, c_2, c_5, c_{11}) , that is, P is domination-free and (c_1, c_2, c_5, c_{11}) -free. Then both P^1 and P^2 are also domination-free and $(c_1, c_2, c_6, c_9, c_{10})$ -free. In other words, they are counter-examples to $(c_1, c_2, c_6, c_9, c_{10})$. Indeed, c_9 or c_{10} in P^1 or P^2 could appear only from c_{11} in P . Moreover, c_6 could appear only from c_5 and c_1 and c_2 only from c_2 and c_1 , respectively.

Similarly, if P is a counter-example to $(c_1, c_2, c_6, c_9, c_{10})$ then both P^1 and P^2 are counter-examples to (c_1, c_2, c_5, c_{11}) . Indeed, if P is $(c_1, c_2, c_6, c_9, c_{10})$ - and domination-free then P^1 and P^2 are both (c_1, c_2, c_5, c_{11}) - and domination-free. Indeed, we can just repeat the above arguments. In particular, c_{11} in P^1 or P^2 can appear only from c_9 or c_{10} in P .

In Section ??, we will prove $(c_1, c_2, c_6, c_9, c_{10})$ and thus (c_1, c_2, c_5, c_{11}) will follow. Let us note that we have no direct prove of the latter DE-theorem.

Let us recall four hypergraphs related to domination: theorems and example for general and tie-transitive cases.

$$\begin{aligned} E_{DE} &= \{(c_1), (c_2), (c_3, c_5, c_6), (c_4, c_5, c_6), (c_5, c_9), (c_5, c_{10}), (c_6, c_{11})\}; \\ E'_{DE} &= \{(c_1), (c_2), (c_3, c_5, c_6), (c_4, c_5, c_6)\}; \\ T_{DE} &= \{(c_1, c_2, c_3, c_4, c_9, c_{10}, c_{11}), (c_1, c_2, c_5, c_6), (c_1, c_2, c_5, c_{11}), (c_1, c_2, c_6, c_9, c_{10})\}; \\ T'_{DE} &= \{(c_1, c_2, c_3, c_4), (c_1, c_2, c_5), (c_1, c_2, c_6)\}. \end{aligned}$$

It is not difficult to verify that semi-inversions respect these four hypergraphs. In particular, two semi-inversions of the example (c_6, c_{11}) result of (c_5, c_9) and (c_5, c_{10}) , respectively, and vice versa, all four semi-inversions of the latter two examples result in (c_6, c_{11}) ; see Figure ?. Also, as we already demonstrated, DE-theorems $(c_1, c_2, c_6, c_9, c_{10})$ and (c_1, c_2, c_5, c_{11}) are transformed one to the other.

1.14 Inverse configurations

Let us apply two distinct semi-inversions together, that is, given a configuration $P = \{P_{x_1}, P_{x_2}\} \mid x_1 \in X_1, x_2 \in X_2\}$, let us inverse all pre-orders: P_{x_1} for all rows $x_1 \in X_1$

and P_{x_2} for all columns $x_2 \in X_2$), denote the obtained configuration P^d and call it the inversion of P .

Obviously, $(P^1)^2 = (P^2)^1 = P^d$; in other words, two semi-inversions of different types commute and their composition is the inversion. It is also clear that inversion is involutive, too, $(P^d)^d = P$.

It is easy to verify that $c_7^d = c_8$ and $c_9^d = c_{10}$, while all remaining 2-squares are self-dual; $c_i^d = c_i$ for $i \in \{1, 2, 3, 4, 5, 6, 11, 12, 13, 14, 15\}$.

Let us notice that inversions respect all targets, except NE. Indeed, inversions respect domination, since semi-inversions do so. For instance, let P be DE-example (c_5, c_9) . Then $P^1 = P_2 = (c_6, c_{11})$, while $P^d = (c_5, c_{10})$. Indeed, all three are DE-examples.

It is also clear that a (weak, semi-weak, or strict) cycle in P is transformed to the corresponding but oppositely directed cycle in P^d .

Finally, it is easy to see that for a strong cycle the above might be not true. For example, (c_5, c_9) is a NE-example, while (c_5, c_{10}) and (c_6, c_{11}) are not. Also four NE-theorems contain c_9 , while none contains c_{10} .

Let us notice that hypergraphs E'_{NE} and T'_{NE} do not contain 2-squares c_7, c_8, c_9 , and c_{10} .

Yet, it is easy to verify that the remaining sixteen hypergraphs remain unchanged if we substitute c_8 for c_7 , respectively, c_{10} for c_9 and vice versa.

In particular, hypergraph T_{St} contains edges

$(c_1, c_2, c_5, c_7, c_{11})$, $(c_1, c_2, c_5, c_8, c_{11})$, and $(c_1, c_2, c_5, c_{13}, c_7, c_{12})$, $(c_1, c_2, c_5, c_{13}, c_8, c_{12})$, and $(c_1, c_2, c_5, c_{13}, c_9)$, $(c_1, c_2, c_5, c_{13}, c_{10})$

that form three pairs of equivalent ST-conjectures. Similarly, T'_{St} contains edges

(c_1, c_2, c_5, c_9) , (c_1, c_2, c_5, c_{10}) , and $(c_1, c_2, c_5, c_7, c_{12})$, $(c_1, c_2, c_5, c_8, c_{12})$

that form two pairs of equivalent tie-transitive St-conjectures.

1.15 Every large configuration contains c_4, c_{12} , or c_{15}

First, let us note that an arbitrary large, or even infinite, configuration might have all its 2-squares of the same type: c_4 , or c_{12} , or c_{15} .

Proposition 6 *There is a configuration P such that sets X_1 and X_2 are both infinite and each 2-square of P is of type c_4 (respectively, c_{12} or c_{15}).*

Proof Let us fix an arbitrary (strict) order $P_i^>$ over X_i and let $P_i^=$ denote the pre-order in which all elements of X_i are equal; $i = 1, 2$. Then, let us introduce three configurations $P^{>, >}$, $P^{>, =}$, and $P^{=, =}$ on $X_1 \times X_2$ as follows:

$$P_{x_i}^{>, >} = P_{3-i}^{>} \text{ and } P_{x_i}^{=, =} = P_{3-i}^{=} \text{ for all } x_i \in X_i \text{ and } i = 1, 2;$$

$$P_{x_1}^{>, =} = P_2^{>} \text{ for all } x_1 \in X_1, \text{ while } P_{x_2}^{>, =} = P_1^{=} \text{ for all } x_2 \in X_2.$$

It is easy to verify that each 2-square of $P^{>, >}$ (respectively, of $P^{>, =}$ or $P^{=, =}$ is of type c_4 (respectively, c_{12} or c_{15}).

We will show that every sufficiently large configuration contains c_4, c_{12} , or c_{15} ; in other words, in this case the hypergraphs of examples and theorems are $E_\infty = (c_4), (c_{12}), (c_{15})$ and $T_\infty = \{(c_4, c_{12}, c_{15})\}$, respectively. First, let us consider the tie-free configurations.

Proposition 7 *Every 3×7 configuration P with no ties contains c_4 .*

Proof A set of cardinality k admits $k!$ distinct orders. Since $7 > 6 = 3!$, by the pigeonhole principle, among seven columns of P there are two with the same order. Let us fix such two columns and consider the obtained 3×2 subconfiguration. It contains c_4 , since $3 > 2! = 2$.

Proposition 8 *Every 4×76 configuration contains c_4 , or c_{12} , or c_{15} .*

Proof Let N_4 denote the number of pre-orders over a set of cardinality 4. A simple counting shows that $N_4 = 75$. Indeed, there are $4! = 24$ strict pre-orders and $3 \times 4 \times 3 = 36$, $4 \times 2 + 3 \times 2 = 14$, and 1 pre-orders with one, two, and three equalities, respectively.

Again, by the pigeonhole principle, among $N_4 + 1 = 76$ columns there are two with the same pre-order. Let us fix two such columns and consider the obtained 4×2 subconfiguration. By the same arguments, it contains a 2-square of type c_4, c_{12} , or c_{15} .

Let us notice that there are arbitrarily large $n \times n$ configurations that contain, e.g., only c_1, c_{12}, c_{15} ; see example in Figure ?? (e) for $n = 3$ and $n = 4$.

1.16 Examples and conjectures for e-configurations

In this Section, instead of pre-orders $\mathcal{P} : \mathcal{Y}^\epsilon \rightarrow \{<, >, =\}$ we will consider equivalence relations $\mathcal{R} : \mathcal{Y}^\epsilon \rightarrow \{=, \neq\}$, where the equality $=$ is transitive. In other words, \mathcal{R} is just a partition of Y . Respectively, given $X = X_1 \times X_2$, e-configurations $R = \{R_{x_1}, R_{x_2} \mid x_1 \in X_1, x_2 \in X_2\}$ substitute for configurations and, instead of fifteen 2-squares $C = \{c_1, \dots, c_{15}$ in Figure ??, we obtain six 2-squares $B = \{b_1, \dots, b_6\}$ in Figure ??.

Our target set will be configurations that contain a constant line (CL), row or column.

Remark 9 *More targets are of interest; for example, read-once line, that is, a row or column whose all entries are pairwise non-equal. Furthermore, let us define a cross as a row and column. Then a constant, read once, or mixed cross are possible targets, too. However, in this paper we consider only constant line targets, because of its relation to dominance-solvability.*

Two e-configurations without constant lines are given in Figure ??. It is easy to see that the first one represents b_2 , while the second one contains only two 2-squares, b_4 and b_6 .

Thus, we obtain the hypergraph of CL-examples $E_{CL} = \{(b_2), (b_4, b_6)\}$. Dualizing it we obtain the hypergraph of CL-conjectures $T_{CL} = \{(b_2, b_4), (b_2, b_6)\}$.

In Section ??, we will prove both conjectures. Furthermore, in Section ??, we will see that (b_2, b_6) is, in fact, equivalent with the important DE-theorem $(c_1, c_2, c_3, c_4, c_9, c_{10}, c_{11})$.

Theorem 9 *A (b_2, b_4) - or (b_2, b_6) -free e-configuration contains a constant line.*

2 CL-theorems; proof of Theorem ??

2.1 A partition of X defined by the first row and column

Let us consider the first row $x_1^1 \in X_1$ and column $x_2^1 \in X_2$, and partition all rows $X_1 = X_1^- \cup X_1^\neq$ and columns $X_2 = X_2^- \cup X_2^\neq$ as follows:

$X_1^- = \{x_1^i \in X_1 \mid x_{i,1} = x_{1,1}\}$, with respect to equivalence relation $R_{x_2^1}$ in the first column, and $X_1^\neq = X_1 \setminus X_1^-$;

$X_2^- = \{x_2^j \in X_2 \mid x_{1,j} = x_{1,1}\}$, with respect to equivalence relation $R_{x_1^1}$ in the first row, and $X_2^\neq = X_2 \setminus X_2^-$.

Furthermore, we will say that an entry $x_{i,j}$ is of type 1 (respectively, 2) if only the first (respectively, the second) of the next two equations holds:

$$x_{i,j} = x_{1,j} \text{ with respect to } R_{x_2^j} \text{ or } x_{i,j} = x_{i,1} \text{ with respect to } R_{x_1^i}.$$

Furthermore, if none or both hold, we say that $x_{i,j}$ is of type 0 or 3, respectively. Examples are given in Figure ??.

Remark 10 *Let us notice that it is not necessary to specify with respect to which row or column a given equality or inequality is considered. Indeed, any two distinct entries belong to at most one line, row or column. Obviously, with respect to it an equality or inequality between these two entries is considered and, by definition, they are incomparable if do not belong to a line. This observation holds for both, pre-orders in configurations and equivalence relations in e-configurations.*

Lemma 1 *A 2-square formed by rows i, i' and columns j, j' is b_2 whenever entries $x_{i,j}$ and $x_{i',j'}$ are of type 1, while $x_{i,j'}$ and $x_{i',j}$ are of type 2.*

Proof . Indeed, $x_{i,j} = x_{1,j}$, since $x_{i,j}$ is of type 1, while $x_{i',j} \neq x_{1,j}$, since $x_{i',j}$ is of type 2. Hence, $x_{i,j} \neq x_{i',j}$. Similar arguments show that $x_{i,j} \neq x_{i,j'}$, $x_{i',j'} \neq x_{i,j'}$, and $x_{i',j'} \neq x_{i',j}$. Thus, the considered 2-square is b_2 .

