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Recent Developments on Graphs of Bounded Clique-width

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RRR 06-2007, February 2007

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RRR 06-2007, FEBRUARY 2007

RECENT DEVELOPMENTS ON GRAPHS OF BOUNDED CLIQUE-WIDTH

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Abstract. Whether the clique-width of graphs in a certain class of graphs is bounded or not is an important question from an algorithmic point of view, as many problems that are NP-hard in general admit polynomial-time solutions when restricted to graphs of bounded clique-width. Over the last few years many classes of graphs have been shown to have bounded clique-width. For many others, this parameter has been proved to be unbounded. This paper provides a survey of recent results addressing this problem.

1 Introduction

Clique-width is a relatively young notion generalizing another important graph parameter, *tree-width*, studied in the literature for decades. The notion of clique-width generalizes that of tree-width in the sense that graphs of bounded tree-width have bounded clique-width.

The importance of these graph invariants is partly due to the fact that numerous problems that are NP-hard in general admit polynomial-time solutions when restricted to graphs of bounded clique-width (see e.g. [17, 19, 43]). A partial list of these problems includes: deciding whether a graph contains a Hamiltonian path or a Hamiltonian cycle, computing the maximum size of a cut, finding the chromatic number, finding the minimum size of a maximal matching, finding the minimum size of a dominating set of vertices/edges, determining whether a graph contains a nonempty cubic subgraph, and various partitioning problems (partitioning into cliques/triangles/complete bipartite subgraphs/perfect matchings/forests) [24, 34, 53].

The notion of clique-width was introduced in the early 1990s [18], and since then the literature on clique-width has been growing at a fast pace. Naturally, many questions regarding this graph parameter have been addressed, both from a theoretical and an algorithmic point of view. How to compute the clique-width of a graph? How difficult this problem is? How to recognize graphs of clique-width at most k? Is the clique-width of graphs in a certain class bounded or not?

The question about the complexity of computing the clique-width has only been settled very recently. In [25], Fellows et. al. showed that the problem "given a graph G and an integer k, decide whether the clique-width of G is at most k" is NP-complete. For specific values of k, polynomial-time algorithms have so far been found only for $k \leq 3$ (see e.g. [15]), while for higher values the complexity remains unknown.

It is therefore desirable to identify classes of graphs with bounded clique-width. Over the last few years, many classes of graphs have been shown to possess this nice property. For many others, it has been shown that the clique-width is unbounded. In this paper, we survey recent results on this topic and discuss some related open problems.

All graphs in this paper are undirected, without loops and multiple edges. We use standard graph terminology. In particular, an *independent set* is a subset of pairwise non-adjacent vertices, while a *clique* is a subset of pairwise adjacent vertices. The vertex set of a graph G will be denoted $V(G)$. For a vertex $v \in V(G)$, we denote by $N(v)$ the neighborhood of v (i.e., the set of vertices adjacent to v) and by $\deg(v) := |N(v)|$ the degree of v. The maximum vertex degree in G is denoted $\Delta(G)$ and the complement of G is denoted $\overline{G}$. As usual, P_n, C_n and K_n denote a chordless path, a chordless cycle and a complete graph on n vertices, respectively, and $K_{n,m}$ a complete bipartite graph with parts of size n and m. We say that a graph H is

- an *induced subgraph* of G if H can be obtained from G by deletion of some (possibly none) vertices; the subgraph of G *induced by* $U \subseteq V(G)$ is the graph obtained from G by deleting the vertices from $V(G) \backslash U$ and it will be denoted by $G[U]$;

– a *subgraph* of G if H can be obtained from G by applying a (possibly empty) sequence of vertex and edge deletions;

– a *minor* of G if H can be obtained from G by applying a (possibly empty) sequence of vertex deletions, edge deletions and edge contractions (an edge contraction is the operation of substituting two adjacent vertices u and v by a new vertex adjacent to every vertex in $(N(u) \cup N(v)) \setminus \{u, v\}$).

With a slight abuse of terminology, we say that G contains H as an induced subgraph (subgraph, minor) if H is isomorphic to an induced subgraph (subgraph, minor) of G.

We present the necessary preliminaries in Section 2. Section 3 is devoted to results related to graph classes with unbounded clique-width, while those of bounded clique-width are the topic of Section 4.

2 Preliminaries

In this section, we recall different types of graph classes, introduce various width parameters of graphs, and discuss several graph operations. Whenever appropriate, we connect these notions to clique-width.

