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ON ACYCLIC, OR TOTALLY TIGHT, TWO-PERSON GAME FORMS [1]

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Abstract. Let $g : X_1 \times X_2 \to A$ be a two-person game form, whose rows X_1 and columns X_2 are the sets of strategies of players 1 and 2, while A is the set of outcomes. A row $x_1 \in X_1$ (column $x_2 \in X_2$) is called constant if $g(x_1, x_2) \equiv a$ for some outcome $a \in A$ and for all columns $x_2 \in X_2$ (respectively, for all rows $x_1 \in X_1$). It is known that g is Nash-solvable if and only if it is tight [9, 10]. We strengthen the concept of tightness as follows: g is called *totally tight* (TT) if every its 2×2 subform g' is tight.
The following implications hold. If game form g is (o) TT then it is (i) tight and, hence, Nash-solvable; moreover, g is (ii) dominance-solvable and (iii) acyclic. Implication $(o) \Rightarrow (i)$ results immediately from [20] and [9], while $(o) \Rightarrow (iii)$ was recently shown by Kukushkin [14]. Moreover, $(o) \Leftrightarrow (iii)$, since the inverse implication $(o) \Leftarrow (iii)$ is trivial.
In this paper we derive (ii) and (iii) from (o) using the following fundamental property of the TT game forms. A game form g is called *reducible* if it has a constant line, row or column. We show that all irreducible TT game forms have the same effectivity function, namely, the $\binom{3}{2}$-majority one defined as follows. There are three outcomes $a_1, a_2, a_3 \in A$ such that each player is effective for all three subsets of cardinality two: $\{a_1, a_2\}$, $\{a_1, a_3\}$, and $\{a_2, a_3\}$. Somewhat surprisingly, even under this (very strong) restriction, it seems difficult to characterize the TT two-person game forms explicitly. We obtain some partial results in this direction.

a_1	a_2
a_2	a_1

g'

a_1	a_1
a_2	a_3

g''

a_1	a_1	a_3
a_1	a_2	a_2
a_3	a_2	a_3

g'''

Figure 1: Tight and not tight game forms

1 Introduction

1.1 Game forms and games

A (two-person) *game form* is a mapping $g : X_1 \times X_2 \to A$, where X_1 (rows) and X_2 (columns) are the strategies of players 1 and 2, while A is the set of outcomes. In this paper we restrict ourselves by finite two-person game forms, that is, we assume that the above three sets, X_1, X_2, and A are finite. Three examples are given in Figure 1.

Furthermore, let $u : \{1, 2\} \times A \to \mathbb{R}$ be a *utility (or payoff)* function. Given a player $i \in \{1, 2\}$ and an outcome $a \in A$, the value $u(i, a)$ is interpreted as the profit of the player i in case the outcome a is realized. The pair (g, u) defines a normal form (bimatrix) game.

A payoff u is called *zero-sum* if $u(1, a) + u(2, a) = 0$ for each $a \in A$. In this case (g, u) is a *matrix game.*

1.2 Nash equilibrium and Nash-solvability

The elements of the direct product $X = X_1 \times X_2$ are called *situations.* Given a game (g, u), a situation $x = (x_1, x_2) \in X_1 \times X_2 = X$ is called a *Nash equilibrium (NE)* if

$$u(1, g(x_1, x_2)) \geq u(1, g(x'_1, x_2)) \ \forall\, x'_1 \in X_1 \text{ and } u(2, g(x_1, x_2)) \geq u(2, g(x_1, x'_2)) \ \forall\, x'_2 \in X_2;$$

in other words, if no player can profit until the opponent keeps the strategy unchanged.

A NE of a zero-sum game is called a *saddle point.*

Theorem 1. *(Shapley (1964), [20]). A zero-sum game has a saddle point whenever each of its 2×2 subgames has one.* □

However, for non-zero-sum games the similar statement does not hold; see for example [12] or [4].

A game form g is called *Nash-solvable (NS)* if for each payoff u the obtained game (g, u) has a NE. Respectively, g is called *zero-sum-solvable* if for each zero-sum payoff u the obtained zero-sum game (g, u) has a saddle point.

1.3 Effectivity functions, game forms, and criteria of solvability

Given a game form $g : X_1 \times X_2 \to A$, we say that a player $i \in \{1, 2\}$ is effective for a subset of outcomes $B \subseteq A$ if i has a strategy $x_i \in X_i$ such that $g(x_i, x_{3-i}) \in B$ for every strategy

$x_{3-i} \in X_{3-i}$ of the opponent. In this case we set $E_i(B) = 1$ and $E_i(B) = 0$ otherwise. Thus, two Boolean functions $E_i^g : 2^A \to \{0,1\}$, where $i \in \{1,2\}$, are associated with every game form g. The pair (E_1^g, E_2^g) is called the *effectivity function (EFF)* of g; see for more detail [17, 15, 16, 18].

It is easy to see that equalities $E_1^g(B) = E_2^g(A \setminus B) = 1$ hold for no g, since every row and column in $X_1 \times X_2$ intersect. Yet, $E_1^g(B) = E_2^g(A \setminus B) = 0$ might hold. For example, let us consider the first game form in Figure 1 and fix $B = \{a_1\}$ (or $B = \{a_2\}$). Indeed, in this case $E_1^g(B) = E_2^g(A \setminus B) = 0$, since all rows and columns contain both a_1 and a_2.

A game form g is called *tight* if

$$E_1^g(B) + E_2^g(A \setminus B) \equiv 1 \;\; \forall B \subseteq A. \tag{1}$$

For example, game forms g'' and g''' in Figure 1 are tight, while g' is not.

Given a game form g, let us assign to each outcome $a \in A$ a Boolean variable and denote it for simplicity by the same symbol a. Then, rows and columns of g naturally define two monotone disjunctive normal forms (DNFs) representing E_1^g and E_2^g, respectively:

$$D_1^g = \bigvee_{x_1 \in X_1} \bigwedge_{x_2 \in X_2} g(x_1, x_2), \;\; D_2^g = \bigvee_{x_2 \in X_2} \bigwedge_{x_1 \in X_1} g(x_1, x_2). \tag{2}$$

It is not difficult to verify that a game form g is tight if and only if its two DNFs D_1^g and D_2^g are dual, $D_1^g = (D_2^g)^d$. This equation is just a reformulation of (1). For example, for the three game forms g', g'' and g''' in Figure 1 we have:

$D_1^{g'} = D_2^{g'} = a_1 a_2;\; D_1^{g'} \neq (D_2^{g'})^d = a_1 \vee a_2;$
$D_1^{g''} = a_1 \vee a_2 a_3,\; D_2^{g''} = a_1 a_2 \vee a_1 a_3,\; D_1^{g''} = (D_2^{g''})^d;$
$D_1^{g'''} = D_2^{g'''} = (D_1^{g'''})^d = (D_2^{g'''})^d = a_1 a_2 \vee a_2 a_3 \vee a_3 a_1.$

Hence, g'' and g''' are tight, while g' is not.

Theorem 2. *([9], see also [10] and [3]). The following three properties of a game form are equivalent: tightness, Nash-solvability, and zero-sum-solvability.* □

For the zero-sum case this claim was proved earlier by Edmonds and Fulkerson [6] and independently in [8].

1.4 Totally tight and irreducible game forms; main theorem

We will call a game form g *totally tight (TT)* if each of its 2×2 subforms is tight.

Proposition 3. *A totally tight game form is tight.*

Proof. Let g be a TT game form and g' be an arbitrary its 2×2 subform. By definition, g' is tight and, by Theorem 2, it is zero-sum-solvable. Then, by Theorem 1, g is zero-sum-solvable and, by Theorem 2, g is tight. □

Given a game form $g : X_1 \times X_2 \to A$, a row $x_1 \in X_1$ (column $x_2 \in X_2$) is called constant if there is an outcome $a \in A$ such that $g(x_1, x_2) \equiv a$ for all columns $x_2 \in X_2$ (respectively, for all rows $x_1 \in X_1$). A game form g is called *reducible* if it has a constant line, row or column.

It is easy to verify that a 2×2 game form is reducible if and only if it is tight. For example, in Figure 1, g'' is tight and reducible (its first row is constant), while g' is not tight and not reducible.

Let us remark that, by the above definition, an $m \times n$ game form is reducible whenever $m = 1$ or $n = 1$. Indeed, in this case each column or, respectively, row is constant. Moreover, formally, even a 1×1 game form is reducible, although there is no game form to reduce it to. By convention, let us say that it is reduced to the empty game form.

A game form will be called *totally reducible* if it can be reduced to the empty game form by successive elimination of constant lines.

Proposition 4. *A totally reducible game form is totally tight.*

Proof. The induction by $m + n$ is obvious. □

More generally, given a game form g, let us eliminate successively its constant lines until we obtain an irreducible game form g' (which might be nonempty). It is obvious that g' is well-defined, i.e., unique; moreover, g' is TT if and only if g is TT.

As our main result, we will prove in Section 2 that all such non-empty irreducible TT game forms have the same effectivity function. This, so-called 2-majority, EFF is defined as follows: three are three outcomes $a_1, a_2, a_3 \in A$ such that each player is effective only for the three its subsets of cardinality two: $\{a_1, a_2\}$, $\{a_1, a_3\}$, and $\{a_2, a_3\}$ and, of course, for every superset of such a subset.