2.2 Proof of CL-theorem (b_2, b_4)

Given a (b_2, b_4) -free e-configuration R , we will show that it has a constant line, row or column.

Lemma 2 *Every entry $x_{i,j} \in X_1^\neq \times X_2^\neq$ is of type 3.*

Proof . We know that $x_{1,1} \neq x_{1,j}$ and $x_{1,1} \neq x_{i,1}$; hence, $x_{i,j} = x_{1,j}$ and $x_{i,j} = x_{i,1}$, since otherwise b_2 or b_4 appear.

Lemma 3 *No entry $x_{i,j} \in X_1^- \times X_2^\neq \cup X_1^\neq \times X_2^-$ is of type 0.*

Proof . Without loss of generality, we can assume that $x_{i,j} \in X_1^- \times X_2^\neq$, that is, $x_{1,1} \neq x_{1,j}$ and $x_{1,1} = x_{i,1}$; hence, b_4 appears whenever $x_{i,j} \neq x_{1,j}$ and $x_{i,j} \neq x_{i,1}$.

Lemma 4 *If an entry $x_{i,j} \in X_1^\neq \times X_2^-$ is of type 1 then $x_{i',j}$ is of type 3 for each row $x_{1,i'} \in X_1^-$. Respectively, if $x_{i,j} \in X_1^- \times X_2^\neq$ is of type 2 then $x_{i,j'}$ is of type 3 for each column $x_{2,j'} \in X_2^-$.*

Proof . Since the above two claims are symmetric, without loss of generality, we can restrict ourselves by the first one. We know that $x_{i,j} \neq x_{i,1}$, since $x_{i,j}$ is of type 1 and $x_{i,1} \neq x_{i',1}$, since $x_{i,1} \in X_1^\neq$, while $x_{i',1} \in X_1^-$. Hence, $x_{i',1} = x_{i',j}$ and $x_{i,j} = x_{i',j}$, since otherwise the 2-square formed by rows i, i' and columns $1, j$ is b_2 or b_4 . The first and second equations imply that $x_{i',j}$ is of type 1 and 2, respectively. Thus, $x_{i',j}$ is of type 3.

Lemma 5 *There exists either a column of type 1 or row of type 2 in $X_1^\neq \times X_2^-$.*

Proof . Assume indirectly that in $X_1^\neq \times X_2^-$ there is an entry of type 1 in every row and an entry of type 2 in every column. In other words, $X_1^\neq \times X_2^-$ contains an alternating n -cycle. By a permutation of rows and columns of $X_1^\neq \times X_2^-$, we can bring this cycle to canonical form, where entries $x_{1,1}, x_{2,2}, \dots, x_{n,n}$ are of type 1, while entries $x_{1,2}, x_{2,3}, \dots, x_{n-1,n}, x_{n,1}$ are of type 2. As in Lemma , we show that $x_{1,1} \neq x_{1,2}, x_{1,2} \neq x_{2,2}, \dots, x_{n-1,n} \neq x_{n,n}, x_{n,n} \neq x_{n,1}, x_{n,1} \neq x_{1,1}$. In case $n = 2$ the corresponding four entries form a 2-square b_2 by Lemma . Yet, the case $n > 2$ we can reduce to $n = 2$ by induction. To do so, let us consider three entries of the cycle, $x_{1,1}, x_{1,2}$, and $x_{2,2}$, and add to them also $x_{2,1}$. As we already know, $x_{1,1} \neq x_{1,2} \neq x_{2,2}$. Hence, $x_{1,1} = x_{2,1} = x_{2,2}$, since otherwise b_2 or b_4 would appear. Thus, we obtain a $(n - 1)$ -cycle if substitute $x_{1,1}, x_{1,2}$, and $x_{2,2}$ by $x_{2,1}$. Reducing n to 2 we get b_2 and a contradiction.

CL-theorem (b_2, b_4) immediately follows from the above Lemmas. Indeed, a column of type 1 in $X_1^\neq \times X_2^-$ is extended to the column of type 1 in $X_1 \times X_2^-$ by Lemma ??, while a row of type 2 in $X_1^\neq \times X_2^-$ is extended to the row of type 2 in $X_1^\neq \times X_2$ by Lemma ??. Thus, in all cases $X_1 \times X_2$ has a constant line, row or column.

2.3 Proof of CL-theorem (b_2, b_6)

Given a (b_2, b_6) -free e-configuration R , we will show that it has a constant line, row or column.

Lemma 6 *Every entry $x_{i,j} \in X_1^\neq \times X_2^\neq$ is of type 1 or 2.*

Proof . We know that $x_{1,1} \neq x_{1,j}$ and $x_{1,1} \neq x_{i,1}$; hence, from two equations, $x_{i,j} = x_{1,j}$ and $x_{i,j} = x_{i,1}$, exactly one holds, since otherwise b_2 or b_6 would appear.

Furthermore, by Lemma ??, a 2-square formed by rows $i, i' \in X_1^\neq$ and columns $j, j' \in X_2^\neq$ is b_2 whenever entries $x_{i,j}$ and $x_{i',j'}$ are of type 1, while $x_{i,j'}$ and $x_{i',j}$ are of type 2. Hence, no such 2-square exists in $X_1^\neq \times X_2^\neq$.

Hence, $X_1^\neq \times X_2^\neq$ contains either a column of type 1 or a row of type 2. To show this we just repeat the proof of Lemma ??.

Since the above two options are symmetric, without loss of generality we can assume that row i of $X_1^\neq \times X_2^\neq$ is of type 2, that is, $i \in X_1^\neq$ and $x_{i,j'}$ is of type 2, whenever $j' \in X_2^\neq$. We will show that $x_{i,j}$ is of type 2 for all $j \in X_2^\neq$, too; in other words, the whole row i in $X_1^\neq \times X_2^\neq$ is of type 2. Let us consider two rows, 1 and $i \in X_1^\neq$, and three columns, 1, $j \in X_2^\neq$ and $j' \in X_2^\neq$. We know that $x_{1,1} = x_{1,j} \neq x_{1,j'}$, $x_{1,1} \neq x_{i,1}$, and $x_{i,1} = x_{i,j'}$. Then, $x_{1,j'} \neq x_{i,j'}$, since $x_{i,j'}$ is of type 2 and also because otherwise b_6 would appear.

If $x_{1,j} = x_{i,j}$. then $x_{i,1} = x_{i,j}$, since otherwise b_6 would appear. If $x_{1,j} \neq x_{i,j}$. then $x_{i,j} = x_{i,j'}$, since otherwise b_2 would appear. In both cases, by transitivity, we conclude that $x_{i,j} = x_{i,1} = x_{i,j'}$ and hence, $x_{i,j}$ is of type 2, too. Since, this holds for arbitrary $j \in X_2^\neq$, we conclude that the whole row i is of type 2.

3 Proof of Theorems ?? and ??

3.1 On games with no ties

Obviously, the concepts of weak or semi-weak cycle make no sense if there are no ties. Thus, in this case only three from our five main concepts remain: NS, DS, and (strict) acyclicity. Let us also recall that acyclicity or DS imply NS.

Furthermore, there are only four tie-free configurations: c_1, c_2, c_3 , and c_4 . Among them c_1 has no NE; hence, it has no DE, either, and it has a cycle. In fact, c_1 itself is a strict 2-cycle and it is domination free. Configuration c_2 is domination free, too, although it is acyclic and it has two NE, while c_3 and c_4 are NS, DS, and acyclic. In contrast, configuration (c_2, c_3) in Figure ?? is NE-free, DE-free, and has a (strict) 3-cycle.

Thus, we obtain the following hypergraphs of examples: $E_{NE} = E_{St} = \{(c_1), (c_2, c_3)\}$ and $E_{DE} = \{(c_1), (c_2)\}$. Respectively the dual hypergraphs of conjectures are $T_{NE} = T_{St} = \{(c_1, c_2), (c_1, c_3)\}$ and $E_{DE} = \{(c_1, c_2)\}$. In fact, Theorems ?? and ?? claim that all these conjectures are theorems.

Let us remark that (c_1, c_3) cannot be a DE-theorem, since c_2 is domination free.

3.2 Acyclicity theorem (c_1, c_3)

Given a tie-free configuration P that does not contain c_1 and c_3 , let us prove that P is (strictly) acyclic (and hence, it is NS, too). Let us notice that P contains c_1 or c_3 if and only

if it contains a directed path of length 3 within a 2×2 subconfiguration.

Assume indirectly that P contains a strict n -cycle C_n ; see Figure ???. Without any loss of generality, we can assume that (i) C_n is canonical, (ii) P is of size $n \times n$, and (iii) P contains no strict cycles shorter than n , although it might contain other n -cycles.

Let us consider two pairs of situations: $x_{1,1}, x_{2,1}$ and $x_{2,1}, x_{2,2}$. With respect to P , there are four possible order relations between them:

- (1) $x_{1,1} < x_{2,1}$ and $x_{2,1} < x_{2,2}$; (2) $x_{1,1} > x_{2,1}$ and $x_{2,1} > x_{2,2}$;
 (3) $x_{1,1} > x_{2,1}$ and $x_{2,1} < x_{2,2}$; (4) $x_{1,1} < x_{2,1}$ and $x_{2,1} > x_{2,2}$,

which are given in Figure ??. In cases 1 (respectively, (3) and (4)) forbidden c_1 (respectively, c_3) appears, while in case 2 we get a shortcut, that is, there exists a strict cycle shorter than n , in contradiction to (iii). Thus, if P is (c_1, c_3) -free then it is acyclic (and hence, NS, too).

3.3 Acyclicity theorem (c_1, c_2)

We have to show that a configuration P is acyclic (and hence, NS, too) whenever P does not contain $(c_1$ and $c_2)$.

Assume indirectly that P contains C_n , as in Figure ??. First, let us consider $x_{2,1}$. Again, it is easy to see that c_1 and a shortcut appear in cases 1 and 2, respectively, and, without loss of generality, we can assume that case 3 takes place; see Figure ??. Next, let us consider $x_{1,3}$ and the similar four cases:

- (1) $x_{1,2} < x_{1,3}$ and $x_{1,3} < x_{2,3}$; (2) $x_{1,2} > x_{1,3}$ and $x_{1,3} > x_{2,3}$;
 (3) $x_{1,2} > x_{1,3}$ and $x_{1,3} < x_{2,3}$; (4) $x_{1,2} < x_{1,3}$ and $x_{1,3} > x_{2,3}$.

Again, cases 1 and 2 result in a shortcut and c_1 , respectively. Case 3 is not possible, either. Indeed, we get a shortcut if $x_{2,1} > x_{2,3}$ and c_2 if $x_{2,1} < x_{2,3}$. Hence, case 4 takes place.

Moreover, it is easy to see that $x_{1,1} < x_{1,3}$ and $x_{2,1} < x_{2,3}$, since otherwise a shortcut appears. We can “propagate this pattern along the main diagonal”. To do so let us consider the sequence of situations

$$x_{2,1}, x_{1,3}, x_{3,2}, x_{2,3}, \dots, x_{i+1,i}, x_{i,i+2}, \dots, x_{n,n-1}, x_{n-1,1}, x_{1,n}. \quad (2)$$

Here and, in the sequel, we standardly set $n+1 = 1$.