2.1 Graph classes

The clique-width of a graph cannot be less than the clique-width of any of its induced subgraphs (see e.g. [20]). This simple but important observation enables us to restrict our attention to graph classes with the following nice property: whenever they contain a graph G, they contain all induced subgraphs of G. Such classes are called *hereditary*. Many classes of theoretical or practical importance are hereditary, which includes, among others,

1. *planar graphs*;
2. *bipartite graphs*;
3. *graphs of bounded vertex degree*;
4. *forests*, i.e., graphs without cycles;
5. *split graphs*, i.e., graphs partitionable into a clique and independent set [26];
6. *transitively orientable graphs*;
7. *threshold graphs*;
8. *perfect graphs*;
9. *interval graphs*;
10. *chordal bipartite graphs*, i.e., bipartite graphs without chordless cycles of length at least six;
11. *line graphs*. A graph G is a line graph if the vertices of G are in a one-to-one correspondence with the edges of some other graph H, and two vertices of G are adjacent if and only if the respective edges of H have a vertex in common. We denote this by $G = L(H)$.

A representative family of hereditary classes are those containing with every graph G all subgraphs of G (not necessarily induced). Such classes are called *monotone*. In the above list, the first four classes are monotone, while the others are not. An important and well studied subfamily of monotone classes are *minor closed* classes, i.e., those containing with every graph G all minors of G. Among classes listed above only 1 and 4 are minor closed.

Classes that are not hereditary can be extended to minimal hereditary classes containing them. For instance, for the class of trees such an extension is the class of forests, and for the class of cubic graphs such an extension is the class of all graphs of vertex degree at most 3.

An important property of hereditary classes is that these and only these classes admit a uniform description in terms of forbidden induced subgraphs, which provides a systematic way to investigate various problems associated with graph classes. If a graph G does not contain induced subgraphs from a set Z, we say that G is Z-free. The set of all Z-free graphs will be denoted by $Free(Z)$. With this notation the above statement about induced subgraph characterization of hereditary classes can be reformulated as follows: a class of graphs X is hereditary if and only if $X = Free(Z)$ for some set Z.

2.2 Width parameters

In what follows, we recall three important and interrelated graph parameters: tree-width, clique-width and rank-width.

2.2.1 Tree-width

To introduce the notion of tree-width [45, 46], let us use one of its equivalent definitions. Given an arbitrary graph $G = (V, E)$, a *triangulation* of G is a chordal graph $H = (V, F)$ such that $E \subseteq F$. The *tree-width* of G, denoted $\mathrm{tw}(G)$, is

$$\min\{\omega(H) - 1 \mid H \textit{ is a triangulation of } G\},$$

where $\omega(H)$ is the size of a maximum clique in H.

Graphs of tree-width 0 are empty (edgeless), and graphs of tree-width at most 1 are trees, or more generally, forests. Graphs of tree-width at most k (also known as partial k-trees) appeared in the literature as a generalization of trees.

2.2.2 Clique-width

The *clique-width* of a graph G is the minimum number of labels needed to construct G using the following four operations:

(i) Creation of a new vertex v with label i (denoted by $i(v)$).

(ii) Disjoint union of two labeled graphs G and H (denoted by $G \oplus H$).

(iii) Joining by an edge each vertex with label i to each vertex with label j ($i \neq j$, denoted by $\eta_{i,j}$).

(iv) Renaming label i to j (denoted by $\rho_{i\to j}$).

Every graph can be defined by an algebraic expression using these four operations. For instance, a chordless path on five consecutive vertices a, b, c, d, e can be defined as follows:

$$\eta_{3,2}(3(e) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))))))))).$$

Such an expression is called a *k-expression* if it uses at most k different labels. Thus the clique-width of G, denoted $\mathrm{cw}(G)$, is the minimum k for which there exists a k-expression defining G. For instance, from the above example we conclude that $\mathrm{cw}(P_5) \le 3$. Moreover, it is not hard to see that the clique-width of any tree (and hence of any forest) is at most 3. This observation is in accordance with the fact that graphs of bounded tree-width have bounded clique-width. More formally:

Proposition 1. *For any graph G,* $\mathrm{cw}(G) \le 3 \cdot 2^{\mathrm{tw}(G)-1}$.