Theorem 5. *Every non-empty irreducible totally tight game form g has a 2-majority effectivity function, that is, there are outcomes $a_1, a_2, a_3 \in A$ such that $E_1^g = E_2^g = a_1a_2 \vee a_2a_3 \vee a_3a_1$.*

This result clarifies the structure of a TT game form g "almost completely": g is either totally reducible, or it is reduced to an irreducible game form g' with a 2-majority EFF.

Somewhat surprisingly, even under this restriction it seems difficult to characterize the TT game forms explicitly. In Section 3 we obtain partial results in this direction.

In particular, in Section 4 we prove that every TT game form is (i) acyclic and (ii) dominance-solvable; see the next two subsections for the definitions. Recently, (i) was proved and (ii) conjectured by Kukushkin, [14]. Moreover, both results (i) and (ii) were significantly strengthened and generalized in [1, 2]. In Section 3 we will show that (i) and (ii) follow easily from Theorem 5; for completeness, we will also prove them from scratch.

1.5 Acyclic game forms

Given positive integral m, n and k such that $2 \leq k \leq \min(m, n)$, a $m \times n$ bimatrix game (g, u), and k distinct strategies of each player, $x_1^1, \ldots, x_1^k \in X_1$ and $x_2^1, \ldots, x_2^k \in X_2$, we say that these strategies form a *(strict improvement) cycle* C_k if

$u(2, g(x_1^1, x_2^1)) < u(2, g(x_1^1, x_2^2))$, $u(1, g(x_1^1, x_2^2)) < u(1, g(x_1^2, x_2^2))$,
$u(2, g(x_1^2, x_2^2)) < u(2, g(x_1^2, x_2^3)), \ldots,$
$u(2, g(x_1^{n-1}, x_2^{n-1})) < u(2, g(x_1^{n-1}, x_2^n))$, $u(1, g(x_1^{n-1}, x_2^n)) < u(1, g(x_1^n, x_2^n))$,
$u(2, g(x_1^n, x_2^n)) < u(2, g(x_1^n, x_2^1))$, $u(1, g(x_1^n, x_2^1)) < u(1, g(x_1^1, x_2^1))$.

A game that have no cycles is called *acyclic*. It is both obvious and well-known that every acyclic game has a NE. A game form g will be called *acyclic* if for any payoff u the obtained game (g, u) is acyclic. It is clear that each acyclic game form is Nash-solvable and, hence, it is tight.

Proposition 6. *(Kukushkin (2007), [14]). A totally tight game form is acyclic.*

In Section 4 we will derive this statement from Theorem 5 and also give an independent proof.

1.6 Dominance-solvable game forms

Given a game (g, u) and two strategies $x_i, x'_i \in X_i$ of a player $i \in \{1, 2\}$, we say that x'_i is dominated by x_i if $u(i, g(x_i, x_{3-i})) \geq u(i, g(x'_i, x_{3-i}))$ for every strategy $x_{3-i} \in X_{3-i}$ of the opponent; in other words, if player i cannot profit by substituting x'_i for x_i if the opponent keeps the same (arbitrary) strategy.

Let us eliminate successively dominated strategies of players 1 and 2. Game (g, u) is called *dominance-solvable* if this procedure results in a 1×1 terminal subgame. It is well-known and easy to see the obtained situation is a NE; see, for example, [15] and [16], Chapter 5.

Although, in general, the result might depend on the order in which dominated strategies are eliminated, yet, there are simple conditions under which the above procedure and concept of domination are well-defined; namely, when utility functions $u_i : A \to \mathbb{R}$ of both players $i = 1$ and $i = 2$ are injective; in other words, when $u(1, a) = u(1, a')$ if and only if $u(2, a) = u(2, a')$ for all $a, a' \in A$; see [15] and [16], Chapter 5, again.

A game form g is called *dominance-solvable* if for any payoff u the obtained game (g, u) is dominance-solvable. Obviously, a dominance-solvable game form is Nash-solvable and, hence, it is tight.

Proposition 7. *A totally tight game form is dominance-solvable.*

In Section 4 we will derive this statement from Theorem 5.

1.7 Hereditary properties and eliminating constant lines

Given a game form $g : X_1 \times X_2 \to A$ (respectively, a game (g, u)) and a pair of subsets $X'_1 \subseteq X_1$, $X'_2 \subseteq X_2$, restricting g to $X'_1 \times X'_2 \subseteq X'_1 \times X'_2$, we define the subform g' of g (respectively, subgame (g', u) of (g, u)).

a_1	a_1	a_1
a_1	a_1	a_2
a_1	a_2	a_1

g

a_1	a_1
a_1	a_2
a_2	a_1

g'

a_1	a_2
a_2	a_1

g''

Figure 2: Eliminating constant lines

A property P of a game (g, u) (respectively, of a game form g) is called *hereditary* if P holds for any subgame $(g', u))$ of $(g, u))$ (respectively, for any subform g' of g) whenever P holds for (g, u) (respectively, for g) itself.

For example, acyclicity of a game, as well as acyclicity or total tightness of a game form are hereditary properties, by definition. This simple fact will play an important role in our arguments. In contrast, tightness of a game form might disappear even after elimination of a constant line. For example, game form g in Figure 2 is tight:

$E_1^g = E_2^g = a_1; \;\; E_1^g = (E_2^g)^d = a_1,$

while g' (obtained from g by eliminating its first, constant, column) is not tight:

$E_1^{g'} = a_1, \; E_2^{g'} = a_1 a_2; \;\; a_1 = (E_1^{g'})^d \neq E_2^{g'} = a_1 a_2.$

In particular, Nash- or zero-sum-solvability is a non-hereditary property.

It is also not difficult to construct two game forms g, g' and a utility function u such that, again, g' is obtained from g by eliminating a constant line and game (g, u) is dominance-solvable, while (g', u) is not. Indeed, let us take an arbitrary (not dominance-solvable) game (g', u'), add to g' a constant strategy x_i to get g, and extend u' to u so that each strategy x_i' in g' is dominated by x_i. Then clearly, the obtained game (g, u) is dominance-solvable, even if (g', u) was not. For example, let us consider the last two game forms g' and g'' in Figure 2 and define a utility function u such that $(u(1, a_1) > (u(1, a_2)$, while $(u(2, a_1) < (u(1, a_2)$. Then it is easy to check that game (g', u) is dominance-solvable, while (g'', u) is not.

However, both game forms g' and g'' are not dominance-solvable. Indeed, let us replace u by u' such that $u'(1, a_1) < (u'(1, a_2)$, while $u'(2, a_1) > (u(1, a_2)$. Then, both games (g', u') and (g'', u') are not dominance-solvable.

It would be interesting to construct a dominance-solvable game form g with a constant line such that game form g' obtained by elimination of this line is not dominance-solvable. Let us recall that g and g' in Figure 2 provide a similar example for Nash-solvability.

Anyway, dominance-solvability is a non-hereditary property of games.

For game forms it is not hereditary, either. Indeed, let us consider the game form g and its subform g' on Figure 3.

It is easy to see that g is dominance-solvable, while g' is not.

Now, let us restrict ourselves by a hereditary class of game forms (or games) $\mathcal{G}$ and show that it is much simpler to verify dominance-solvability within $\mathcal{G}$ than in general.

a_1	a_1	a_2	a_2
a_3	a_4	a_3	a_4

g

a_1	a_2
a_4	a_3

g'

Figure 3: Tightness and dominance-solvability are not hereditary

A game (g, u) is called *domination-free* if it contains no two comparable strategies. In its turn, a game form g is called *domination-free* if for some u the obtained game (g, u) is domination-free.

Obviously, a domination-free game or game form is not dominance-solvable. However, the converse is not always true, of course.

Proposition 8. *Every $g \in \mathcal{G}$ is dominance-solvable if and only if every $g \in \mathcal{G}$ is not domination-free.*

Proof. "Only if part" is obvious, since a domination-free game cannot be dominance-solvable. To prove "if part" let us assume that each $g \in \mathcal{G}$ is not domination-free and show that it is dominance-solvable, too. We know that for every u the obtained game (g, u) has a dominated strategy. Let us eliminate such a strategy. Since class $\mathcal{G}$ is hereditary, the obtained subform g' is not domination-free, either, and we can eliminate one more strategy, etc. Thus, every $g \in \mathcal{G}$ is dominance-solvable. □

Standardly, a domination-free game form g (game (g, u)) is called *locally minimal* if, by eliminating any line of g, we obtain a subform g' (subgame (g', u)) that is no longer domination-free.

Proposition 9. *A locally minimal domination-free game or game form cannot have a constant line.*

Proof. Let game form g (game (g, u)) contain a constant line whose elimination results in the subform g' (subgame (g', u)). It is easy to verify that g' (respectively, (g', u)) is domination-free whenever g (respectively, (g, u)) is domination-free. (Let us remark that the converse is not true, since some lines of g could be dominated by the considered constant line.)

Now, it is clear that a locally minimal domination-free game form g (game (g, u)) has no constant line, since otherwise, by eliminating such a line from g, we would obtain a smaller domination-free subform g' (subgame (g', u)), in contradiction with local minimality of g. □

1.8 Totally tight, acyclic, Nash- and dominance-solvable game forms; main diagram

The next statement will summarize most of the above observations.