Adjusting the above arguments, by induction on i , it is easy to show that:

$x_{i,i-1} < x_{i,i}$, $x_{i,i-1} < x_{i-1,i-1}$, and $x_{i,i-1} < x_{i,i-1}$, where $i = 2, \dots, n$, while
 $x_{i,i+2} > x_{i,i+1}$, $x_{i,i+2} > x_{i+1,i+2}$, and $x_{i,i+2} > x_{i,i}$, where $i = 1, \dots, n-1$.

In particular, $x_{n+1,n+1} = x_{1,1} > x_{1,n} = x_{n+1,n}$.

Then, by the second round of induction, now on j , we can extend these inequalities as follows:

$x_{i,i-j} < x_{i,i}$, $x_{i,i-j} < x_{i-j,i-j}$, and $x_{i,i-j} < x_{i,i+1}$,
 where $i = 2, \dots, n$ and $j = 1, \dots, i-1$, while

$x_{i,i+j} > x_{i,i+1}$, $x_{i,i+j} > x_{i+j-1,i+j}$, and $x_{i,i+j} > x_{i,i}$,
 where $i = 1, \dots, n-1$ and $j = 1, \dots, n-i+1$.

In particular, $x_{1,1} < x_{1,n}$ and we get a contradiction; see Figure ??.

Let us remark that for the first induction the situations should be ordered in accordance with (??), while for the second one they can appear in an arbitrary order unless j decreases.

3.4 Dominance-solvability theorem (c_1, c_2)

Let us consider 2-squares c_1, c_2, c_3 and c_4 in Figure ?? and notice that in c_1 and c_2 both pairs of opposite sides are directed oppositely, in c_4 both are directed similarly, while in c_3 one oppositely and one similarly.

We want to prove that every (c_1, c_2) -free configuration P is DS. Yet, it will suffice to show that P cannot be domination free, since (c_1, c_2) -free configurations form a hereditary class. Let us assume indirectly that P is domination free and prove that it must contain c_1 or c_2 .

Given a $m \times n$ domination free configuration $P = (P_1, P_2)$, without loss of generality let us assume that $x_{1,1} < \dots < x_{1,n}$ with respect to P_2 . For all i, j such that $1 \leq i < j \leq m$ there is a $k \in [m] = \{1, \dots, m\}$ such that $x_{k,i} > x_{k,j}$, since otherwise column j dominates column i . In particular, $x_{k,1} > x_{k,n}$ for some $k \in [m]$. Then, for every $\ell \in \{2, \dots, n-1\}$ we have $x_{k,1} > x_{k,\ell}$ or $x_{k,\ell} > x_{k,n}$, by transitivity of P_2 , while, in contrast, $x_{1,1} < x_{1,\ell}$ and $x_{1,\ell} < x_{1,n}$. Hence, P contains c_1 or c_2 whenever there are $\ell', \ell'' \in [n]$ such that $x_{1,\ell'} > x_{k,\ell'}$, while $x_{1,\ell''} < x_{k,\ell''}$ with respect to P_1 . Yet, if there are no such $\ell', \ell'' \in [n]$ then either row 1 dominates row k , or vice versa, with respect to P_1 .

4 NE-theorems; proof of Theorems ?? and ??

4.1 Nash-solvability and strong acyclicity

Here we will prove Theorems ?? and ?? that imply all conjectures from $T_{NE} = \{(c_1, c_2, c_5), (c_1, c_2, c_6, c_9), (c_1, c_3, c_5), (c_1, c_3, c_9, c_2), (c_1, c_3, c_9, c_4), (c_1, c_3, c_9, c_6)\}$; in other words, for every edge $t \in T_{NE}$, a configuration P is NS whenever P is t -free, that is, it contains no 2-squares from t .

Suppose that P is a counterexample. Since, for any t , the class of t -free configurations is hereditary, we can assume that P is a (locally) minimal t - and NE-free configuration. Then, by Theorem ??, P is of size $n \times n$ and it contains strong improvement n -cycle C_n (in canonical form, as we can assume without loss of generality); see Section ?. We will show in this case P cannot be t -free, thus, getting a contradiction.

As we already mentioned, we can restrict ourselves by the locally minimal NE-free configurations. By Theorem ?? each such configuration P is of size $n \times n$ for some $n \geq 2$ and P contains a strong improvement cycle C_n . Without loss of generality we can assume that $C_n = C_n^0$ is canonical. In particular,

$$x_{i,i+1} \geq x_{i,j}, x_{i,i+1} > x_{j,i+1}, x_{j,j} \geq x_{i,j}, x_{j,j} > x_{j,i+1}, \text{ for } j > i + 1. \quad (3)$$

Furthermore, if $n = 2$ then 2-square c_1 is a unique NE-free configuration (in fact, c_1 is a strong 2-cycle). Hence, we will assume that $n \geq 3$. Additionally, we assume that P is t -free and consider successively the following subsets t : (c_2, c_5) , (c_3, c_5) , (c_2, c_6, c_9) , and (c_3, c_9) . Theorem ??) will follow, since in the first three cases we get a contradiction. For $t = (c_3, c_9)$ we will characterize the corresponding configurations explicitly, thus proving (Theorem ??).

4.2 Locally minimal NE-free and (c_2, c_5) -free configurations

Let us consider C_n^0 in Figure ?? (where $n = 7$). By definition, $x_{i,i} > x_{i,j}$ (with respect to P_2) whenever $j \neq i$; in particular, $x_{i,i} > x_{i,i-1}$ for $i \in [n] = \{1, \dots, n\}$, where standardly, $0 = n$. Then, $x_{i,i} \geq x_{j,i}$ (with respect to P_1); in particular, $x_{i,i} \geq x_{i+1,i}$ for $i \in [n] = \{1, \dots, n\}$, where standardly, $n+1 = 1$. Moreover, the latter n inequalities are also strict, since otherwise c_5 would appear.

By similar arguments we show that $x_{i,i+1} > x_{i,i+2}$ and $x_{i,i+1} > x_{i-1,i+1}$ for $i = 1, \dots, n-1$; see Figure ??.

Next, let us notice that $x_{i,i} = x_{i-2,i}$ for $i = 2, \dots, n$. Indeed, $x_{i,i} \geq x_{i-2,i}$, since C_n is a string cycle, and c_2 would appear in case $x_{i,i} > x_{i-2,i}$.

Furthermore, $x_{i,i+2} \geq x_{i,i+3}$ for $i = 1, \dots, n-3$, since otherwise $x_{i,i+2}, x_{i,i+3}, x_{i+2,i+2}, x_{i+2,i+3}$ would form a c_5 .

Next, let us notice that $x_{i,i+3} = x_{i+1,i+3}$ for $i = 1, \dots, n-3$. Indeed, $x_{i,i+3} \leq x_{i+3,i+3} = x_{i+1,i+3}$, and if $x_{i,i+3} < x_{i+1,i+3}$ then $x_{i,i+1}, x_{i,i+3}, x_{i+3,i+1}, x_{i+3,i+3}$ would form a c_2 , by (??).

Similarly, by induction on j , we show that $x_{i,i+j} \geq x_{i,i+j+1}$ and $x_{i,i+j} = x_{i+1,i+j}$ for $1 \leq i \leq n-3$ and $2 \leq i+j \leq n-1$.

In particular, $x_{n,n} = x_{n-2,n} = x_{n-3,n} = \dots = x_{2,n} = x_{1,n}$ in contradiction with the strict inequality $x_{n,n} > x_{1,n}$ obtained before.

4.3 Locally minimal NE-free and (c_2, c_6, c_9) - or (c_3, c_5) -free configurations

These two cases are easy. Let us consider C_n^0 in Figure ?? (a) and (b) (where $n = 3$). By definition, in both cases $x_{2,2} > x_{2,1}$ $x_{1,1} \geq x_{2,1}$. In case (a) we already got a contradiction, since four above situations form c_3 or c_5 .

In case (b) we have to proceed a little further. Clearly, $x_{2,3} \geq x_{2,1}$, $x_{1,2} \geq x_{1,3}$, $x_{2,3} > x_{1,3}$, and again we get a contradiction, since situations $x_{1,1}, x_{1,3}, x_{2,1}, x_{2,3}$ form c_9 if two equalities hold, c_6 if exactly one, and c_2 if none.

4.4 Locally minimal NE-free and (c_3, c_9) -free configurations

Let us consider C_n^0 in Figure ?? (a) and (b) (where $n = 7$ and $n = 8$). By ??, for all $i \in [n]$ we have:

$$x_{i,i} > x_{i,i+1}, x_{i,i} > x_{i,i-1}, x_{i,i} \geq x_{i+1,i}, x_{i,i} \geq x_{i-2,i}; \tag{4}$$

$$x_{i,i+1} > x_{i+1,i+1}, x_{i,i+1} > x_{i-1,i+1}, x_{i,i+1} \geq x_{i,i+2}, x_{i,i+1} \geq x_{i,i-1}. \tag{5}$$

Furthermore, it is not difficult to show that $x_{i,i} = x_{i+1,i}$ and $x_{i,i+1} = x_{i,i+2}$, since otherwise c_3 appears, while $x_{i,i} > x_{i-2,i}$ and $x_{i,i+1} > x_{i,i-1}$, since otherwise c_9 appears; see Figure ?? (a). Standardly, we prove all four claims by induction introducing situations in the following order:

$$x_{2,1}, x_{1,3}, \dots, x_{i,i-1}, x_{i-1,i+1}, \dots, x_{n,n-1}, x_{n-1,1}, x_{1,n}, x_{n,2}.$$

Furthermore, $x_{1,1} = x_{2,1} \geq x_{4,1}$ unless $n < 5$; moreover, $x_{2,1} = x_{4,1}$, since otherwise situations $x_{2,1}, x_{4,1}, x_{2,4}$, and $x_{4,4}$ form c_3 .

Similarly, we prove that $x_{1,3} = x_{1,5}$ unless $n < 5$.

Then let us recall that $x_{4,5} \geq x_{4,1}$ and conclude that $x_{4,5} > x_{4,1}$, since otherwise situations $x_{1,1}, x_{4,1}, x_{1,5}$, and $x_{4,5}$ form c_9 .

In general, it is not difficult to prove by induction that

$$x_{i,i} = x_{i,i+1} = x_{i,i+3} = \dots = x_{i,i+2j-1}, \text{ while } x_{i-1,i} > x_{i,i} > x_{i,i+2j}; \tag{6}$$

$$x_{i,i+1} = x_{i,i+2} = x_{i,i+4} = \dots = x_{i,i+2j}, \text{ while } x_{i,i} > x_{i,i+1} > x_{i,i+2j+1}; \tag{7}$$

In both cases each sum is taken (n) (in particular, $n = 0$) and $1 \leq j < n/2$ (in particular, j takes values 1, 2, and 3 for $n = 7$ and $n = 8$).

If $n > 1$ is odd we immediately get a contradiction, since in this case, by (??), $x_{1,1} = x_{n-1,1}$, while, by (??), $x_{1,1} > x_{n-1,1}$ for all $n > 1$. Yet, for each even n , the family F_n of all locally minimal NE-free and (c_3, c_9) -free configurations is not empty.

Up to an isomorphism, F_2 (respectively, F_4) consists of a unique configuration: c_1 in Figure ?? (respectively, (c_2, c_4, c_5, c_6) in Figure ??) (c). Two larger examples, from F_6 and F_8 , are given in Figures ?? (d) and (e), respectively.

We already know that each configuration $P \in F_{2k}$ must satisfy (??) - (??). Yet, P has one more important property:

$$x_{i,i+2j+1} \neq x_{i,i+2j'+1}, x_{i+2j,i} \neq x_{i+2j',i} \tag{8}$$

for all $i \in [n]$ and for all positive distinct $j, j' < n/2$. Indeed, it is easy to see that otherwise c_9 appears; see Figure ??(d).