This proposition has been proved by Corneil and Rotics in [16], improving the inequality $\mathrm{cw}(G) \le 2^{\mathrm{tw}(G)+1} + 1$ shown by Courcelle and Olariu in [20]. In the same paper, Courcelle and Olariu revealed many more interesting relations between tree- and clique-width, among which the following results will be of particular interest.

Proposition 2. *There exist integer functions f, f_H, where f_H is associated with each graph H, such that for any graph G,*

(a) $\mathrm{tw}(G) \le f(\Delta(G), \mathrm{cw}(G))$,

(b) $\mathrm{tw}(G) \le f_H(\mathrm{cw}(G))$ *if G does not contain a fixed graph H as a minor.*

We also mention another interesting fact from [29] that relates the clique-width of graphs without large complete bipartite subgraphs with their tree-width.

Proposition 3. *Let G be a graph that does not contain the complete bipartite graph $K_{t,t}$ as a subgraph. If* $\mathrm{cw}(G) \le k$, *then* $\mathrm{tw}(G) \le 3k(t-1) - 1$.

We conclude this subsection with the following important result from [30] relating the tree-width of a graph G with the clique-width of its line graphs $L(G)$.

Proposition 4. $(\mathrm{tw}(G) + 1)/4 \le \mathrm{cw}(L(G)) \le 2\mathrm{tw}(G) + 2$.

2.2.3 Rank-width

The *rank-width* of a graph G has been recently introduced in [43] and can be defined as follows. A *cut in G* is a partition of the vertex set of G into two disjoint subsets. The *cut-rank* of a cut $C = (A, B)$, denoted $\mathrm{cutrk}_G(C)$, is the linear rank over GF(2) of the $|A| \times |B|$ submatrix of the adjacency matrix of G corresponding to the set of pairs (a, b) with $a \in A$ and $b \in B$.

If T is a tree of maximum degree 3 with at least two vertices and L is a bijection from the set of vertices of G to the set of leaves of T, then (T, L) is a *rank-decomposition* of G. For each edge e of T, the two components of $T - e$ define a cut C_e in G. The *width* of the edge e is $\text{cutrk}_G(C_e)$. The *width* of (T, L) is the maximum width of all the edges of T. The *rank-width* of G, denoted $\text{rw}(G)$, is the minimum width of all rank-decompositions of G. If $|V(G)| \leq 1$, then we define $\text{rw}(G) = 0$. The following proposition has been proved in [43].

Proposition 5. *For any graph G, $\text{rw}(G) \leq \text{cw}(G) \leq 2^{\text{rw}(G)+1} - 1$. Furthermore, there is a polynomial-time algorithm to convert any rank-decomposition of G of width k into a $(2^{k+1} - 1)$-expression defining G.*

According to this theorem, the clique-width is bounded if and only if the rank-width is bounded. Moreover, for graphs of bounded clique-width an algebraic expression using a bounded number of labels can be constructed in polynomial time.

2.3 Graph operations

In the study of clique-width we may obviously restrict our attention to connected graphs, since an expression defining a disconnected graph can be obtained from expressions defining its connected components with the $\oplus$-operation. Moreover, as was shown in [5, 36], we can further restrict ourselves to 2-connected graphs, since in every graph G there is a block H whose clique-width is at least $\text{cw}(G) - 2$. This observation alone allows to establish boundedness of the clique-width in some classes of graphs, such as trees, or more generally, *block-cactus* graphs. By definition, a block in a block-cactus graph is either a complete graph or a cycle, each of which can be constructed using at most 4 labels. Therefore, the clique-width of block-cactus graphs is at most 6.

Another important observation refers to the operation of complementation (see [20]): for any graph G, $\text{cw}(\overline{G}) \leq 2\text{cw}(G)$. Two related operations are *local complementation* and *bipartite complementation*.

The local complementation has been introduced in the context of rank-width. It consists in complementing the subgraph induced by the neighborhood of a vertex v, in which case we call it local complementation *centered* at v. It is known (see [42]) that

Proposition 6. *Local complementation does not change the rank-width of a graph.*

Let $G = (V, E)$ be a bipartite graph with a bipartition $V = V_1 \cup V_2$ of its vertex set. The *bipartite complement of* G is the bipartite graph $\widetilde{G} = (V, (V_1 \times V_2) - E)$. Observe that for this definition, the bipartite graph must be given together with a bipartition of its vertex set. In general, a bipartite graph can admit several bipartitions. Different bipartitions may have different bipartite complements. However, every connected bipartite graph has a unique bipartition. In the following proposition proved in [37], we must distinguish between different bipartitions of the same bipartite graph if they have different bipartite complements.