Theorem 10. *The following twelve properties of a game form g are equivalent: every 2×2 subform of g is (1) tight, (2) Nash-solvable, (3) zero-sum-solvable, (4) dominance-solvable, (5) acyclic; furthermore, every subform of g is (1′) tight, (2′) Nash-solvable, (3′) zero-sum-solvable, (4′) dominance-soilvable, (5′) acyclic; finally, g itself is (6) acyclic, and (7) totally tight.*

Thus, total tightness and acyclicity are equivalent. In Section 4, we will prove that total tightness implies dominance-solvability. Furthermore, it is well-known that dominance-solvability implies Nash-solvability, see, for example, [16], and that Nash-solvability is equivalent to tightness [9, 10]. Let us also mention that total tightness implies tightness, by Proposition 3. Relations between main classes of two-person game forms are summarized by the diagram given in Figure 4

| tight | $\Leftrightarrow$ | Nash-solvable | $\Leftarrow$ | dominance-solvable | $\Leftarrow$ | acyclic | $\Leftrightarrow$ | totally tight |

Figure 4: Relations between main classes of two-person game forms

Let us recall that a game form g is tight if and only if the corresponding monotone Boolean functions E_1^g and E_2^g are dual. In Section 2, we will prove Theorem 5: if g is totally tight then $E_1^g = E_2^g = a_1a_2 \vee a_2a_3 \vee a_3a_1$. However, the inverse does not hold and we cannot characterize TT game forms explicitly. In Section 4, we obtain partial results in this direction.

Remark 1. *Let us notice that an important necessary condition for acyclicity (or for total tightness) of a two-person game form is given in terms of its effectivity function. Somewhat surprisingly, many properties of game forms can be characterized in such terms. For example, a two-person game form g is Nash-solvable if and only if it is tight, that is, its effectivity function is self-dual. More example can be found in [11].*

1.9 Dominance-solvable but not totally tight game forms

Every TT game form is dominance-solvable (see Section 4), yet not vise versa; see, for example, g in Figure 3.

More generally, it is well-known that a game form g is dominance-solvable whenever it is obtained from a positional game form with perfect information Gale (1953); see also Chapter 5 of [16]. However, in this case g is acyclic (or equivalently, totally tight) if and only if all positions of each player belong to a single play in the corresponding tree. This result was obtained in 2002 by Kukushkin; see Theorem 1 of [13]. Let us mention that both result hold for the n-person case, not only for $n = 2$.

Also, there are many dominance-solvable but not TT game forms related to veto-voting; see manuscript [7]. Since it is not that easy to find out, we reproduce two examples in Figure 5, for completeness.

	a_3	a_2	a_1
a_3	1	1	2
a_2	1	3	3
a_1	2	3	2

g_3

	a_3 a_4	a_2 a_4	a_2 a_3	a_1 a_4	a_1 a_3	a_1 a_2
a_4	1	1	1	2	2	3
a_3	**1**	1	4	2	4	**4**
a_2	1	3	4	3	4	3
a_1	**2**	3	4	2	2	**3**

g_4

	a_4 a_5	a_3 a_5	a_3 a_4	a_2 a_5	a_2 a_4	a_2 a_3	a_1 a_5	a_1 a_4	a_1 a_3	a_1 a_2
a_4 a_5	1	1	1	1	1	1	2	2	2	3
a_3 a_5	1	1	1	1	1	4	2	2	4	4
a_3 a_4	1	1	5	1	5	5	2	5	5	5
a_2 a_5	1	1	1	3	3	4	3	3	4	3
a_2 a_4	1	1	5	3	5	5	3	5	5	3
a_2 a_3	**1**	4	5	**4**	5	4	4	5	4	4
a_1 a_5	**2**	2	2	**3**	3	4	2	2	2	3
a_1 a_4	2	2	5	3	5	5	2	2	2	3
a_1 a_3	2	4	5	4	5	4	2	2	4	4
a_1 a_2	3	4	5	3	3	4	3	3	4	3

g_5

Figure 5: Dominance-solvable but not TT veto-voting schemes

Three game forms in Figure 5 are veto-voting schemes of two voters (players) 1 and 2 and p candidates (outcomes) $\{a_1, \ldots, a_p\}$ in g_p, where $p = 3, 4, 5$. In g_3 each voter can veto one candidate; in g_4 voters 1 and 2 can veto one and two candidates, respectively; finally, in g_5 they can veto two candidates each. (To save space in Figure 5, we frequently substitute for a_j just its subscript j.)

The following veto-voting scheme is considered. All candidates sit at a round table in an arbitrary order, say, $\{a_1, \ldots, a_p\}$. Two voters (secretly) choose one or two candidates they are going to veto. Then they reveal their strategies and put one veto-card against each chosen candidate. If each candidate got at most one veto-card then only one remains "unvetoed" and (s)he is elected. Yet, since voting is secret, it might happen that some candidate(s) are "over-vetoed", that is, got more than one veto-card. In this case we shift the superfluous veto-card(s) counterclockwise, one by one in an arbitrary order, until no "over-vetoed" candidate left. Then, as we already mentioned, only one remains "unvetoed" and (s)he is elected. It is not difficult to see that this **counterclockwise** election rule is well defined, that is, the elected candidate (the winner) does not depend on the order in which the superfluous veto-cards were shifted.

The obtained three game forms are given in Figure 5. It is shown in [7] that all these game forms are dominance-solvable. Yet, it is easy to verify that only g_3 is totally tight, while g_4 and g_5 are not.

1.10 On totally tight n-person game form

Concepts of tightness and total tightness naturally generalize the case of n-person game forms; see Section 5. It was shown in [10] that already for $n = 3$ tightness is no longer necessary or sufficient for Nash-solvability. Obviously, the same example gives a not TT 3-person game form that is dominance-solvable and acyclic. Indeed, both acyclicity or dominance-solvability imply Nash-solvability and total tightness implies tightness.

However, it is an important open question whether each TT n-person game form is Nash-solvable. Although total tightness is a very strong property, still the positive answer would be of interest, since no good sufficient condition for Nash-solvability is known for n-person game forms, yet.

In contrast, it is not difficult to construct TT not dominance-solvable or TT acyclic 3-person game forms; see Section 5.

2 Proof of Theorem 5

Let g be a totally tight game form. By Proposition 3 g is tight, that is, the corresponding two monotone Boolean functions E_1^g and E_2^g are dual. Yet, Theorem 5 claims much more, namely, all TT game forms generate the same self-dual pair: $E_1^g = E_2^g = a_1a_2 \vee a_2a_3 \vee a_3a_1$.

2.1 Game correspondences and associated game forms

A *game correspondence* is defined as a mapping $G : X_1 \times X_2 \to 2^A$. In other words, to each situation $(x_1, x_2) \in X_1 \times X_2$ we assign a subset of outcomes $G(x_1, x_2) \subseteq A$. If $|G(x_1, x_2)| = 1$ for all situations $(x_1, x_2) \in X_1 \times X_2$, we obtain a game form.

In general, with a given game correspondence G we associate $k = \prod_{(x_1,x_2)\in X_1\times X_2} |G(x_1, x_2)|$ game forms $g \in G$, by choosing for each situation $(x_1, x_2) \in X_1 \times X_2$ an outcome $g(x_1, x_2) \in G(x_1, x_2)$. Let us notice that $k = 0$ whenever $G(x_1, x_2) = \emptyset$ for at least one situation. We will say that $g \in G$ is *associated with* G and call G *(totally) tight* if $k > 0$ and at least one $g \in G$ is (totally) tight.

2.2 Game correspondences associated with pairs of dual monotone DNFs or Boolean functions

First, let us recall the following two well-known properties of dual monotone Boolean functions that will be instrumental for our analysis.

a_1/a_3	a_1	a_3
a_1	a_1/a_2	a_2
a_3	a_2	a_2/a_3

Figure 6: $\binom{3}{2}$ majority voting; two TT game form is associated with this game correspondence

Lemma 11. *(see, for example, [5], Part I, Chapter 4).*

(i) Every two dual implicants α of E and β of E^d have at least one variable in common.

(ii) Given a prime implicant α of E and a variable x of α, there is a prime implicant β of E^d such that x is the only common variable of α and β.

Given arbitrary monotone (that is, negation-free) DNFs $D_1 = \vee_{x_1 \in X_1} B_{x_1}$ and $D_2 = \vee_{x_2 \in X_2} B_{x_2}$ over the set of variables A, let us define a game correspondence $G = G^{D_1,D_2} : X_1 \times X_2 \to 2^A$ by setting $G(x_1, x_2) = B_{x_1} \cap B_{x_2}$ for every situation $(x_1, x_2) \in X_1 \times X_2$; see, for example, G^{D_1,D_2} in Figure 6, where $D_1 = D_2 = a_1a_2 \vee a_2a_3 \vee a_3a_1$.