Let us denote by G_n the family of all configurations satisfying (??) - (??). We already know that $F_n \subseteq G_n$ and $F_n = G_n = \emptyset$ if $n > 1$ is odd. Let us show that $F_n = G_n$ for even n . Obviously, G_4 consists of a inique configuration (c_2, c_4, c_5, c_6) and $G_2 = \{c_1\}$; see Figure ?? (c). Examples of configurations from G_6 and G_8 are given in Figures ?? (d) and (e). It is not difficult to verify that each configuration of G_n contains eight 2-squares

$C' = \{c_2, c_4, c_5, c_6, c_1, c_7, c_8, c_{13}\}$ whenever $n \geq 6$; they are shown in Figure ?? (d). Moreover, it is clear that c_{12} appears, too, when $n \geq 10$.

On the other hand, no configuration $P \in G_n$ contains $c_9, c_{10}, c_{11}, c_{14}$, or c_{15} . Indeed, it is easy to check that no 2-square in P can have two adjacent equalities. Also simple case analysis shows that P cannot contain c_3 . Indeed, c_3 has a unique NE and no situation in P can play this role.

Thus, P can contain only nine 2-squares of $C'' = C' \cup \{c_{12}\}$. In particular, each $P \in G_n$ is (c_3, c_9) -free; in other words, $G_n \subseteq F_n$ and, hence, $G_n = F_n$ for all (even) n . This implies Theorem ?? and provides an explicit characterization for family F_n of locally minimal NE-free and (c_3, c_9) -free configurations.

Remark 11 *Interestingly, for even n each configuration $P \in F_n = G_n$ contains the same set of nine 2-squares C'' if $n \geq 10$; for $P \in G_8$ there are two options: C' or C'' (see example in Figure ?? (e), where c_{12} does not appear); for $P \in G_6$ only C' is possible; furthermore, G_4 consists of a unique configuration (c_2, c_4, c_5, c_6) in Figure ?? (c) and G_2 only of c_1 ; finally, $F_n = G_n$ is empty if $n > 1$ is odd.*

5 All weak and semi-weak acyclicity theorems

Here we will prove all semi-weak and weak acyclicity conjectures:

$$T_{SW} = \{(C', c_2), (C', c_3, c_4, c_6), (C', c_3, c_4, c_7, c_8), (C', c_3, c_6, c_7), (C', c_3, c_6, c_8), (C', c_3, c_6, c_9, c_{10}), (C', c_3, c_7, c_8, c_9, c_{10})\};$$

$$T_{We} = \{(C'', c_2), (C'', c_3, c_6, c_9, c_{10}), (C'', c_3, c_7, c_8, c_9, c_{10})\};$$

where $C' = \{c_1, c_5, c_{11}, c_{13}, c_{14}\}$ and $C'' = C' \cup \{c_{15}\}$ are the sets of all semi-weak and, respectively, weak 2-cycles. In this section we will omit C' and C'' , yet, remember that C' (respectively, C''), must be added to each semi-weak (respectively, weak) acyclicity theorem.

5.1 Several common assumptions

Again, in all cases we will assume indirectly that the considered configuration $P = (P_1, P_2)$ contains a (semi-) weak n -cycle C_n . Without loss of generality we will also assume that $C_n = C_n^0$ is the canonical cycle, that is,

$$x_{1,1} \geq x_{1,2} \geq x_{2,2} \geq \dots \text{geq} x_{i-1,i} \geq x_{i,i} \geq x_{i,i+1} \geq \dots \geq x_{n-1,n} \geq x_{n,n} \geq x_{n,1} \geq x_{1,1}, \quad (9)$$

where even and odd inequalities are taken with respect to pre-orders P_1 and P_2 , respectively; see Figures ?? - ?. Moreover, in case of a semi-weak cycle we will always assume that $x_{1,1} < x_{n,1}$ is a strict inequality.

Furthermore, we can assume that configuration P is the minimal counterexample to a considered conjecture $t \in T_{St}$. From this assumption we immediately conclude that P is of

size $n \times n$ and that C_n^0 is the shortest cycle in P ; in other words, P can have more (semi-) weak n -cycles but no (semi-) weak n' -cycle such that $n' < n$. From this we derive the next claim.

Lemma 7 *None of the following four chains of non-strict inequalities can take place:*

$$\begin{aligned} x_{i-1,i} \geq x_{j,i} \geq x_{j,j+1}, \quad x_{j,j} \geq x_{j,i} \geq x_{i,i}, \quad j > i; \\ x_{i,i} \geq x_{i,j} \geq x_{j,j}, \quad x_{j-1,j} \geq x_{i,j} \geq x_{i,i+1}. \end{aligned}$$

In all cases we assume that $j \neq i - 1$ and $j \neq i$

Proof Indeed, in both cases a shortcut (i.e., a (semi-) weak n' -cycle with $n' < n$) appears. Let us notice that if both inequalities in a chain are, in fact, equalities then the obtained shorter cycle might be weak, while we might need a semi-weak one, when the corresponding case is considered. However, a semi-weak cycle exists, too, since we can treat these two equalities as non-strict inequalities in the opposite direction. All other cases are even more obvious.

Several examples are given in Figure ??.

Finally, we will assume that P is C' -, or respectively, C'' -free since 2-squares $c_1, c_5, c_{11}, c_{13}, c_{14}$ are semi-weak and c_{15} is a weak cycle. (Let us recall that every semi-weak cycle is a weak cycle, as well.)

5.2 Weak and semi-weak theorems

$$(c_2), (c_3, c_6, c_9, c_{10}), (c_3, c_7, c_8, c_9, c_{10})$$

We will prove these three claims together, since they hold for both, semi-weak and weak, cases, modulo $C''' = C' \cup \{c_{15}\}$.

Standardly (see Figure ??, from Lemma ?? (and transitivity) we derive the next two alternatives:

- (i) $x_{i,i} < x_{i+1,i} > x_{i+1,i+1}$ or (ii) $x_{i,i} > x_{i+1,i} < x_{i+1,i+1}$, and
- (iii) $x_{i,i+1} < x_{i,i+2} > x_{i+1,i+2}$ or (iv) $x_{i,i+1} > x_{i,i+2} < x_{i+1,i+2}$.

It is not difficult to verify (see Figure ??) that

(a) if (i) and (iii) take place then either a shortcut, or $(c_2$ and $c_3)$, or $(c_2$ and c_6 and $c_7)$, or $(c_2$ and $c_{10})$ appear;

(b) if (ii) and (iv) take place then either a shortcut, or $(c_2$ and $c_3)$, or $(c_2$ and c_6 and $c_8)$, or $(c_2$ and $c_9)$ appear.

Let us notice that every two sets, one from $\{(c_2), (c_3, c_6, c_9, c_{10}), (c_3, c_7, c_8, c_9, c_{10})\}$ and another from $\{(c_2, c_3), (c_2, c_6, c_7), (c_2, c_6, c_8), (c_2, c_9), (c_2, c_{10})\}$ intersect. Hence, assuming that P is (c_2) -, or (c_3, c_6, c_9, c_{10}) -, or $(c_3, c_7, c_8, c_9, c_{10})$ -free configuration, we conclude that either (i,iv) or (ii,iii) hold.

Since these two cases are symmetric, let us assume, without loss of generality, that (i,iv) holds, as in Figure ???. It is not difficult to show that this pattern propagates. Indeed, let us assume that, in contrast to (i,iv), we have $x_{i,i} > x_{i+2,i} < x_{i+1,i+2}$, as in Figure ??. Again, simple case analysis shows that 2-squares (c_2, c_3) , or (c_2, c_6, c_7) , or (c_2, c_6, c_8) , or (c_2, c_9) , or (c_2, c_{10}) appear, in contradiction to each of the three assumptions that P is (c_2) -, or (c_3, c_6, c_9, c_{10}) -, or $(c_3, c_7, c_8, c_9, c_{10})$ -free. Hence, inequality $x_{i,i} < x_{i+2,i} > x_{i+1,i+2}$ must hold, in agreement with (i). Similarly, we conclude that $x_{i,i+1} > x_{i,i+3} < x_{i+4,i+3}$, and then, by induction, that $x_{i,i+1} > x_{i,i+j} < x_{i+j+1,i+j}$, in agreement with (iv). However, this contradicts transitivity, since $x_{1,2} > x_{1,n} > x_{1,1} > x_{1,2}$.

Remark 12 *Let us notice that the weak 2-square c_{15} did not appear in the above arguments. Hence, (c_2) , (c_3, c_6, c_9, c_{10}) , and $(c_3, c_7, c_8, c_9, c_{10})$ are both, weak and semi-weak acyclicity theorems. Since there are no other weak acyclicity conjectures, the studying of this concept is complete. Yet, we will prove a few more semi-weak acyclicity conjectures. Clearly, these proofs cannot extend the weak acyclicity case. In other words, assuming that C_n^0 is a weak n -cycle, we will show all $2n$ non-strict inequalities in it are, in fact, equalities.*

5.3 Four remaining semi-weak theorems

$$(c_3, c_4, c_6), (c_3, c_4, c_7, c_8), (c_3, c_6, c_7), (c_3, c_6, c_8)$$

We will prove all four theorems, simultaneously. Let us assume indirectly that a $n \times n$ configuration P is (c_3, c_4, c_6) -, or (c_3, c_4, c_7, c_8) -, or (c_3, c_6, c_7) -, or (c_3, c_6, c_8) - free and P contains a canonical semi-weak n -cycle C_n^0 . Moreover, we can assume that P contains no semi-weak n' -cycle for any $n' < n$. We will get a contradiction showing that C_n^0 must consist of $2n$ equalities; hence, it is weak but not semi-weak.

Again Lemma ??? imply that one of the four given above options (i,ii,iii,iv) holds. Let us consider all four combinations (i,iii), (ii,iv), (i,iv), (ii,iii) given in Figure ???.

The following observation is obvious, yet, it will be frequently applied.

Lemma 8 *For all four combinations, c_3 appears whenever C_n^0 has two successive strict inequalities.*

Thus, in this case we always get a contradiction, since c_3 is forbidden by all four considered assumptions.

If C_n^0 has no two successive strict inequalities then without loss of generality we can assume that $x_{1,1} = x_{1,2} > x_{2,2} = x_{2,3}$, as in Figure ??. It is easy to verify that P contains 2-squares (c_4, c_6) , (c_4, c_7, c_8) , (c_6, c_8) , (c_6, c_7) in cases (i,iii), (ii,iv), (i,iv), (ii,iii), respectively; see Figure ??. It is easy to check that each of these four sets and every "assumption set" (c_3, c_4, c_6) , (c_3, c_4, c_7, c_8) , (c_3, c_6, c_7) , (c_3, c_6, c_8) , intersect. Thus, all four considered theorems are proved by these contradictions.

6 Strict acyclicity theorems

Here, we will prove six St-conjectures:

$$\{(c_1, c_2, c_4, c_5), (c_1, c_2, c_5, c_6), (c_1, c_2, c_6, c_9, c_{10}), (c_1, c_3, c_4, c_9, c_{10}), (c_1, c_3, c_5), (c_1, c_3, c_6, c_9, c_{10})\} \subset T_{St}.$$

6.1 Several common assumptions and theorem (c_1, c_3, c_5)

In all cases we will assume indirectly that the considered configuration $P = (P_1, P_2)$ contains a strict n -cycle C_n . Without loss of generality we will also assume that $C_n = C_n^0$ is the canonical cycle, that is,

$$x_{1,1} > x_{1,2} > x_{2,2} > \dots > x_{i-1,i} > x_{i,i} > x_{i,i+1} > \dots > x_{n-1,n} > x_{n,n} > x_{n,1} > x_{1,1}, \quad (10)$$

where even and odd inequalities are taken with respect to P_1 and P_2 , respectively; see Figures ?? - ??.

Furthermore, we can always assume that P is the minimal counterexample to a considered conjecture from T_{St} . From this assumption we immediately conclude that P is of size $n \times n$ and that C_n^0 is the shortest cycle in P ; in other words, P can have more strict n -cycles but no strict n' -cycle, where $n' < n$. From this, in turn, we derive the next claim.