Proposition 7. *If G is a bipartite graph, then* $\text{cw}(\widetilde{G}) \leq 4\text{cw}(G)$.

We now introduce two new complementation operations that can be viewed as generalizations of those mentioned above.

- *Subgraph complementation* is the operation of complementing the edges of an induced subgraph of a graph G;
- *Bipartite subgraph complementation* is the operation of complementing the edges between two disjoint subset of vertices of a graph G.

Let us remark that edge additions and deletions are just special cases of these two operations. Similarly, contracting an edge of a graph is equivalent to deleting a vertex and applying a particular bipartite subgraph complementation to the resulting graph.

We also note that a special case of bipartite subgraph complementation, when the two parts of a bipartite subgraph partition the vertex set of the graph, was studied in the literature under the name *Seidel switching* [50].

The importance of these operations is due to the fact that they do not change the clique-width of a graph "too much". Without giving any specific bound on the size of this change, we simply state the following.

Proposition 8. *For a class of graphs X and a nonnegative integer k, denote by $X^{(k)}$ the class of graphs obtained from graphs in X by applying at most k subgraph complementations or bipartite subgraph complementations. Then the clique-width of graphs in $X^{(k)}$ is bounded if and only if it is bounded for graphs in X.*

Proof. Necessity is trivial. For sufficiency, we may assume without loss of generality that $k = 1$.

Given a subset $U \subseteq V(G)$, the operation of complementing the edges of the subgraph $G[U]$ can be viewed as a sequence of the following three operations: introducing a new vertex u with $N(u) = U$, local complementation centered at u, and deleting u. We know that deletion of a vertex cannot increase the clique-width. Moreover, it has been shown in [36] that addition of a new vertex at most doubles the clique-width. Together with Propositions 5 and 6 this proves the statement.

For complementing the edges between two disjoint subsets U and W, we introduce three new vertices u, v, w with $N(u) = U$, $N(w) = W$ and $N(v) = U \cup W$, apply local complementations centered at u, v and w, and delete these three vertices. The conclusion follows analogously. □

Two last remarks of this section deal with graphs of bounded vertex degree. First, let us make the following observation (see e.g. [38]).

Proposition 9. *Subdivision of an edge of a graph G does not change the tree-width of G.*

This proposition together with Propositions 1 and 2 lead to the following conclusion.

Proposition 10. *Let X be a class of graphs of bounded vertex degree and X^* a class of graphs obtained from graphs in X by edge subdivisions. Then the clique-width of graphs in X^* is bounded if and only if it is bounded for graphs in X.*

Finally, Propositions 1, 2 and 4 together imply the following conclusion.

Proposition 11. *Let X be a class of graphs of bounded vertex degree and $L(X)$ the class of line graphs of graphs in X. Then the clique-width of graphs in $L(X)$ is bounded if and only if it is bounded for graphs in X.*

3 Graphs of unbounded clique-width

We partition the results of this section into two parts: those that deal with graphs of unbounded vertex degree, and those where degree is bounded.

3.1 Graphs of unbounded vertex degree

The first result exploits some quantitative characteristics of hereditary classes of graphs. For a class of graphs X, let X_n denote the number of labeled graphs with n vertices (i.e., graphs with vertex set $\{1, \ldots, n\}$) in the class X. Scheinerman and Zito showed in [49] that if X is a hereditary class, then the growth of X_n is far from arbitrary. Specifically, the rates of the growth constitute discrete layers. In [49], the authors distinguish five such layers: constant, polynomial, exponential, factorial and superfactorial. Independently, a similar result has been obtained in [1]. Moreover, the latter paper provides the first three layers with complete structural characterizations. The structure of graphs belonging to classes in the first three layers is extremely simple and leads to an immediate conclusion that all such classes are of bounded clique-width. So let us turn our attention to classes in higher layers.

A class of graphs X is said to be *factorial* if X_n satisfies the inequalities $n^{c_1 n} \leq X_n \leq n^{c_2 n}$ for some constants c_1 and c_2. If there is no constant c such that $X_n \leq n^{cn}$, then X is said to be *superfactorial*. Examples of factorial classes are planar, permutation, line graphs. Classes of bipartite, co-bipartite and split graphs are superfactorial, since the number of n-vertex graphs in these classes is at least $2^{n^2/4}$. The following theorem proved in [5] shows that no superfactorial class can be of bounded clique-width.