Lemma 12. *([10], see also [19]). If D_1 and D_2 are dual then game correspondence $G(D_1, D_2)$ is tight. In particular, in this case $G(x_1, x_2) \neq \emptyset$ for all $(x_1, x_2) \in X_1 \times X_2$; moreover, all associated game forms $g \in G$ have the same Boolean functions E_1^g and E_2^g defined by DNFs D_1 and D_2, respectively. Conversely, if at least one game form $g \in G^{D_1,D_2}$ is tight then DNFs D_1 and D_2 are dual.*

Proof. It follows immediately from Lemma 11 (i) and (ii). □

Let us recall that, by definition, G is TT if at least one $g \in G$ is TT. However, in contrast with tightness, it does not imply that all $g \in G$ are TT. Let us consider, for example, game correspondence G in Figure 6. Only two game forms associated with G are TT (they are given in Figure 6, while it is easy to verify that the remaining six are not TT.

Given a DNF D, let D^0 denote the corresponding irredundant DNF, that is, disjunction of all prime (irreducible) implicants of D.

Lemma 13. *Game correspondence G^{D_1,D_2} is TT if and only if $G^{D_1^0,D_2^0}$ is TT.*

Proof. The "only if part" immediately follows, since total tightness is a hereditary property of game forms and game correspondences.

Lemma 14. *A subcorrespondence G' of G is TT whenever G is TT.* □

Let us prove the "if part". By assumption, there is a TT game form $g^0 \in G^0 = G^{D_1^0,D_2^0}$. Let us extend it to a TT game form $g \in G = G^{D_1,D_2}$ as follows. For $i = 1, 2$ to each strategy $x_i \in X_i$ in G assign a strategy $x_i^0 \in X_i$ in G^0 such that $B_{x_i^0} \subseteq B_{x_i}$. Then for each situation $x = (x_1, x_2)$ of G choose the same outcome as for $x^0 = (x_1^0, x_2^0)$ in g^0. Obviously, the obtained extension g of g^0 is totally tight, too. □

a_1	a_2	a_2
a_3	a_2/a_3	a_2
a_3	a_3	a_4

G

a_1	a_2
a_3	a_4

g'

Figure 7: No TT game form is associated with this game correspondence

2.3 Totally tight Boolean functions

Thus, we can restrict ourselves by dual pairs of irredundant DNFs. In other words, keeping in mind the characterization of TT game forms, we will take as the input a monotone Boolean function E rather than a game form g. Given E, we set $E_1 = E$ and $E_2 = E^d$, consider the corresponding irredundant DNFs D_1^0 and D_2^0 and game correspondence $G = G^E = G^{D_1^0, D_2^0}$. We will call E TT if G is TT, or in other words, if there is a TT $g \in G$. By construction, E is TT if and only if E^d is TT. Let us consider several examples.

If $E = E_1 = a_1a_2 \vee a_3a_4$ then $E^d = E_2 = a_1a_3 \vee a_1a_4 \vee a_2a_3 \vee a_2a_4$. It is easy to see that every two prime implicants, one of E and the other of E^d, have exactly one variable in common. (This is a characteristic property of so-called monotone *read-once* Boolean functions; see [5], Chapter 12.) In other words, game correspondence G^E is, in fact, a game form, since $|G^E(x_1, x_2)| = 1$ for every situation $(x_1, x_2) \in X_1 \times X_2$. This gameform g is shown on Figure 3. However, this game form is not TT, since it has a 2×2 subform g' that is not tight, see Figure 3.

In general, G^E is a game form, $G^E = g^E$, if and only if E is read-once. It is not difficult to show that in this case E is TT if and only if g^E is totally reducible; see Proposition 4. (This is a characteristic property of so-called monotone *threshold* Boolean functions; see [5], Part II, Chapter 10.) Yet, we are looking for irreducible TT game forms.

As another example, consider

$$E = E_1 = a_1a_2 \vee a_2a_3 \vee a_3a_4 \text{ and } E^d = E_2 = a_1a_3 \vee a_3a_2 \vee a_2a_4.$$

It is easy to check that G^E is not TT, since it contains a 2×2 subform g'; see Figure 7.

A case analysis might be needed for more difficult examples.

Let $E = E\binom{5}{3} := \vee_{\{i,j,k\} \subseteq \{1,2,3,4,5\}} a_i a_j a_k$, where i, j, and k are pairwise distinct triplets; in other words, $E = 1$ if and only if at least 3 out of its 5 variables are equal to 1. To show that G^E is not TT let us consider its subcorrespondence G given in Figure 8 (where, to save space, we substitute only the subscript $j \in \{1,2,3,4,5\}$ for a_j. Let us choose an arbitrary game form $g \in G$. Due to obvious symmetry, we can fix a_1 from $\{a_1, a_2, a_3\}$, without any loss of generality. Yet, in this case G already contains a 2×2 subconfiguration G' that is clearly not TT; see Figure 8. Hence, g cannot be TT and, by Lemma 14, G and G^E are not TT, either.

The following Lemma is instrumental in characterizing TT Boolean functions.

Given E, let us choose two of its distinct prime implicants and denote by $B, B' \subseteq A$ the corresponding two set of variables. Obviously, $B \setminus B' \neq \emptyset$ and $B' \setminus B \neq \emptyset$.

	123	145	245	345
123	123	1	2	**3**
145	1	145	45	45
245	**2**	45	245	**45**
345	3	45	45	345

G

1	**3**
2	**45**

G'

Figure 8: $\binom{5}{3}$ majority voting; no TT game forms are associated with this game correspondence

B	a_1	a_2	b_1	b_2
B'	b_3	b_4	a_3	a_4

a_1	a_2	b	b
b'	b'	a_3	a_4

$\mathbf{a_1}$	a_2	a_2	$\mathbf{a_2}$
$\mathbf{a_3}$	a_3	a_3	$\mathbf{a_4}$

Figure 9: $|B \setminus B'| = 1$ or $|B' \setminus B| = 1$

Lemma 15. *If E is totally tight then $|B \setminus B'| = 1$ or $|B' \setminus B| = 1$.*

Proof. Let us assume indirectly that $|B \setminus B'| \geq 2$ and $|B' \setminus B| \geq 2$, say, $a_1, a_2 \in B \setminus B'$ and $a_3, a_4 \in B' \setminus B$, where $a_1, a_2, a_3, a_4 \in A$ are four pairwise distinct outcomes, yet, E is TT.

By Lemma 11 (ii), there are four prime implicants of E^d whose sets of variables B_1, B_2, B_3, B_4 are such that $B_1 \cap B = \{a_1\}$, $B_2 \cap B = \{a_2\}$, $B_3 \cap B' = \{a_3\}$, $B_4 \cap B' = \{a_4\}$.

Let us fix a game form $g \in G^E$ and consider the corresponding 2×4 subform g' in g; it is given in Figure 9, where the first (second) row is assigmed to B (respectively, to B') and it contains a_1 and a_2 (respectively, a_3 and a_4). The remaining four outcomes $b_1, b_2, b_3, b_4 \in A$ are not necessarily pairwise distinct, yet, $\{b_1, b_2\} \cap \{a_3, a_4\} = \{b_3, b_4\} \cap \{a_1, a_2\} = \emptyset$, since $b_1, b_2 \in B$ and $b_3, b_4 \in B'$; see Figure 9.1.

By assumption, Boolean function E and game correspondence G^E is TT. Hence, we can assume that the associated game form $g \in G^E$, and its subform g' are TT, too. Then $b_1 = b_2$ and $b_3 = b_4$, otherwise the first or the last two columns of g' form a not tight subform. Let us set $b_1 = b_2 = b$ and $b_3 = b_4 = b'$; see see Figure 9.2. Yet, b (respectively, b') cannot be equal to both a_1 and a_2 (respectively, a_3 and a_4), since they are distinct. Without loss of generality, let us assume that $b \neq a_1$ and $b' \neq a_4$; see Figure 9.3. Then the first and last columns of g' form a not tight subform (even if $b = b'$) and we obtain a contradiction. □

2.4 Irreducible TT Boolean functions are self-dual

There is a simple characterization of reducibility of a game form in Boolean terms.

Lemma 16. *Game correspondence G^E contains a constant row (column) whose every entry is an outcome $a \in A$ if and only if $E = a \vee E'$ (respectively, $E^d = a \vee E''$). In both cases, every associated game form $g \in G^E$ is reducible.*

Proof. It follows immediately from the definitions. □

Thus, we can reformulate Theorem 5 as follows: If E is TT then either $E = a \vee E'$ or $E^d = a \vee E''$ or $E = E^d = a_1a_2 \vee a_2a_3 \vee a_3a_1$. In first two cases we will call E *reducible.*

Lemma 17. *If E is TT and irreducible then every two of its prime implicants have a variable in common.*

Proof. Let us assume indirectly that there are two prime implicants of E with disjoint set of variables $B, B' \subseteq A$. By Lemma 16, if E is TT then $|B| = 1$ or $|B'| = 1$, in other words, E is reducible and we get a contradiction. □

Lemma 18. *If E is TT and irreducible then it is self-dual, $E = E^d$.*

Proof. It is both obvious and well-known (see, for example, [5]) that E is dual-minor, $E \leq E^d$ if and only if every two prime implicants of E have a variable in common. Thus, by Lemma 17, if E is irreducible and TT then it is dual-minor, $E \leq E^d$. Yet, E is irreducible and TT if and only if E^d is irreducible and TT. To show this, it would suffice just to rename players 1 and 2. Hence, E and E^d are both dual-minor: $E \leq E^d$ and $E^d \leq (E^d)^d = E$. Thus, $E = E^d$, that is, E is self-dual. □

We will show that all self-dual functions, but one, are not TT.