Lemma 9 *Neither of the following two chains of nonstrict inequalities can take place:*

$$x_{i,i} \geq x_{j,i} \geq x_{j,j}, \quad x_{j,j+1} \geq x_{j,i} \geq x_{i-1,i}.$$

Proof In both cases a shortcut (i.e., a strict n' -cycle with $n' < n$) appears.

Several examples are given in Figure ?. Let us note that in all cases we make use of transitivity:

$$y > y' \ \& \ y' \geq y'' \Rightarrow y > y'', \quad y \geq y' \ \& \ y' > y'' \Rightarrow y > y''$$

Finally, we will assume that P is c_1 -free, since c_1 itself is a strict cycle.

For the beginning let us notice that theorem (c_1, c_3, c_5) trivially follows from the above assumption. Indeed, $x_{i,i} \geq x_{i+1,i} \geq x_{i+1,i+1}$ cannot take place, by Lemma ?. Hence, $x_{i,i} < x_{i+1,i}$ or $x_{i+1,i} < x_{i+1,i+1}$. In both cases we obtain a directed 3-path that implies c_1 or c_3 or c_5 .

6.2 Strict acyclicity theorem (c_1, c_2, c_5, c_6)

By Lemmas ??, a shortcut appears whenever $x_{i,i} \geq x_{i+1,i} \geq x_{i+1,i+1}$; on the other hand, c_1 appears whenever $x_{i,i} < x_{i+1,i} < x_{i+1,i+1}$; Finally, c_5 appears whenever $x_{i,i} < x_{i+1,i} = x_{i+1,i+1}$ or $x_{i,i} = x_{i+1,i} < x_{i+1,i+1}$; see Figure ?? (a). Thus, only two options remain:

- (i) $x_{i,i} > x_{i+1,i} < x_{i+1,i+1}$ and (ii) $x_{i,i} < x_{i+1,i} > x_{i+1,i+1}$.

The same arguments show that similar two options take place for $x_{i,i+1}$, $x_{i,i+2}$, and $x_{i+1,i+2}$:

- (iii) $x_{i,i+1} > x_{i,i+2} < x_{i+1,i+2}$ and (iv) $x_{i,i+1} < x_{i,i+2} > x_{i+1,i+2}$.

It is easy to see that if (i,iii) or (ii, iv) take place then c_2 , or c_6 , or a shortcut will appear; see Figure ?? (a). Thus, either (i,iv) or (ii,iii) take place. Without any loss of generality we can assume the latter case; see Figure ?? (a).

Furthermore, a simple case analysis shows that c_2 , or c_6 , or a shortcut take place whenever $x_{i,i+1} \geq x_{i,i+3}$ and one of the four forbidden 2-squares appears whenever $x_{i+2,i+3} \geq x_{i,i+3}$; see Figure ?? (b). Thus, we conclude that $x_{i+2,i+3} < x_{i,i+3}$ and $x_{i,i+1} < x_{i,i+3}$ (with respect to P_2 and P_1 , correspondingly).

This pattern propagates. Similarly, we show that c_2 , or c_6 , or a shortcut takes place whenever $x_{i,i+1} \geq x_{i,i+4}$ and one of the four forbidden 2-squares appears whenever $x_{i+3,i+4} \geq x_{i,i+4}$; see Figure ?? (b). Hence, $x_{i+3,i+4} < x_{i,i+4}$ and $x_{i,i+1} < x_{i,i+4}$; etc. Finally, we conclude that $x_{1,n} > x_{n-1,n}$ in contradiction to transitivity, since $x_{n-1,n} > x_{n,n} > x_{1,n}$; see Figure ?? (b).

6.3 Strict acyclicity theorem $(c_1, c_2, c_6, c_9, c_{10})$

Standardly (see Figure ??, from Lemma ?? we derive that

- (i) $x_{i,i} \geq x_{i+1,i} < x_{i+1,i+1}$ or (ii) $x_{i,i} < x_{i+1,i} \leq x_{i+1,i+1}$, and
 (iii) $x_{i,i+1} \geq x_{i,i+2} < x_{i+1,i+2}$ or (iv) $x_{i,i+1} < x_{i,i+2} \leq x_{i+1,i+2}$.

It is easy to see that if (i,iii) or (ii, iv) take place then c_2 , or c_6 , or c_9 , or c_{10} , or a shortcut will appear; see Figure ???. Thus, either (i,iv) or (ii,iii) take place. Without any loss of generality we can assume the latter case; see Figure ???.

The non-strict inequalities propagate along the rows. Indeed, c_1 , or c_2 , or c_6 , or c_9 appear whenever $x_{i,i+1} < x_{i,i+3}$; hence, $x_{i,i+1} \geq x_{i,i+3}$; see Figure ???. Similarly, c_1 , or c_2 , or c_6 , or c_9 appear whenever $x_{i,i+1} < x_{i,i+4}$; see Figure ???. etc. Finally, we conclude that $x_{1,2} \geq x_{1,n}$ in contradiction to transitivity, since $x_{1,n} > x_{1,1} > x_{1,2}$; see Figure ???.

Let us remark that implicitly we frequently make use of the following implication: c_2 , or c_6 , or c_9 appear whenever for some $i \neq i'$ and $j \neq j'$ we have:

$$x_{i,j} < x_{i',j'}, x_{i,j} < x_{i',j}, x_{i',j'} \leq x_{i',j}, x_{i',j'} \leq x_{i,j'}.$$

6.4 Strict acyclicity theorem ($c_1, c_3, c_6, c_9, c_{10}$)

First, let us notice that each 2-square with a directed 3-path is either c_1 , or c_3 , or c_5 and only c_5 can appear, since c_1 and c_3 are forbidden. Then, standardly (see Figure ??, from Lemma ?? we derive that

- (i) $x_{i,i} < x_{i+1,i} = x_{i+1,i+1}$ or (ii) $x_{i,i} = x_{i+1,i} < x_{i+1,i+1}$, and
- (iii) $x_{i,i+1} < x_{i,i+2} = x_{i+1,i+2}$ or (iv) $x_{i,i+1} = x_{i,i+2} < x_{i+1,i+2}$.

It is easy to see that if (i,iii) or (ii, iv) take place then c_2 , or c_6 , or c_9 , or c_{10} , or a shortcut will appear; see Figure ?. Thus, either (i,iv) or (ii,iii) take place. Without any loss of generality we can assume the latter case; see Figure ?.

Furthermore, if $x_{1,4} \geq x_{3,4}$ then either a shortcut or c_6 appear. Hence, $x_{1,4} < x_{3,4}$. Then, $x_{1,4} \leq x_{1,2}$, since otherwise we get a shortcut again. By similar arguments, this pattern propagates: $x_{1,5} < x_{4,6}$, otherwise a shortcut or c_6 appear; $x_{1,5} \leq x_{1,2}$, etc. Finally, we conclude that $x_{1,n} \leq x_{1,2}$, in contradiction to transitivity, since $x_{1,n} = x_{1,1} > x_{1,2}$.

6.5 Strict acyclicity theorem ($c_1, c_3, c_4, c_9, c_{10}$)

Again, each 2-square with a directed 3-path results in c_5 , since c_1 and c_3 are forbidden. Then, standardly (see Figure ??), from Lemma ?? we derive the same four options:

- (i) $x_{i,i} < x_{i+1,i} = x_{i+1,i+1}$ or (ii) $x_{i,i} = x_{i+1,i} < x_{i+1,i+1}$, and
- (iii) $x_{i,i+1} < x_{i,i+2} = x_{i+1,i+2}$ or (iv) $x_{i,i+1} = x_{i,i+2} < x_{i+1,i+2}$.

However, unlike the previous section, now combinations (i,iv) and (ii, iii) are impossible, since c_4 appears. Thus, either (i,iii) or (ii,iv) take place. Without any loss of generality we can assume the latter case; see Figure ?.

Furthermore, let us notice that $x_{i,i} < x_{i,i+2}$ and $x_{i,i+1} < x_{i+2,i+1}$, since otherwise c_{10} or a shortcut appears; see Figure ?.

Then, let us show that $x_{i,i} < x_{i+2,i}$, too. Indeed, otherwise, either a shortcut, or c_1 , or c_3 , or c_{10} appear; see Figure ?. Now, it is easy to see that $x_{i+2,i} = x_{i+2,i+1}$, since c_3 or c_4 appear in case of a strict inequality. These pattern propagates. Similarly, we can show that $x_{i,i} < x_{i+3,i}$ and $x_{i+3,i} = x_{i+3,i+2}$, etc. In gwneral, $x_{i,i} < x_{i+j,i}$ and $x_{i+j,i} = x_{i+j,i+j-}$ for $1 < j \leq n - i$. In particular, $x_{n,1} = x_{n,n-1}$ in contradiction to transitivity, since $x_{n,n-1} = x_{n,n} > x_{n,1}$.

Let us remark that in fact we already got a contradiction one step before. Indeed, $x_{n-1,1} = x_{n-1,n-2} = x_{n-1,n-1}$ with respect to P_2 and $x_{n-1,1} = x_{n,1}$ with respect to P_1 . Hence, $x_{n-1,1}$, $x_{n,1}$, $x_{n-1,n-1}$, and $x_{n,n-1}$ form a c_{10} .

Let us also notice that in the considered case (i,iii) we made no use of c_9 , while c_{10} would not appear in case of (ii,iv).

6.6 Strict acyclicity theorem (c_1, c_2, c_4, c_5)

Now each 2-square with a directed 3-path results in c_3 , since c_1 and c_5 are forbidden. Again standardly (see Figure ??), from Lemma ?? we derive similar four options:

- (i) $x_{i,i} < x_{i+1,i} > x_{i+1,i+1}$ or (ii) $x_{i,i} > x_{i+1,i} < x_{i+1,i+1}$, and
- (iii) $x_{i,i+1} < x_{i,i+2} > x_{i+1,i+2}$ or (iv) $x_{i,i+1} > x_{i,i+2} < x_{i+1,i+2}$.

First, let us assume that (ii) and (iv) take place for all i , as in Figure ?? A. Now, it is easy to see that $x_{i+1,i} = x_{i,i+2}$ and $x_{i,i+2} = x_{i+2,i+2}$, since otherwise c_2 or a shortcut would appear; see Figure ?? A.

Then, we will show that $x_{i,i} > x_{i+2,i} < x_{i+2,i+2}$.

First, let us assume indirectly that $x_{i,i} \leq x_{i+2,i}$; see Figure ?? B. Then, $x_{i+2,i} > x_{i+1,i}$, by transitivity. Furthermore, if $x_{i+2,i} < x_{i+2,i+1}$ then quadruple $x_{i,i}, x_{i,i+1}, x_{i+2,i}, x_{i+2,i+1}$ form c_1 or c_5 ; yet, if $x_{i+2,i} > x_{i+2,i+1}$ then $x_{i+1,i}, x_{i+1,i+1}, x_{i+2,i}, x_{i+2,i+1}$ form c_2 ; finally, if $x_{i+2,i} = x_{i+2,i+1}$ then $x_{i+1,i}, x_{i+1,i+3}, x_{i+2,i}, x_{i+2,i+3}$ form c_5 ; see Figure ?? B. Thus, we conclude that $x_{i,i} < x_{i+2,i}$;

Now, it remains to notice that $x_{i+2,i} > x_{i+2,i+2}$, since otherwise a shortcut would take place; see Figure ?? B.

Similar arguments show that this pattern propagates. As the next step, let us show that $x_{i+3,i} < x_{i,i}$. Indeed, if $x_{i+3,i} \geq x_{i,i}$ then c_1 or c_2 or c_5 would appear; see Figure ?? B. Then, if $x_{i+3,i} \geq x_{i+3,i+3}$ then a shortcut would take place; etc.

Finally, we get a contradiction, since quadruple $x_{1,1}, x_{n-1,1}, x_{n-1,1}, x_{n-1,n}$ forms c_2 ; see Figure ?? B.

The consider case is fully symmetric with the case when (i) and (iii) take place for all i .