Theorem 1. *If a class of graphs X is superfactorial, then the clique-width of graphs in X is unbounded.*

As an immediate consequence of Theorem 1 we conclude that the clique-width of bipartite, co-bipartite and split graphs is unbounded (a direct proof of unboundedness of the clique-width of split graphs has been obtained in [41]). These simple observations can be significantly improved by combining Theorem 1 with some combinatorial results.

The first example comes from the well-known results on the maximum number of edges in bipartite graphs containing no $K_{p,p}$. Denoting the class of $K_{p,p}$-free bipartite graphs by

X^p, we have (see, e.g., [6, 23]):

$$c_1 n^{2-\frac{2}{p+1}} < \log_2 X_n^p < c_2 n^{2-\frac{1}{p}} \log_2 n.$$

In particular, the class of C_4-free bipartite graphs (i.e., X^2) is superfactorial and hence is not of bounded clique-width due to Theorem 1.

The result for C_4-free bipartite graphs has been improved in [33] in the following way. For each odd k, the authors present an infinite family of n-vertex bipartite graphs of girth (the length of a smallest cycle) at least $k+5$ that have at least $2^{t-k-2}n^{1+\frac{1}{k-t+1}}$ edges, where $t = \lfloor \frac{k+2}{4} \rfloor$. Consequently, for each odd $k \geq 1$, $(C_4, \ldots, C_{k+3})$-free bipartite graphs constitute a superfactorial class and hence are not of bounded clique-width.

Another example deals with chordal bipartite graphs, i.e., bipartite graphs containing no induced cycles of length more than four. Spinrad has shown in [51] that the number of chordal bipartite graphs is $\Omega(2^{\Omega(n \log^2 n)})$. Thus, the chordal bipartite graphs form a superfactorial class and hence are not of bounded clique-width.

The conclusion for chordal bipartite graphs has been strengthened in [9] by showing that the clique-width is unbounded for bipartite permutation graphs. This also improves the result for permutation graphs, where the clique-width has been proved to be unbounded in [28]. We believe that any further improvement of this result is impossible. However, the problem is open.

Open Problem 1. *Determine whether the class of bipartite permutation graphs is a minimal hereditary class of unbounded clique-width or not.*

To simplify the study of this problem, let us show that we can restrict our attention to a specific family of subclasses of bipartite permutation graphs. To this end, we introduce the following definitions and notations.

A *chain graph* is a bipartite graph such that the vertices in each part of the graph can be ordered under inclusion of their neighborhoods. In a chain graph, a vertex ordering "<" is called *increasing* if $x < y$ implies $N(x) \subseteq N(y)$, and *decreasing* if $x < y$ implies $N(y) \subseteq N(x)$.

Denote by $B_{n,m}$ the graph with nm vertices which can be partitioned into m independent sets $V_1 = \{v_{1,1}, \ldots, v_{1,n}\}$, ..., $V_m = \{v_{m,1}, \ldots, v_{m,n}\}$ so that for each $i = 1, \ldots, n-1$ and for each $j = 1, \ldots, n$, the vertex $v_{i,j}$ is adjacent to the vertices $v_{i+1,1}, v_{i+1,2}, \ldots, v_{i+1,j}$ and there are no other edges in the graph. In other words, every two consecutive independent sets induce in $B_{n,m}$ a chain graph. Figure 1 shows an example with $m = n = 5$.

The following theorem from [9] shows that for any positive integers n and m, the graph $B_{n,m}$ is bipartite permutation.

Theorem 2. *A connected graph G is bipartite permutation if and only if the vertex set of G can be partitioned into independent sets $D_0, D_1, \ldots, D_q$ so that*

(a) any two vertices in non-consecutive sets are non-adjacent,

(b) any two consecutive sets D_{j-1} and D_j induce a chain graph, denoted by G_j,

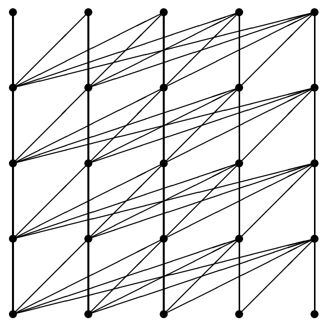

Figure 1: Graph $B_{5,5}$

(c) *for each* $j = 1, 2, \ldots, q-1$*, there is an ordering of vertices in set* D_j*, which is decreasing for* G_j *and increasing for* G_{j+1}*.*

This characterization of bipartite permutation graphs has been used in [39] to prove the following theorem.