For example, let us consider classical Fano function associated with the Fano projective plane:

$E = a_1a_2a_3 \vee a_3a_4a_5 \vee a_5a_6a_1 \vee a_0a_1a_4 \vee a_0a_2a_5 \vee a_0a_3a_6 \vee a_2a_4a_6$.

It is well-known and not difficult to verify that $E = E^d$. However, rows $\{a_1, a_2, a_3\}$, $\{a_3, a_4, a_5\}$ and columns $\{a_0, a_1, a_4\}$, $\{a_0, a_2, a_5\}$ form a 2×2 game form that is not tight.

2.5 A totally tight self-dual Boolean functions is a 2-wheel

As another example, let us consider the so-called k-wheel defined for $k \geq 2$ by formula

$$E_k = a_0a_1 \vee a_0a_2 \vee \ldots \vee a_0a_k \vee a_1a_2 \ldots a_k.$$

Again, it is well-known and easy to check that $E_k = E_k^d$ for any $k \geq 2$. Game correspondences, G^{E_k} are given in Figure 10 for $k = 2, 3$, and in general. (Again, to save space we substitute for an outcome a_j only its subscript j.) Let us fix an arbitrary $g \in G^{E_k}$. Due to obvious symmetry, without loss of generality, we can choose a_k from $\{a_1, a_2, \ldots, a_k\}$. Yet, then a 2×2 not tight subform g' appears in g whenever $k \geq 3$; see Figure 10.

Yet, as we already know, 2-wheel E_2 is TT. There are two associated with G^{E_2} TT game forms; see Figure 6 (in which $i + 1$ is substituted for $i = 0, 1$ and 2).

Furthermore, we can strengthen Lemma 18 as follows.

Lemma 19. *If E is TT and irreducible then it is a 2-wheel.*

	01	02	12
01	01	0	1
02	0	02	2
12	1	2	12

	01	02	03	123
01	01	**0**	0	**1**
02	0	02	0	2
03	0	0	03	3
123	1	**2**	3	12**3**

	01	02	$\cdots$	$0k$	$12k$
01	01	**0**	$\cdots$	0	**1**
02	0	02		0	2
$\vdots$			$\ddots$		$\vdots$
$0k$	0	0		$0k$	k
12...k	1	**2**	$\cdots$	k	12**k**

Figure 10: 2-wheel, 3-wheel, and k-wheel

Proof. Let us fix a prime implicant of E with the largest set of variables, which we will denote, without loss of generality, by $B = \{a_1, \dots, a_k\} \subseteq A$. Since E is irreducible, $k \geq 2$.

By Lemma 18, E is self-dual, $E = E^d$. Then, by Lemma 11 (ii), for every $j = 1, \dots, k$ function E contains a prime implicant with the set of variables B_j such that $B \cap B_j = \{a_j\}$. Furthermore, by Lemma 15, $|B \setminus B_j| = 1$ or $|B_j \setminus B| = 1$.

Let us assume that $k \geq 3$. Then $|B \setminus B_j| \geq 2$. Hence, $|B_j \setminus B| = 1$, that is, $B_j = \{a_j, b_j\}$ for each $j = 1, \dots, k$. Moreover, by Lemma 11 (i) all b_j must coinside, that is, $B_j = \{a_0, a_j\}$ for each $j = 1, \dots, k$. In other words, E is a k-wheel with $k \geq 3$. Yet, as we already know, in this case E is not TT. Hence, $k = 2$, that is, every prime implicant of E has exactly two variables; in other words, $E = a_1 a_2 \vee a_0 a_1 \vee a_0 a_2$ is the 2-wheel. □

Thus, all TT irreducible game forms have the same EFF, the 2-wheel. This completes the proof of Theorem 5. □

3 Characterizing totally tight game forms

3.1 Canonical partition of a totally tight game form

Let g be a TT game form. We know that $E_1^g = E_2^g = a_1 a_2 \vee a_2 a_3 \vee a_3 a_1$. Yet, the corresponding DNFs $D_1 = D_1^g$ and $D_2 = D_2^g$ might be redundant. Let us consider partitions

$$X_i = X_i^{12} \cup X_i^{13} \cup X_i^{23} \cup X_i^{123} \cup X_i^{1234} \text{ for } i \in \{1, 2\},$$

where the first four sets of lines, rows ($i = 1$) and columns ($i = 2$), consist of outomes $\{a_1, a_2\}$, $\{a_1, a_3\}$, $\{a_2, a_3\}$, and $\{a_1, a_2, a_3\}$, respectively, while X_i^{1234} is the set of lines that contain an outcome $a \notin \{a_1, a_2, a_3\}$. Let us notice that $X_i^{12} \neq \emptyset$, $X_i^{13} \neq \emptyset$, and $X_i^{23} \neq \emptyset$, while X_i^{123} and X_i^{1234} might be empty.

3.2 Subform $(X_1^{12} \cup X_1^{13} \cup X_1^{23}) \times (X_2^{12} \cup X_2^{13} \cup X_2^{23})$

It is easy to see that

$g(x_1, x_2) = a_1$ when $x_1 \in X_1^{12}, x_2 \in X_2^{13}$ or $x_1 \in X_1^{13}, x_2 \in X_2^{12}$;

$g(x_1, x_2) = a_2$ when $x_1 \in X_1^{12}, x_2 \in X_2^{23}$ or $x_1 \in X_1^{23}, x_2 \in X_2^{12}$;

$g(x_1, x_2) = a_3$ when $x_1 \in X_1^{13}, x_2 \in X_2^{23}$ or $x_1 \in X_1^{23}, x_2 \in X_2^{13}$.

As we already mentioned in Section 2.2, only the following two assignments are feasible in the main diagonal, see Figure 6,

(i) $g(x_1, x_2) = a_1$ when $x_1 \in X_1^{13}, x_2 \in X_2^{13}$, $g(x_1, x_2) = a_2$ when $x_1 \in X_1^{12}, x_2 \in X_2^{12}$, and $g(x_1, x_2) = a_3$ when $x_1 \in X_1^{23}, x_2 \in X_2^{23}$;

(ii) $g(x_1, x_2) = a_3$ when $x_1 \in X_1^{13}, x_2 \in X_2^{13}$, $g(x_1, x_2) = a_1$ when $x_1 \in X_1^{12}, x_2 \in X_2^{12}$, and $g(x_1, x_2) = a_2$ when $x_1 \in X_1^{23}, x_2 \in X_2^{23}$.

It is not difficult to verify that any mixture of (i) and (ii) would contradict total tightness of g. Due to symmetry, we can fix either (i) or (ii) without any loss of generality. From now on, we will assume that (i) holds, as in Figures 6 and 11, where we substitute only subscript j for a_j.

3.3 Subforms $X_1^{1234} \times (X_2^{12} \cup X_2^{13} \cup X_2^{23})$ and $(X_1^{12} \cup X_1^{13} \cup X_1^{23}) \times X_2^{1234}$

Let us show that $g(x_1, x_2) = a_1$ when $x_1 \in X_1^{1234}$ and $x_2 \in X_2^{13}$.

The last inclusion implies that $g(x_1, x_2)$ equals either a_1 or a_3. Let us assume indirectly that $g(x_1, x_2) = a_3$. Then, $g(x_1, x_2) = a_1$ when $x_1 \in X_1^{12} \cup X_1^{13}$ and $x_2 \in X_2^{1234}$, otherwise g is not TT; see Figure 12. Furthermore, from total tightness of g we also derive that equalities $g(x_1, x_2) = a_2$ and $g(x_1, x_2) = a_3$ hold simultaneously when $x_1 \in X_1^{1234}$ and $x_2 \in X_2^{23}$; see Figure 12. The obtained contradiction proves our claim. By similar arguments, we show that

$g(x_1, x_2) = a_2$ when $x_1 \in X_1^{1234}$ and $x_2 \in X_2^{12}$,

$g(x_1, x_2) = a_3$ when $x_1 \in X_1^{1234}$ and $x_2 \in X_2^{23}$;

$g(x_1, x_2) = a_1$ when $x_1 \in X_1^{13}$ and $x_2 \in X_2^{1234}$,

$g(x_1, x_2) = a_2$ when $x_1 \in X_1^{12}$ and $x_2 \in X_2^{1234}$,

$g(x_1, x_2) = a_3$ when $x_1 \in X_1^{23}$ and $x_2 \in X_2^{1234}$.

The results are summarized in Figure 11. Let us notice that lines X_1^{1234} and X_2^{1234} are filled in accordance with majority rule, that is, each entry of the last line is the most frequent outcome in the corresponding orthogonal line.

Let us also notice the following important corollary: if a line contains an outcome $a \notin \{a_1, a_2, a_3\}$ then this line must contain a_1, a_2, and $a_3\}$, too. For example, no line can consist of outcomes a_1, a_2, a_4, a_5 only.

3.4 Further partition of sets X_1^{123} and X_2^{123}

From total tightness of g we can also derive the following implication. If $g(x_1, x_2) = a_3$ for some $x_1 \in X_1^{123}$ and $x_2 \in X_2^{13}$ then $g(x_1, x_2') = a_2$ (respectively, $g(x_1, x_2') = a_3$) for the same x_1 and arbitrary $x_2' \in X_2^{12}$ (respectively, $x_2' \in X_2^{23}$).