Finally, it remains to notice that c_4 appears whenever combinations (i,vi) or (ii,iii) take place; see Figure ?? C. An alternative way to the same conclusion is shown in Figure ?? D.

7 DE-theorems

Here we will prove all four DE-theorems: $(c_1, c_2, c_3, c_4, c_9, c_{10}, c_{11})$, (c_1, c_2, c_5, c_6) , (c_1, c_2, c_5, c_{11}) , and $(c_1, c_2, c_6, c_9, c_{10})$. In Section ??, it was shown that the last two statements are equivalent. For this reason, we will prove only $(c_1, c_2, c_6, c_9, c_{10})$. In fact we have no independent proof for (c_1, c_2, c_5, c_{11}) .

7.1 General remarks

Let us recall that every class K of games (configurations) defined by a family, finite or infinite, of forbidden subgames (subconfigurations) is hereditary, that is, if a game (configuration) belongs to K then every its subgame (subconfiguration) also belongs to K . Hence, to prove that every game U in K is dominance solvable, it will be enough to show that U is not domination-free, that is, it has at least one pair of strategies such that one is dominated

by the other. Let us assume indirectly that U is domination-free. Since class K is hereditary, we can without any loss of generality we can strengthen this assumption and assume that U is locally minimal domination-free game, that is, after deleting any strategy from U , a pair of dominating strategies appears in the obtain subgame.

A strategy of a player, 1 or 2, is called *constant* if the result of the opponent does not depend on his strategy. It is easy to see that in a locally minimal domination-free game U no player has a constant strategy. Indeed, deleting such a strategy from U we obtain a domination-free game whenever U was a domination-free. Hence, U cannot be locally minimal.

7.2 Proof of DE-theorem $(c_1, c_2, c_3, c_4, c_9, c_{10}, c_{11})$

As we just demonstrated, it is enough to show that every $(c_1, c_2, c_3, c_4, c_9, c_{10}, c_{11})$ -free game U has a constant strategy. Yet, this trivially follows from CL-theorem (b_2, b_6) . Indeed, each pre-order \mathcal{P} uniquely defines an equivalence relation \mathcal{R} . To get \mathcal{R} we just substitute symbol \neq for each symbol $>$ or $<$ in \mathcal{P} . Hence, each configuration P uniquely defines an e-configuration $R = R(P)$.

It is easy that P has a constant strategy if and only if $R(P)$ has a constant line.

Obviously, a game U is $(c_1, c_2, c_3, c_4, c_9, c_{10}, c_{11})$ -free if and only if the corresponding e-configuration R is (b_2, b_6) -free. Indeed, 2-squares (c_1, c_2, c_3, c_4) all define b_2 , while c_9, c_{10}, c_{11} all define b_6 . Moreover, there is no other 2-square that defines b_2 or b_6 .

By Theorem ??, every (b_2, b_6) -free e-configuration contains a constant line. Hence, every $(c_1, c_2, c_3, c_4, c_9, c_{10}, c_{11})$ configuration has a constant strategy.

Remark 13 *Let us note that CL-theorem (b_2, b_6) is much stronger than the corresponding DE-theorem $(c_1, c_2, c_3, c_4, c_9, c_{10}, c_{11})$. For example, the latter would hold even without transitivity assumption for relations $>$ and $<$.*

7.3 Proof of DE-theorem (c_1, c_2, c_5, c_6)

7.3.1 Several remarks on (c_1, c_2, c_5, c_6) -free configurations

Given a (c_1, c_2, c_5, c_6) -free configuration P , standardly we assume that P is locally minimal domination-free; in particular P has no constant strategies.

Furthermore, P is strictly acyclic. Indeed, in Section ??, we proved that (c_1, c_2, c_5, c_6) is a St-theorem, that is, (c_1, c_2, c_5, c_6) -free configuration has no strict improvement cycle.

Let us notice that all 2-squares, except c_1, c_2, c_3, c_5 , and c_6 , have the following characteristic property $p12356$: if two parallel lines oriented oppositely, then the remaining two parallel lines are equalities. For example, if $x_{1,1} > x_{1,2}$, while $x_{2,1} < x_{2,2}$ then either $x_{1,1} = x_{2,1}$ and $x_{2,1} = x_{2,2}$.

Furthermore, all 2-squares, except c_1, c_2, c_5 , and c_6 , have a similar but slightly weaker characteristic property $p1256$: if two parallel lines oriented oppositely, then the remaining

two parallel lines are oriented similarly. For example, if $x_{1,1} > x_{1,2}$, while $x_{2,1} < x_{2,2}$ then either $x_{1,1} > x_{2,1}$ and $x_{2,1} > x_{2,2}$, or $x_{1,1} < x_{2,1}$ and $x_{2,1} < x_{2,2}$, or $x_{1,1} = x_{2,1}$ and $x_{2,1} = x_{2,2}$.

Indeed, it is easy to see that 2-squares $c_4, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{14}$, and c_{15} do not have oppositely directed parallel lines, while c_{13} and c_3 have two, yet, the other two are equalities or similar inequalities, respectively.

Thus, $p1256$ (respectively, $p12356$) holds for a configuration P if and only if P is (c_1, c_2, c_5, c_6) -free (respectively, $(c_1, c_2, c_3, c_5, c_6)$ -free). (Naturally, we say that P satisfies $p1256$ or $p12356$ if every 2-square of P does.)

Making use of these two reformulations, we reduce DE-theorem (c_1, c_2, c_5, c_6) to the seemingly weaker DE-theorem $(c_1, c_2, c_3, c_5, c_6)$ and prove the latter one.

7.3.2 Proof of DE-theorem $(c_1, c_2, c_3, c_5, c_6)$

Let us assume indirectly that configuration P is locally minimal domination-free and $(c_1, c_2, c_3, c_5, c_6)$ -free and prove that there are two options: (i) P contains a constant row or (ii) each column in P is constant. In both cases we get a contradiction, since no locally minimal domination-free configuration can have a constant line.

Then, by $p12356$, if $x_{i,j} > x_{i,j'}$, while $x_{i',j} < x_{i',j'}$ then either $x_{i,j} = x_{i',j}$ and $x_{i,j'} = x_{i',j'}$. For domination-free configurations this claim can be strengthened. Indeed, since P is domination-free, no distinct columns $x_2^j, x_2^{j'} \in X_2$ can dominate each other, that is, there are two distinct rows x_1^i and $x_1^{i'}$ such that the above equations and inequalities hold. Since, equations are transitive, we conclude that $x_{i,j} = x_{i',j}$ and $x_{i,j'} = x_{i',j'}$ for any two rows i and i' such that $x_{i,j} \neq x_{i,j'}$ and $x_{i',j} \neq x_{i',j'}$.

Given x_2^j and $x_2^{j'}$, let us denote by $X_1(j, j')$ the set of all such rows, that is,

$$X_1(j, j') = \{x_1^i \in X_1 \mid x_{i,j} \neq x_{i,j'}\} \subseteq X_1.$$

Let us fix a column $x_2^{j_0} \in X_2$ and consider the family F of subsets $X_1(j_0, j) \subseteq X_1$ for all columns x_2^j .

It is easy to see that these subsets cover the whole set X_1 , since otherwise X_1 contains a constant row. Indeed, if row x_i does not belong to the considered union then $x_{i,j_0} = x_{i,j}$ for any column $x_2^j \in X_2$.

Now, let us delete from F all subsets that are not inclusion-maximal, choose a unique representative from each subfamily of equal inclusion-maximal subsets, and denote the obtained Sperner set-family by F' . Obviously, the union is still X_1 , that is, $\cup_{X_1(j_0, j) \in F} X_1(j_0, j) = \cup_{X_1(j_0, j) \in F'} X_1(j_0, j) = X_1$.

Now let us consider two more columns $x_2^{j_1}$ and x_2^j such that $X_1(j_0, j_1), X_1(j_0, j) \in F'$. Since family F' is Sperner, both differences $X_1(j_0, j_1) \setminus X_1(j_0, j)$ and $X_1(j_0, j) \setminus X_1(j_0, j_1)$ are not empty. Let us choose two rows $x_1^{i'} \in X_1(j_0, j_1) \setminus X_1(j_0, j)$ and $x_1^{i''} \in X_1(j_0, j) \setminus X_1(j_0, j_1)$. By definition, $x_{i',j_1} \neq x_{i',j_0}$ and $x_{i'',j} \neq x_{i'',j_0}$, while $x_{i',j} = x_{i',j_0}$ and $x_{i'',j_1} = x_{i'',j_0}$. Hence, by transitivity, $x_{i',j_1} \neq x_{i',j}$ and $x_{i'',j_1} \neq x_{i'',j}$. These arguments imply that $X_1(j_1, j) \supseteq X_1(j_0, j_1) \Delta X_1(j_0, j)$, where Δ denotes symmetric difference, and also that $x_{i',j_1} = x_{i'',j_1}$ for any $i', i'' \in X_1(j_0, j_1) \cup X_1(j_0, j)$. Repeating these arguments for all columns $x_2^j \in X_2^j$ such that $X_1(j_0, j) \in F'$ we derive that column $x_2^{j_1}$ is constant.

7.3.3 Proof of DE-theorem (c_1, c_2, c_5, c_6)

Since DE-theorem $(c_1, c_2, c_3, c_5, c_6)$ is already proved, we can assume without loss of generality that P contains c_3 . We also assume indirectly that P is locally minimal domination-free and (c_1, c_2, c_5, c_6) -free. The last assumption implies that P is strictly acyclic, that is, there is no strict improvement cycle in P .

Let $X' \subseteq X = X_1 \times X_2$ be the set of all entries that belong to a 2-square c_3 . By strict acyclicity, there is an entry $x_{i,j} \in X'$ minimal in X' both in row i and in column j , that is,

$$x_{i,j} \leq x_{i,j'} \forall x_{i,j'} \in X'; \quad x_{i,j} \leq x_{i',j} \forall x_{i',j} \in X'.$$

Indeed, otherwise X' would contain a strict improvement cycle.

Without loss of generality, we can assume that $i = j = 1$ and four entries $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}$ form c_3 , as in Figure ??.

Since $x_{1,1} < x_{2,1}$, $x_{1,2} < x_{2,2}$, and P is domination-free, there exists a column 3 such that $x_{3,1} > x_{3,2}$.

Furthermore, since $x_{1,2} < x_{2,2}$, while $x_{1,3} > x_{2,3}$, by property *p1256*, there are only three options:

- (i) $x_{1,2} = x_{1,3}$ and $x_{2,2} = x_{2,3}$, or
- (ii) $x_{1,2} > x_{1,3}$ and $x_{2,2} > x_{2,3}$, or
- (iii) $x_{1,2} < x_{1,3}$ and $x_{2,2} < x_{2,3}$.

Similarly, since $x_{1,1} < x_{2,1}$, while $x_{1,3} > x_{2,3}$, by property *p1256*, there are only three options:

- (j) $x_{1,1} = x_{1,3}$ and $x_{2,1} = x_{2,3}$, or
- (jj) $x_{1,1} > x_{1,3}$ and $x_{2,1} > x_{2,3}$, or
- (jjj) $x_{1,1} < x_{1,3}$ and $x_{2,1} < x_{2,3}$.

It is easy to verify that c_2 appears in case (i). Furthermore, (ii) implies (jj), by transitivity, and (jj) is in contradiction with row-minimality of $x_{1,1}$. Hence, (iii) holds and implies (jjj); see Figure ??.

In the obtained 2×3 subconfiguration of P columns 1 and 2 are dominated by column 3. Hence, there is a row, say 3, such that $x_{3,1} > x_{3,3}$. Since $x_{1,1} < x_{1,3}$ and $x_{2,1} < x_{2,3}$, by *p1256*, we conclude that $x_{1,3} < x_{3,3}$. Hence, by transitivity, $x_{2,3} < x_{3,3}$ and, by *p1256* again, $x_{2,1} < x_{3,1}$, as in Figure ??.

If $x_{3,2} < x_{1,2}$ then, by *p1256* again, $x_{3,1} < x_{3,2} < x_{3,3}$, while $x_{3,3} < x_{3,1}$, in contradiction with transitivity. Hence, $x_{3,2} \geq x_{1,2}$; see Figure ??.

Thus, in the considered 3×3 subconfiguration of P , row 1 is dominated by row 3. Hence, there exists a column, say 4, such that $x_{3,4} < x_{1,4}$; see Figure ??.

By *p1256* and transitivity, we derive that $x_{1,3} < x_{1,4}$, $x_{3,3} < x_{3,4}$, $x_{3,1} < x_{3,4}$. $x_{2,1} \leq x_{2,4}$.

Thus, in the considered 3×4 subconfiguration of P , column 1 is dominated by column 4. Hence, there exists a row, say 4, such that $x_{4,4} < x_{4,1}$; see Figure ??.

Again, by *p1256* and transitivity, we derive that $x_{3,1} < x_{4,1}$, $x_{3,4} < x_{4,4}$, $x_{1,4} < x_{4,4}$, $x_{1,2} \leq x_{4,2}$ and $x_{1,3} \leq x_{4,3}$.

Thus, in the considered 4×4 subconfiguration of P , row 1 is dominated by row 4. Hence, there exists a column, say 5; etc. Since the above pattern propagates infinitely, while P is finite and strictly acyclic, we get a contradiction.

7.3.4 An alternative proof; partial results

Strict acyclicity implies that there is an entry $x_{i,j} \in X$ minimal simultaneously in its row x_1^i and column x_2^j . Indeed, if each entry can be strictly improved for one of the players then obviously P has a strict improvement cycle (since X is finite). Without loss of generality, we will assume that $i = j = 1$.

Similar to Section ??, let us introduce two partitions $X_1 = X_1^= \cup X_1^>$ and $X_2 = X_2^= \cup X_2^>$, where

$$\begin{aligned} X_1^= &= \{x_1^i \mid x_{i,1} = x_{1,1}\} \text{ and } X_2^= = \{x_2^j \mid x_{1,j} = x_{1,1}\}; \\ X_1^> &= \{x_1^i \mid x_{i,1} > x_{1,1}\} \text{ and } X_2^> = \{x_2^j \mid x_{1,j} > x_{1,1}\}; \end{aligned}$$

see Figure ?? A. It is easy to see that both partitions are non-trivial. Indeed, $X_1^=$ and $X_2^=$ are not empty, since they both contain $x_{1,1}$; if $X_1^>$ or $X_2^>$ is empty then, respectively, x_2^1 or x_1^1 is a constant strategy, which contradicts to the local minimality of P .

We will prove that the following inequalities hold:

$$x_{i,j} \geq x_{i,1}, \quad x_{i,j} \geq x_{1,j} \quad \forall x_{i,j} \in X_1^> \times X_2^>; \quad (11)$$

see Figure ?? A. In other words, we were able to prove DE-conjecture (c_1, c_2, c_5, c_6) in all cases but (??). It seems that in this case a (c_1, c_2, c_5, c_6) -free DE-example might exist.

We will prove (??) indirectly. Let us assume that $x_{i,1} > x_{i,j}$ for some $x_{i,j} \in X_1^> \times X_2^>$. We also know that $x_{i,1} > x_{1,1}$ and $x_{1,j} > x_{1,1}$; see Figure ?? B. If $x_{i,j} \leq x_{1,j}$ then c_2 or c_6 will appear. Hence, $x_{i,j} > x_{1,j}$.

However, strategy x_1^1 is not dominated by x_1^i . Hence, there exists k such that $x_{1,k} > x_{i,k}$.

Let us suppose that $x_2^k \in X_2^=$, as in Figure ?? C. We know also that $x_{1,j} > x_{1,k}$, since $x_2^k \in X_2^=$, while $x_2^j \in X_2^>$. Hence, $x_{i,j} > x_{i,k}$, since otherwise, if $x_{i,j} \leq x_{i,k}$, then quadruple $x_{1,j}, x_{1,k}, x_{i,j}, x_{i,k}$ would form c_1 or c_5 . Then, by transitivity, $x_{i,1} > x_{i,k}$. We know also that $x_{1,1} = x_{1,k}$, since $x_2^k \in X_2^=$. Thus, $x_{1,1}, x_{1,k}, x_{i,1}, x_{i,k}$ form c_6 and we get a contradiction from which we conclude that $x_2^k \in X_2^>$, as in Figure ?? D.

Moreover, it is not difficult to show that $x_{1,k} \geq x_{1,j}$. Indeed, let us assume indirectly that $x_{1,k} < x_{1,j}$. If $x_2^k \in X_2^=$, as in Figure ?? C, we already got a contradiction. Yet, in case $x_2^k \in X_2^>$, as in Figure ?? D, we can repeat exactly the same arguments and come to a similar conclusion; only quadruple $x_{1,1}, x_{1,k}, x_{i,1}, x_{i,k}$ form c_2 instead of c_6 ; cf Figures ?? C and D. However, both c_2 and c_6 are forbidden and we can conclude that $x_{1,k} \geq x_{1,j}$.

However, it is easy to show that equality $x_{1,k} = x_{1,j}$ cannot hold either. Indeed, if it holds then the equality $x_{i,k} \geq x_{i,j}$ must hold, too, since otherwise quadruple $x_{1,j}, x_{1,k}, x_{i,j}, x_{i,k}$ would form c_5 or c_6 ; see Figure ?? E.

Thus, $x_{1,k} > x_{1,j}$, as Figure ?? E. Then $x_{i,k} > x_{i,j}$, too, since otherwise quadruple $x_{1,j}, x_{1,k}, x_{i,j}, x_{i,k}$ would form c_2 or c_6 ; see Figure ?? F.

Similarly, $x_{i,k} > x_{i,1}$, since otherwise quadruple $x_{1,1}, x_{1,k}, x_{i,1}, x_{i,k}$ would form c_2 or c_6 ; see Figure ?? F.

Thus, $x_{i,k} > x_{i,1}$ and $x_{1,k} > x_{1,1}$. Since X is domination-free, there should be some ℓ such that $x_{\ell,k} < x_{\ell,1}$, as in Figure ?? G or H.

First, let us assume that $x_1^\ell \in X_1^-$, as in Figure ?? G. Let us notice that in this case $x_{i,1} > x_{\ell,1} = x_{1,1}$. Furthermore, $x_{i,k} > x_{\ell,k}$, since otherwise c_1 or c_5 would appear. Then, by transitivity, we conclude that $x_{1,k} > x_{\ell,k}$ and quadruple $x_{1,1}, x_{1,k}, x_{\ell,1}, x_{\ell,k}$ forms c_6 , see Figure ?? G.

Now, let us assume that $x_1^\ell \in X_1^>$ and $x_{i,k} \geq x_{\ell,k}$, as in Figure ?? H. Then quadruple $x_{1,1}, x_{1,k}, x_{\ell,1}, x_{\ell,k}$ forms c_2 . Hence, $x_1^\ell \in X_1^>$ and $x_{i,k} < x_{\ell,k}$, as in Figure ?? I.

Etc. Obviously, the sequence $x_{1,1} < x_{1,j} < x_{i,j} < x_{i,k} < x_{\ell,k} < \dots$ will never end. Yet, X is finite and we obtain a contradiction.

7.4 Proof of DE-theorem ($c_1, c_2, c_6, c_9, c_{10}$)

In Section ??, it was shown that $(c_1, c_2, c_6, c_9, c_{10})$ is a strict acyclicity theorem.

Hence, in every $(c_1, c_2, c_6, c_9, c_{10})$ -free configuration P there is an entry $x_{i,j}$ minimal in its row i and column j simultaneously. Indeed, if each entry can be improved in its row or column then obviously a strict improvement cycle would exist. Without loss of generality we may assume that $i = j = 1$. Furthermore, let us assume indirectly that P is domination-free.

Lemma 10 *If $x_{1,j} > x_{i,j}$ then $x_{i,j} \geq x_{i,1}$.*

Proof Let us consider Figure ?? and assume indirectly that $x_{i,j} < x_{i,1}$. Then we obtain:

c_2 if $x_{1,1} < x_{1,j}$ and $x_{1,1} < x_{i,1}$,

c_9 if $x_{1,1} = x_{1,j}$ and $x_{1,1} = x_{i,1}$, and

c_6 if $x_{1,1} = x_{1,j}$, $x_{1,1} < x_{i,1}$ or $x_{1,1} < x_{1,j}$, $x_{1,1} = x_{i,1}$;

in contradiction with our assumption that P is $(c_1, c_2, c_6, c_9, c_{10})$ -free.

Lemma 11 *If $x_{i,1} > x_{i,j_1}$ then $x_{i,j_1} \geq x_{1,j_1}$. Moreover, there is a column j_2 such that $x_{1,j_2} > x_{i,j_2} > x_{i,j_1}$.*

Proof Consider Figure ?? again. The first inequality, $x_{i,j_1} \geq x_{1,j_1}$, immediately follows from Lemma ??.

Furthermore, there is a column j_2 such that $x_{1,j_2} > x_{i,j_2}$, otherwise row 1 would be dominated by row i , in contradiction with our assumption that P is domination-free.

Again, $x_{i,j_2} \geq x_{i,1}$, by Lemma ??, and $x_{i,j_2} > x_{i,j_1}$, by transitivity.

For each column j_1 there is a row i_1 such that $x_{1,j_1} > x_{i_1,j_1}$, otherwise column 1 would be dominated by column j_1 . Then, by Lemma ??, $x_{i_1,j_1} \geq x_{1,j_1}$ and there is a column j_2 such that $x_{1,j_2} > x_{i_1,j_2} > x_{i_1,j_1}$. For column j_2 , in its turn, there is a row i_2 such that $x_{1,j_2} > x_{i_2,j_2}$, otherwise column 1 would be dominated by column j_2 . Then, by Lemma ?? again, $x_{i_2,j_2} \geq x_{1,j_2}$ and there is a column j_3 such that $x_{1,j_3} > x_{i_2,j_3} > x_{i_2,j_2}$, etc.; see Figure ?. This strict improvement chain propagates infinitely, in contradiction with strict acyclicity of P .

8 Tie-transitive case

Here we consider our five targets assuming tie-transitivity of the considered bimatrix games (and hence, of their configurations, as well). The obtained results are related to game forms rather than to games.

8.1 Tie-transitive DE-conjectures (c_1, c_2, c_5) , (c_1, c_2, c_6) , and DE-theorem (c_1, c_2, c_3, c_4)

We will start with DE. In this case we have only one DE-theorem, (c_1, c_2, c_3, c_4) , which, in fact, is not new. As we already mentioned, this statement is equivalent to dominance solvability of totally tight game forms; it was conjectured in [?] and proved in [?].

Tie transitive DE-conjectures (c_1, c_2, c_5) and (c_1, c_2, c_6) remain open. Perhaps, some DE-examples are missing. Yet, such examples (if any) must be of size at least 4×5 , since all game forms of size at most 4×4 were verified by a computer code. All found DE-examples are of size 2×2 or 3×3 , there is no one of size 2×3 , 3×4 , and 4×4 .

Let us note that DE-conjectures (c_1, c_2, c_5) and (c_1, c_2, c_6) are equivalent. Indeed, by the semi-inversion, c_1 is transformed to c_2 , c_5 is transformed to c_6 , and vice versa; that is, $c_1^1 = c_2^2 = c_2$ and $c_2^1 = c_5^5 = c_1$, $c_5^1 = c_6^6 = c_6$ and $c_6^1 = c_5^5 = c_5$. Thus, conjectures (c_1, c_2, c_5) and (c_1, c_2, c_6) are transformed one to the other and hence, they are equivalent.