	X_2^{13}	X_2^{12}	X_2^{23}	X_1^{123}	X_1^{1234}
X_1^{13}	1	1	3	13	1
X_1^{12}	1	2	2	12	2
X_1^{23}	3	2	3	23	3
X_1^{123}	13	12	23	123	123
X_1^{1234}	1	2	3	123	1234

Figure 11: Structure of a TT game form; approximation I

	X_2^{13}	X_2^{12}	X_2^{23}	X_2^{1234}
X_1^{13}	1	1	3	1
X_1^{12}	1	2	2	2
X_1^{23}	3	2	3	3
X_1^{1234}	1	2	3	4

1	1	3	1
1	2	2	1
3	2	3	
3		2/3	4

1	1	3	
1	2	2	2
3	2	3	2
1/3	1		4

1	1	3	3
1	2	2	
3	2	3	3
	1/2	2	4

Figure 12: Contradictions

	X_2^{13}	X_2^{12}	X_2^{23}	X_2^{3123}	X_2^{1223}	X_2^{1123}	X_2^{0123}	X_2^{123}
X_1^{13}	1	1	3	3 1	1	1	1	1
X_1^{12}	1	2	2	2	2	1 2	2	2
X_1^{23}	3	2	3	3	2 3	3	3	3
X_1^{3123}	3 1	2	3	3	2	3	2 3	3
X_1^{1223}	1	2	2 3	2	2	1	1 2	2
X_1^{1123}	1	1 2	3	3	1	1	1 3	1
X_1^{013}	1	2	3	2 3	1 2	1 3	1 2 3	1 2 3
X_1^{1234}	1	2	3	3	2	1	1 2 3	1 2 3 4

Figure 13: Structure of a TT game form; approximation II

Indeed, let us assume indirectly that $g(x_1, x_2') = a_1$ (respectively, $g(x_1, x_2') = a_2$) and choose an arbitrary $x_1' \in X_1^{12}$. It is easy to verify that rows x_1, x_1' and columns x_2, x_2' result in a 2×2 game form that is not tight. Hence, g is not TT and we get a contradiction.

Let $X_1^{3123} \subseteq X_1$ denote the set of rows such that $g(x_1, x_2) = a_3$ for each $x_1 \in X_1^{3123}$ and some $x_2 \in X_2^{13}$. In other words, subform $g' : X_1^{3123} \times X_2^{13} \to A$ takes only two values a_1, a_3 and a_3 appears in every its row. (In the next section we will show that a_1 appears in every row, too.) Since g is TT, g' is also TT, that is, every 2×2 subform of g' is tight. Hence, by permutations of rows and columns we can transform g' to a form such that in every its row or column outcomes a_3 go first, while a_1 (if any) follow; see Figure 13.

Thus, by symmetry, we obtain two partitions:

$X_i^{123} = X_i^{3123} \cup X_i^{1223} \cup X_i^{1123} \cup X_i^{0123}$ for rows, $i = 1$, and columns, $i = 2$.

First, we substitute $i = 2$ for $i = 1$ to define subset of columns $X_2^{3123} \subseteq X_2^{123}$ Then, we introduce subsets X_i^{1223} and X_i^{1123} for $i \in \{1, 2\}$, similarly to X_i^{3123}, using the cyclic shift of outcomes $a_3 \to a_2 \to a_1$. Finally, we define $X_i^{0123} \subseteq X_i^{123}$ as the set of rows $(i = 1)$ or columns ($i = 2$) such that $g(x_i, x_{3-i}) = a_1$ (respectively, $= a_2$ and $= a_2$ for every $x_i \in X_i^{0123}$ and $x_{3-i}) \in X_{3-i}^{13}$ (respectively, $\in X_{3-i}^{12}$ and $\in X_{3-i}^{23}$). The obtained two partitions

$$X_i = X_i^{12} \cup X_i^{13} \cup X_i^{23} X_i^{3123} \cup X_i^{1223} \cup X_i^{1123} \cup X_i^{0123} \cup X_i^{1234}$$

for rows, $i = 1$, and columns, $i = 2$, are given in Figure 13. Let us notice again that the last five sets might be empty, while the first three cannot.

3.5 Other refinements given by Figure 13

By definition, lines X_i^{13}, X_i^{12}, and X_i^{23} consist of outcomes $\{a_1, a_3\}$, $\{a_1, a_2\}$, and $\{a_2, a_3\}$, respectively. Furthermore, from the previous section, we already know for $i = 1$ and $i = 2$ the entries of the subforms

$$\{X_{3-i}^{13} \cup X_{3-i}^{12} \cup X_{3-i}^{23}\} \times \{X_i^{13} \cup X_i^{12} \cup X_i^{23} \cup X_i^{3123} \cup X_i^{1223} \cup X_i^{1123} \cup X_i^{0123} \cup X_i^{1234}\}.$$

In particular, subform $X_{3-i}^{1234} \times X_i^{13}$ (respectively, $X_{3-i}^{1234} \times X_i^{12}$ and $X_{3-i}^{1234} \times X_i^{23}$) contains a unique outcome a_1 (respectively, a_2 and a_3).

Now, let us compute $X_{3-i}^{1234} \times \{X_i^{3123} \cup X_i^{1223} \cup X_i^{1123}\}$.

For example, let us consider rows $X_1^{13} \cup X_1^{1234}$ and columns $X_2^{3123} \cup X_2^{1234}$. By definition, in every row of the subform $X_1^{1234} \times X_2^{1234}$ there is an outcome $a \notin \{a_1, a_2, a_3\}$. Also the subform $X_1^{13} \times X_2^{3123}$ contains a row whose every entry is a_3 (we will call it a_3-row). These two observations together with total tightness imply that $g(x_1, x_2) = a_3$ for all $x_1 \in X_1^{1234}$ and $x_2 \in X_2^{3123}$. By symmetry, we fill subforms $X_{3-i}^{1234} \times \{X_i^{3123} \cup X_i^{1223} \cup X_i^{1123}\}$ for $i = 1, 2$, as in Figure 13.

Let us also recall that the subform $X_{3-i}^{0123} \times X_i^{13}$ (respectively, $X_{3-i}^{0123} \times X_i^{12}$ $X_{3-i}^{0123} \times X_i^{23}$) contains only outcome a_1 (respectively, a_2 and a_3), by definition of X_{3-i}^{0123}.

Now, let us refine $X_{3-i}^{0123} \times \{X_i^{3123} \cup X_i^{1223} \cup X_i^{1123}\}$.

For example, let us consider rows $X_1^{13} \cup X_1^{0123}$ and columns $X_2^{12} \cup X_2^{3123}$. By definition, subform $X_1^{13} \times X_2^{3123}$ contains a a_3-row. This observation together with total tightness imply that $g(x_1, x_2) = a_2$ or $g(x_1, x_2) = a_3$ for all $x_1 \in X_1^{0123}$ and $x_2 \in X_2^{3123}$. By symmetry, we fill subforms $X_{3-i}^{0123} \times \{X_i^{3123} \cup X_i^{1223} \cup X_i^{1123}\}$. for $i = 1, 2$, as in Figure 13.

Finally, let us consider the central subform $\{X_1^{3123} \cup X_1^{1223} \cup X_1^{1123}\} \times \{X_2^{3123} \cup X_2^{1223} \cup X_2^{1123}\}$.

Let us consider rows $X_1^{13} \cup X_1^{3123}$ and columns $X_2^{13} \cup X_2^{3123}$. By definition, subforms $X_1^{13} \times X_2^{3123}$ and $X_1^{3123} \times X_2^{13}$ contain respectively a a_3-row and a_3-column. This observation and total tightness imply that $X_1^{3123} \times X_2^{3123}$ is a a_3-subform (that is, it contains only one outcome a_3). By symmetry, we conclude that subforms $X_1^{1223} \times X_2^{1223}$ and $X_1^{1123} \times X_2^{1123}$ are a_2- and a_1-subforms, respectively, as shown in Figure 13.

Now, let us consider rows $X_1^{12} \cup X_1^{3123}$ and columns $X_2^{13} \cup X_2^{1223}$. As we already mentioned, subform $X_1^{3123} \times X_2^{13}$ contains a a_3-column. This observation together with total tightness imply that subform $X_1^{3123} \times X_2^{1223}$ contains only outcomes a_2 and a_3.

Also, let us recall that, by definition, each row from X_1^{3123} must contain a_1, since otherwise this row would beloing to X_1^{23}. Yet, we already proved that a_1 can appear only in columns from X_2^{13}. Hence, subform $X_1^{3123} \times X_2^{13}$ contains a a_1-column. (It also contains a a_3-column, as it was shown earlier.)

Let us consider rows $X_1^{23} \cup X_1^{3123}$ and columns $X_2^{13} \cup X_2^{1223}$. We know that subform $X_1^{3123} \times X_2^{13}$ contains a a_1-column, while $X_1^{23} \times X_2^{1223}$ contains a a_2-row. Hence, subform $X_1^{3123} \times X_2^{1223}$ contains only outcomes a_1 and a_2. Thus, it is a a_2-subform.

By symmetry, we conclude that $X_1^{3123} \times X_2^{1223}$ and $X_1^{1223} \times X_2^{3123}$ are a_2-subforms; $X_1^{3123} \times X_2^{1123}$ and $X_1^{1123} \times X_2^{3123}$ are a_3-subforms; $X_1^{1223} \times X_2^{1123}$ and $X_1^{1123} \times X_2^{1223}$ are a_1-subforms.