Remark 14 *Let us mention that five DE-theorems: (c_1, c_2, c_5, c_6) , (c_1, c_2, c_5, c_{11}) , $(c_1, c_2, c_6, c_9, c_{10})$, $(c_1, c_2, c_3, c_4, c_9, c_{10}, c_{11})$ for games, and (c_1, c_2, c_3, c_4) for game forms are proved by different methods and belong to different areas, although all five were formally obtained by dualizing the hypergraphs of computer-generated examples. This approach can be viewed as a part of artificial intelligence.*

8.2 Tie-transitive NE-theorems (c_1, c_2, c_5) , (c_1, c_2, c_6) , and (c_1, c_3)

Let us remark that c_1, c_2, c_5 and c_1, c_2, c_6 are both DE- and NE-conjectures. In case of DE they remain open; yet, for NE we will prove them. In fact, we have already proved that (c_1, c_2, c_5) is a general NE-theorem, hence, it remains NE-theorem in the considered tie-transitive case, too.

We will prove c_1, c_3 and c_1, c_2, c_6 together. Let us assume indirectly that there is a strong (canonical) n -cycle; see Figure ?? . By definition,

$$x_{1,1} > x_{1,2} > x_{2,2} > \dots > x_{n-1,n} > x_{n,n} > x_{1,n} > x_{1,1};$$

$$x_{2,1} < x_{2,2}, x_{3,2} < x_{3,3}, \dots, x_{n,n-1} < x_{n,n}, x_{1,n} < x_{1,1}.$$

In a c_1, c_3 -free configuration a 2-square is c_5 whenever it contains a 3-path. Hence, in this case

$$x_{1,1} = x_{2,1}, x_{2,2} = x_{3,2}, \dots, x_{n-1,n-1} = x_{n,n-1}, x_{n,n} = x_{1,n}.$$

Let us show that these equalities hold for c_1, c_2, c_6 -free configurations, as well. Indeed, in a strong canonical cycle $x_{i,i+1} \geq x_{i,i+2} < x_{i+1,i+2}$ and $x_{i,i} \geq x_{i+1,i}$, as in Figure ?? B. Moreover, $x_{i,i} = x_{i+1,i}$ and $x_{i+1,i} = x_{i+1,2}$, since otherwise c_2 or c_6 would appear; see Figure ?? B.

Thus, in contradiction with tie-transitivity, we obtain:

$$x_{1,1} = x_{2,1} < x_{2,2} = x_{3,2} < \dots < x_{n-1,n-1} = x_{n,n-1} < x_{n,n} = x_{1,n} < x_{1,1}.$$

8.3 Tie-transitive weak acyclicity-theorems

In this case there are only two conjectures:

$$(c_1, c_5, c_{11}, c_{15}, c_2) \text{ and } (c_1, c_5, c_{11}, c_{15}, c_3, c_9, c_{10}).$$

The first one was already proved in general, so it holds in tie-transitive case, as well. To prove the second one, let us consider the canonical weak cycle

$$x_{1,1} \geq x_{1,2} \geq x_{2,2} \geq \dots \geq x_{n-1,n} \geq x_{n,n} \geq x_{1,n} \geq x_{1,1}$$

and recall that either $x_{i,i} < x_{i+1,i} > x_{i+1,i+1}$ or $x_{i,i} > x_{i+1,i} < x_{i+1,i+1}$ and also either $x_{i,i+1} < x_{i,i+2} > x_{i+1,i+2}$ or $x_{i,i+1} > x_{i,i+2} < x_{i+1,i+2}$, since otherwise a shorter weak cycle (shortcut) would appear; see Figure ?? . It is easy to see that, independently on the above two alternatives, the strict and not strict inequalities in the considered canonical cycle must alternate, since otherwise c_3 or c_9 or c_{10} would appear. see Figure ?? . Indeed, c_3 (respectively, c_9 or c_{10}) appear whenever two successive strict inequalities (respectively, equalities) take place. As we already know, such alternating is in contradiction with tie-transitivity.

8.4 Tie-transitive semi-weak acyclicity-theorems

We will prove all eight such theorems:

$$(C', c_2), (C', c_3, c_4, c_6), (C', c_3, c_6, c_7), (C', c_3, c_6, c_8), (C', c_3, c_4, c_7, c_8),$$

$$(C', c_3, c_4, c_7, c_9), (C', c_3, c_4, c_8, c_{10}), (C', c_3, c_9, c_{10}),$$

where $C' = \{(c_1, c_5, c_{11})\}$ is the set of all tie-transitive semi-weak cycles. Let us notice that c_{15} is weak and tie-transitive but not semi-weak, while c_{13} and c_{14} are semi-weak but not tie-transitive.

Let us notice that SW-theorems

$$(C'', c_2), (C'', c_3, c_4, c_6), (C'', c_3, c_6, c_7), (C'', c_3, c_6, c_8)$$

were already proved in general (that is, without tie-transitivity assumption) and $C'' = C' \cup \{c_{13}, c_{14}\}$ substituted for C' . Hence, the corresponding two theorems hold for the tie-transitive case, too. Indeed, c_{13} and c_{14} are forbidden in both cases, in the general one, because they are SW-cycles, in the tie-transitive one, because they are not tie-transitive.

Now, let us prove (C', c_3, c_9, c_{10}) . As we already mentioned, in this case strict inequalities and equalities in the canonical cycle C_n^0 must alternate, since c_3 (respectively, c_9 or c_{10}) appear whenever two successive strict inequalities (respectively, equalities) take place in C_n^0 ; see Figure ???. As above, such alternating is in contradiction with tie-transitivity.

Finally, let us prove (C', c_3, c_4, c_7, c_9) and $(C', c_3, c_4, c_8, c_{10})$ together. Still, in both cases, two successive strict inequalities cannot take place, since otherwise c_3 would appear, which is forbidden in both theorems. Yet, one strict inequality must exist in a semi-weak cycle. Hence, there is a sequence: equality, strict inequality, equality. Let us consider four cases in see Figure ???. In the first two, c_4 appears, which is forbidden in both theorems. In the last two we will have to follow the canonical cycle further. Standardly, if equalities and strict inequalities alternate in it then we get a contradiction with tie-transitivity. Hence, we can extend our sequence, equality, strict inequality, equality, by one more equality; see Figure ???. In each case 3 and 4 we consider two subcases. In cases 3.1, 3.2, and 4.2 we obtain (c_7, c_{10}) , (c_4) , and (c_8, c_9) , respectively. It is easy to see that each of these combinations is a transversal to both (C', c_3, c_4, c_7, c_9) and $(C', c_3, c_4, c_8, c_{10})$; hence, in both cases we get a contradiction.

Yet in case 4.1 we obtain (c_6, c_8, c_{10}) which is disjoint from (C', c_3, c_4, c_7, c_9) . However, it is easy to see that in all three cases, $x_{i-1,i} < x_{i+1,i}$, $x_{i-1,i} > x_{i+1,i}$, and $x_{i-1,i} = x_{i+1,i}$, we obtain c_3 , c_4 , or a shortcut, respectively. (We have to notice that in the last case the obtained shorter cycle remains semi-weak, since $x_{i+1,i} > x_{i+1,i+1}$.) Thus, in case 4.1 we get a contradiction, too.

8.5 Tie-transitive strict acyclicity-theorems

In this case we have the following long list of conjectures:

$$\{(c_1, c_2, c_4, c_5), (c_1, c_2, c_4, c_6), (c_1, c_2, c_5, c_6), (c_1, c_2, c_5, c_7, c_8), (c_1, c_2, c_5, c_7, c_{12}), (c_1, c_2, c_5, c_8, c_{12}), (c_1, c_2, c_5, c_9), (c_1, c_2, c_5, c_{10}), (c_1, c_2, c_5, c_{11}), (c_1, c_2, c_6, c_9, c_{10}), (c_1, c_3, c_4), (c_1, c_3, c_5), (c_1, c_3, c_9, c_{10}), (c_1, c_3, c_{15})\}.$$

Conjectures $\{(c_1, c_2, c_4, c_5), (c_1, c_2, c_5, c_6), (c_1, c_2, c_3, c_5),$ and $(c_1, c_2, c_6, c_9, c_{10})$ from this list were proved in general and, hence, they hold in the tie-transitive case, too.

Here we will prove only one additional conjecture: $\{(c_1, c_3, c_4)\}$. Let us recall that it strengthens theorem (c_1, c_2, c_3, c_4) recently proved in [?] and [?].

Assume indirectly that a tie-transitive (c_1, c_3, c_4) -free configuration P has a strict improvement n -cycle C_n ; see Figure ?? A. Standardly and without loss of generality we can assume that n -cycle $C_n = C_n^0$ is canonical and shortest, that is, P contains no n' -cycle with $n' < n$.

Since c_1 and c_3 are forbidden, we conclude that 2-square is c_5 whenever it contains the 3-path. Furthermore, a shortcut appears whenever

$$x_{i,i} \geq x_{i,i+1} \geq x_{i+1,i+2} \text{ or } x_{i,i+1} \geq x_{i,i+2} \geq x_{i+1,i+2}.$$

Hence, we obtain two standard alternatives, as in Figure ?? A:

- (i) $x_{i,i} < x_{i+1,i} = x_{i+1,i+1}$ or (ii) $x_{i,i} = x_{i+1,i} < x_{i+1,i+1}$, and
 (iii) $x_{i,i+1} < x_{i,i+2} = x_{i+1,i+2}$ or (iv) $x_{i,i+1} = x_{i,i+2} < x_{i+1,i+2}$.

There are four combinations: (i,iv) case 1, (ii, iii) case 2, and (i,iii) or (ii, iv) case 3; see Figure ?? A. In case 1, P contains c_4 and we get a contradiction immediately. In case 2 let us consider two subcases:

Case 2.1 $x_{i,i} < x_{i,i+2}$, $x_{i+1,i} = x_{i+1,i+2}$ or $x_{i,i} = x_{i,i+2}$, $x_{i+1,i} < x_{i+1,i+2}$ and

Case 2.2 $x_{i,i} < x_{i,i+2}$, $x_{i+1,i} < x_{i+1,i+2}$ or $x_{i,i} = x_{i,i+2}$, $x_{i+1,i} = x_{i+1,i+2}$.

It is easy to check that in any other case a shortcut appears. In case 2.1, the non-tie-transitive 2-square c_{14} results in contradiction; see Figure ?? A. In case 2.2 we consider the next two entries $x_{i+2,i+1}, x_{i+2,i+2}$ and two more subcases:

Case 2.2.1 $x_{i+1,i+1} = x_{i+1,i+2} < x_{i+2,i+2}$ and

Case 2.2.2 $x_{i+1,i+1} < x_{i+1,i+2} = x_{i+2,i+2}$ and

In case 2.2.1, entries $x_{i,i+1}, x_{i,i+2}, x_{i+2,i+1}, x_{i+2,i+2}$ form c_4 ; see Figure ?? B. In case 2.2.2 we consider the next two entries $x_{i+2,i+1}, x_{i+2,i+2}$ and two more subcases:

Case 2.2.2.1 $x_{i+1,i+2} = x_{i+1,i+3} < x_{i+2,i+3}$ and

Case 2.2.2.2 $x_{i+1,i+2} < x_{i+1,i+3} = x_{i+2,i+3}$.

In case 2.2.2.1, entries $x_{i+1,i+1}, x_{i+1,i+3}, x_{i+2,i+1}, x_{i+2,i+3}$ again form c_4 , while case 2.2.2.2 is partitioned in two subcases, etc. This pattern propagates in a unique way and we get contradiction with tie-transitivity; see Figure ?? C.

Similarly, we analyse Case 3. Again there are two subcases:

Case 3.1 $x_{i+1,i+1} < x_{i+2,i+1} = x_{i+2,i+2}$ and

Case 3.2 $x_{i+1,i+1} = x_{i+2,i+1} < x_{i+2,i+2}$.

Case 3.1 is reduced to case 2, by transposition of P .

In case 3.2 we propagate the pattern in a unique way, as in Figure ?? C, again in contradiction with tie-transitivity.

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