Also by symmetry, we conclude that subform $X_1^{1223} \times X_2^{23}$ contains an a_2- and a_3-columns, respectively, $X_1^{1123} \times X_2^{12}$) contains a_1- and a_2-columns. Furthermore, subforms $X_1^{13} \times X_2^{3123}$, $X_1^{12} \times X_2^{1123}$, and $X_1^{23} \times X_2^{1223}$ contain respectively, a_1- and a_3-rows, a_1- and a_2-rows, and a_2- and a_3-rows; see Figure 13.

However, TT game forms are not explicitly characterized; see in Figure 13 subforms

$X_{3-i} \times \{X_i^{3123} \cup X_i^{1223} \cup X_i^{1123} \cup X_i^{0123} \cup X_i^{1234}\}$ for $i = 1, 2$.

4 Totally tight game forms are acyclic and dominance-solvable; proofs of Propositions 6 and 7

First, we will show that Propositions 6 and 7 result immediately from Theorem 5; more precisely, from the structure of TT game forms given in Figure 13.

Let us recall that, by definition, total tightness is a hereditary property of game forms, that is, if g is TT and g' is a subform of g then g' is TT, too. In contranst, dominance-solvability (DS) of games and game forms is not hereditery; see example in Figure 3, where g is DS, while g' is not.

Proof of Proposition 7, dominance-solvability. Let us assume indirectly that a TT game form g is not DS. Then there is a payoff (or preference profile) u such that game (g, u) is not DS. Let us eliminate successively dominated strategies from (g, u) in an arbitrary order until we obtain a domination-free subgame (g', u). Yet, game form g' is TT, since g was TT. However, g' might be reducible. Then, let us successivle eliminate constant lines, rows or columns, from g' until we obtain a (unique) irreducible game form g''.

It is easy to see that game (g'', u) is still domination-free, since elimination of a constant line respects this property. Since g'' is TT and irreducible, it is of of type given in Figure 13. Let us recall that sets of rows X_i^{12}, X_i^{13}, and X_i^{23} are not empty for $i = 1, 2$; in contrast, sets X_i^{1123}, X_i^{1223}, X_i^{3123}, X_i^{0123}, and X_i^{1234} might be empty.

Without loss of generality, we can assume that $a_1 > a_2 > a_3$ is the preference of player 1. However, then each row from X_1^{23} is dominated by each row from X_1^{13}; see Figure 13. Hence, game (g'', u) is not domination-free; a contradiction. □

Proof of Proposition 6, acyclicity. Given a TT game form g', assume indirectly that it is not acyclic, i.e., there is a payoff (or preference profile) u such that game (g', u) has a strict improvement n-cycle C_n. Let us consider the corresponding $n \times n$ subform g; obviously, it is TT, too. Moreover, in every line, row or column, of g there is exacly one arc of C_n. Since, a constant line cannot contain such an arc, we conclude that g is irreducible. Furthermore, being TT and irreducible, g is of of type given in Figure 13.

Then let us notice that every row (column) from $X_i^{12} \cup X_i^{13} \cup X_i^{23}$, where $i = 1$ (respectively, $i = 2$), contains exactly two outcomes: $\{a_1, a_2\}$, $\{a_1, a_3\}$, and $\{a_2, a_3\}$. Hence, $u(i, a_{j'}) \neq u(i, a_{j''})$ for all $i \in \{1, 2\}$ and distinct $j', j'' \in \{1, 2, 3\}$. Indeed, otherwise each line of the corresponding set $X_i^{j'j''}$ is constant and, hence, it contains no arc of C_n.

Obviously, the chain of inequalities $u(i, a_1) > u(i, a_2) > u(i, a_3) > u(i, a_1)$ cannot hold, by transitivity. Without loss of generality, let us assume that $u(1, a_1) > u(1, a_3)$. and prove that then $u(2, a_3) > u(2, a_2)$. Assume indirectly that $u(2, a_3) < u(2, a_2)$. Each column of X_2^{13} contains a (unique) arc of C_n. This arc goes from a_1 to a_3 and this a_3 is either in X_1^{23} or in X_1^{3123}; see Figure 13. Where the next arc of C_n can lead to? If a_3 is in X_1^{3123} then it can lead only to a_1 in a column of X_2^{13} again. This column also contain a (unique) arc of C_n that can lead only to a_3, etc. Thus, sooner or later, cycle C_n will come to a_3 in X_1^{23}. Then the next arc can only lead to a_2. Hence, $u(2, a_3) > u(2, a_2)$. Thus, we proved the implication: if $u(1, a_1) > u(1, a_3)$ then $u(2, a_3) > u(2, a_2)$. Exactly the same arguments prove the following chain of similar implications:

$u(1, a_1) > u(1, a_3) \Rightarrow u(2, a_3) > u(2, a_2) \Rightarrow u(1, a_2) > u(1, a_1) \Rightarrow u(2, a_1) > u(2, a_3) \Rightarrow u(1, a_3) > u(1, a_2) \Rightarrow u(2, a_2) > u(2, a_1) \Rightarrow u(2, a_2) > u(2, a_1)$.

Yet, it is easy to notice that they contradict transitivity of both $u(1, *)$ and $u(2, *)$; see inequalities $1, 3, 5$ and $2, 4, 6$, respectively. □

Direct proof of acyclicity. Let us derive Proposition 6 directly from the definition of total tightness. Let us assume, by contradiction, that there exists a totally tight game form g that is not acyclic, and fix a shortest strict improvement cycle C, which is of length $2n$ for some $n > 2$. Using that this is a shortest improving cycle, and that total tightness is a hereditary property, we can assume without loss of generality that both players have exactly n strategies, and that Player 1's arcs in C are from $g(i, i)$ to $g(i+1, i)$, $(i = 1, \dots, n)$, while the Player 2's arcs in C are from $g(i, i-1)$ to $g(i, i)$, $(i = 1, \dots, n)$; otherwise we can delete rows and columns from g in which no moves are made, and permute the remaining ones to obtain a cycle of this form. To simplify notation, row and column indices 0 and $n+1$ are always understood as row and column n and 1.

Recall that a game form is totally tight if and only if every 2×2 subform of it is reducible (contains a constant line). It is immediate from the definition of the strict improvement cycle that consecutive outcomes of C are different, or formally,

$$g(i, i-1) \neq g(i, i) \neq g(i+1, i), \quad i = 1, \dots, n.$$

We claim that the second neighbors among the outcomes of C are also different, that is,

$$g(i, i) \neq g(i+1, i+1) \text{ and } g(i, i-1) \neq g(i+1, i), \quad i = 1, \dots, n, \tag{3}$$

otherwise C would not be shortest: if $g(i, i) = g(i+1, i+1)$, then the reducibility of the submatrix induced by columns i and $i+1$, and rows i and $i+1$ implies that $g(i, i) =$

$g(i+1,i+1) = g(i,i+1)$, and hence we could replace the 4-step segment

$$g(i,i-1) \to g(i,i) \to g(i+1,i) \to g(i+1,i+1) \to g(i+2,i+1)$$

in C by the 2-step segment

$$g(i,i-1) \to g(i,i+1) \to g(i+2,i+1).$$

A similar contradiction can be obtained in the case when $g(i,i-1) = g(i+1,i)$ (the second half of (3)).

Using the reducibility of the subform in rows and columns 1 and 2, we obtain that either $g(1,2) = g(1,1)$ or $g(1,2) = g(2,2)$. Without loss of generality (by symmetry) we can assume that the latter case holds. We claim that then the following equalities hold:

$$g(i,i) = g(i-1,i) \text{ and } g(i+1,i) = g(i+1,i-1), \quad i = 1,\ldots,n. \tag{4}$$

We already assumed $g(2,2) = g(1,2)$, and if $g(i,i) = g(i-1,i)$, then using reducibility of 2×2 subforms and (3) we obtain $g(i+1,i) = g(i+1,i-1)$, and our claim follows by induction. To complete the inductive step, use the reducibility in two subforms: first in the one induced by rows i and $i+1$, and columns $i-1$ and i, and then in the one induced by rows $i-1$ and $i+1$, and columns $i-1$ and i to obtain that on one hand, $g(i+1,i-1)$ is equal to $g(i,i-1)$ or $g(i+1,i)$, but on the other hand it is also equal to $g(i-1,i-1)$ or $g(i+1,i)$. (In the second subform we are also using the difference between second neighbors, (3).) Finally, this and $g(i,i-1) \neq g(i-1,i-1)$ imply that $g(i+1,i-1) = g(i+1,i)$, as claimed. Similar argument shows that once $g(i+1,i) = g(i+1,i-1)$ is established for some i, then it implies $g(i+1,i+1) = g(i,i+1)$. Finally, (4) follows by induction on i.

Now let us return to (3). There must be an $i \in \{1,\ldots,n\}$ such that $u(2,g(i,i)) < u(2,g(i+1,i+1))$, otherwise (3) implies $u(2,g(1,1)) > u(2,g(2,2)) > \cdots > u(2,g(n,n)) > u(2,g(1,1))$, a contradiction. But now we can construct another cycle which is shorter than C by replacing the 4-step segment

$$g(i,i-1) \to g(i,i) \to g(i+1,i) \to g(i+1,i+1) \to g(i+2,i+1)$$

by the 2-step segment

$$g(i,i-1) \to g(i,i+1) \to g(i+2,i+1).$$

This is justified by the inequalities

$$u(2,g(i,i-1)) < u(2,g(i,i)) < u(2,g(i+1,i+1)) = u(2,g(i,i+1))$$

and

$$u(1,g(i,i+1)) = u(1,g(i+1,i+1)) < u(1,g(i+2,i+1)).$$

By obtaining a contradiction, we proved our claim. □

Direct proof of dominance-solvability in case of three outcomes. Suppose the game form has only three different outcomes; let these be denoted by a_1, a_2, and a_3. We may assume that the game form has no identical or constant rows or columns. (Rows and columns are strategies of Player 1 and 2, respectively.) We show that under these assumptions for every preference of the column player there are two columns, one dominating the other.

Suppose that the column player prefers a_1 to a_2 and a_2 to a_3. For any two columns j and k, let us denote by $D(j,k)$ the set of rows in which they differ.

Let us note first that if $j \neq k$, and both $g(i,j)$ and $g(i,k)$ are in $\{a_1, a_3\}$ for every $i \in D(j,k)$, then either the game form is not TT, or one of columns j and k dominates the other. So we can assume that for every pair of distinct columns $\{j,k\}$ there exists an $i \in D(j,k)$ such that $g(i,j) = a_2$ and $g(i,k) \in \{a_1, a_3\}$. Suppose that $g(i,k) = a_1$. (The other case can be handled similarly.) Then from the total tightness of the game form it is immediate that for any $i' \in D(j,k), i \neq i'$ we have

$$(g(i',j), g(i',k)) \notin \{(a_1,a_2), (a_1,a_3), (a_3,a_2)\}. \tag{5}$$

Thus, column k dominates column j, unless there exists an $i' \in D(j,k)$ for which $(g(i',j), g(i',k)) = (a_2, a_3)$. Summing up, for columns j and k we can find rows i and i' such that $(g(i,j), g(i,k)) = (a_2, a_1)$ and $(g(i',j), g(i',k)) = (a_2, a_3)$. Let us say that when this holds for columns j and k, then we orient the pair $\{j,k\}$ from j to k.

The total tightness of the game form implies that if no column dominates another, then each pair of columns can be oriented in exactly one direction. Let G denote the obtained directed complete graph.

If G is acyclic, and j is a source of G, then either $g(i,j) = a_2$ for all i, and hence we have a constant column, or there is a constant row in A. Hence, G must be acyclic, implying that there exists a cyclically oriented triangle $\{j,k,\ell\}$. Then we have an $i \in D(j,k)$ and an $i' \in D(k,\ell)$ satisfying $g(i,j) = 2$, $g(i,k) = 3$, $g(i',k) = 2$, and $A_{i'\ell} = 1$. The last equation and 5 show that $i' \notin D(\ell,j)$, consequently $g(i',j) = g(i',\ell) = 1$. Then the subform induced by rows i and i', and columns j and k is not tight, violating the TT property of the game form. □

5 On totally tight 3-person game forms

Let $I = \{1, \ldots, n\}$ and $A = \{a_1, \ldots, a_p\}$ be sets of players and outcomes, respectively. A mapping $u : I \to a$ is an *n-person payoff or utility function.* Value $u(i,a)$ is interpreted as the profit of the player $i \in I$ in case of the outcome $a \in A$. Furthermore, let X_i be a finite set of strategies of a player $i \in I$. The direct product $\prod_{i \in I} X_i$ is the set of *situations.* A mapping $g : X \to A$ is an *n-person game form.* A pair (g,u) is an *n-person normal form game.*

Given a game (g,u), Nash-equilibrium is defined as a situation $x \in X$ such that $u(i, g(x)) \geq u(i, g(x^i))$ for every $i \in I$ and for each situations $x^i \in X$ such that $x_j = x_{i_j}$ for every $j \neq i$. In other words, when no player can profit by choosing in situation x another strategy if all opponents keep the same strategies.

A game form g is called *Nash-solvable* if for every payoff u the obtained game (g, u) has at least one Nash equilibrium.

A set of situations $x^1, \ldots, x^k \in X$ is a *strict improvement cycle* if for every $j \in \{1, \ldots, k\}$ there is a player $i_j \in I$ such that $x_i^j = x_i^{j+1}$ for every $i \neq i_j$ and $u(i_j, x^j) < u(i_j, x^{j+1})$. Standardly, we assume that $k+1 = 1$. In other words, in situation x^j player i_j, by changing the strategy, can get a better situation x^{j+1}, for every $j \in \{1, \ldots, k\}$. A game form g is called *acyclic* if for every payoff u the obtained game (g, u) has no strict improvement cycle. Clearly, if a game has no Nash equilibrium then it has a strict improvement cycle. Hence, every acyclic n-person game form is Nash-solvable and, hence, it is tight.

Given an n-person game (g, u) and two strategies $x_i, x'_i \in X_i$ of a player $i \in I$, we say that x'_i is dominated by x_i if $u(i, g(x_i, x_{I \setminus \{i\}}) \geq u(i, g(x'_i, x_{I \setminus \{i\}})$ for any strategies $x_{I \setminus \{i\}} \in X_{I \setminus \{i\}} = \prod_{j \in I \setminus \{i\}}$ of the opponents; in other words, if player i cannot profit by substituting x'_i for x_i if all other players keep the same strategies.

Let us eliminate successively dominated strategies of players. Game (g, u) is called *dominance-solvable* if this procedure results in a single-situation terminal subgame, in which each player remains with only one strategy. It is well-known and easy to show that the obtained situation is a NE; see, for example, [16], Chapter 5.

However, in general, the result might depend on the order in which dominated strategies are eliminated. Yet, there are simple conditions under which the above procedurte and concept of domination are well-defined, namely, when utility function $u_i : A \to \mathbb{R}$ of each player $i \in I$ is injective; in other words when $u(i, a) = u(i, a')$ if and only if $u(i', a) = u(i', a')$ for all $i, i' \in I$ and $a, a' \in A$; see again [16].

A game form g is called *dominance-solvable* if for any payoff u the obtained game (g, u) is dominance-solvable. Clearly, each dominance-solvable game form is Nash-solvable and, hence, it is tight.

A non-empty subset of players $K \subseteq I$ is called a *coalition*; $x_K = \{x_i \mid i \in K\}$ is a strategy of a coalition K. Given an n-person game form $g : X \to A$ and a coalition K such that $K \neq \emptyset$ and $K \neq I$, we define a 2-person game form $g : X_K \times X_{I \setminus K} \to A$. Let us notice that two complementary coalitions K and $I \setminus K$ define the same game form (up two renaming of its two players). An n-person game form g is called (totally) tight if all $2^{n-1} - 1$ corresponding 2-person game forms are (totally) tight.

In particular, to a 3-person game form we assign three 2-person game forms corresponding to partitions $\{1, 2, 3\} = \{1\} \cup \{2, 3\} = \{2\} \cup \{3, 1\} = \{3\} \cup \{1, 2\}$.

A Nash-solvable not tight 3-person game form was constructed in [10]. Obviously, the same example gives a not TT 3-person game form that is dominance-solvable and acyclic, since both acyclicity or dominance-solvability imply Nash-solvability and total tightness implies tightness.

Also in [10], a tight but not Nash-solvable 3-person game form was constructed. Thus, already for $n = 3$, tightness is not necessary or sufficient for Nash-solvability. However, it is an important

Open Problem: whether all TT 3-person (n-person) game forms are Nash-solvable.

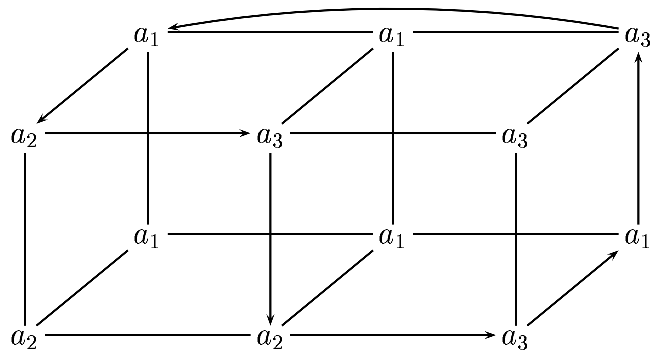

Figure 14: Odd strict improvement cycle in a TT 3-person game form

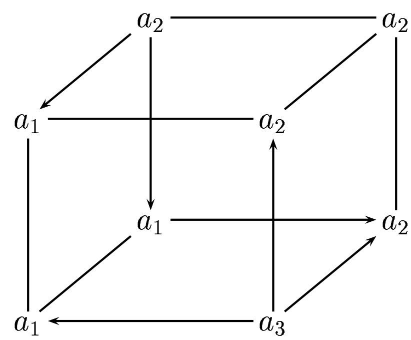

Figure 15: Totally tight but not dominance-solvable 3-person game form

In contrast, it is easy to construct TT 3-person game forms that are not acyclic or dominance-solvable. The corresponding two examples are given in Figures 14 and 15. Let us also remark that the strict improvement cycle in Figure 14 is odd.

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