Theorem 3. *The graph* $B_{n,n}$ *contains all bipartite permutation graphs with* n *vertices as induced subgraphs.*

As a trivial consequence of this theorem we conclude that

Corollary 1. *Every proper hereditary subclass of bipartite permutation graphs is contained in the class of* $B_{n,n}$*-free bipartite permutation graphs for some* n*.*

Now let us show that the above discussion can be equally applied to unit interval graphs. To this end, let us observe that if we create a clique out of each independent set D_j in Theorem 2 (i.e., connect every two vertices of D_j by an edge), then we obtain a unit interval graph. Moreover, every connected unit interval graph admits a partition into cliques $D_1, \ldots, D_q$ satisfying (a),(b),(c). This can be proved by analogy with Theorem 2, which has been proved using the intersection model of bipartite permutation graphs. We advise the reader to use the intersection model of unit interval graphs and leave this proof as an exercise. The same transformation of a bipartite permutation graph into a unit interval graph can be done by a sequence of vertex deletions and local complementations, which implies, in particular, that the clique-width of unit interval graphs is unbounded. Originally, this fact was proved in [27]. By analogy with bipartite permutation graphs we conjecture that the clique-width of graphs in any hereditary subclass of unit interval graphs is bounded by a constant.

Chain graphs play an important role in two more constructions of unbounded clique-width. To describe one of them, let us denote by Γ_n the graph with vertex set $A \cup B \cup C$ and edge set $E_1 \cup E_2 \cup E_3$ defined as follows:

Consider $(n+1)(n+1)$ vertices v_{ij}, $i, j \in \{0, \ldots, n\}$. We shall say that a vertex v_{ij} belongs to row i and column j. Omit the vertex v_{00}, take the vertices of the 0-th column as

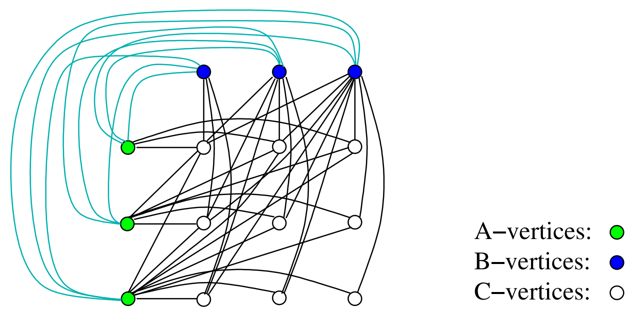

Figure 2: The graph Γ_3

the A-vertices $a_i = v_{i0}$, take the vertices of the 0-th row as the B-vertices $b_i = v_{0i}$ and take the other vertices as the C-vertices $c_{ij} = v_{ij}$, $1 \leq i, j \leq n$.

The edges between A, B and C are defined as follows.

- A and B induce a complete bipartite graph; E_1 is the set of all edges $a_i b_j$, $1 \leq i, j \leq n$.
- Every vertex $a_i \in A$ is adjacent to all vertices in the j-th row, $1 \leq j \leq i$; E_2 is the set of all edges $a_i v_{j\ell}$, $1 \leq j \leq i$, $1 \leq \ell, i \leq n$.
- Every vertex $b_i \in B$ is adjacent to all vertices in the j-th column, $1 \leq j \leq i$; E_3 is the set of all edges $b_i v_{\ell j}$, $1 \leq j \leq i$, $1 \leq \ell, i \leq n$.

Thus, A, B, and C are independent sets, $|A| = |B| = n$, and $|C| = n^2$ such that $A \cup B$ induces a complete bipartite graph, and $A \cup C$ as well as $B \cup C$ induce a chain graph. See Figure 2 for the graph Γ_3. It has been proved in [11] that for each $n \geq 1$, $\mathrm{cw}(\Gamma_n) \geq n$. We shall denote the class of graphs containing Γ_n for each n and all their induced subgraphs by Γ.

To introduce one more construction of unbounded clique-width, we use the graph Γ_n defined above. For each vertex $c \in C$, we split c into two vertices c_a and c_b so that $N(c_a) = N(c) \cap A$ and $N(c_b) = N(c) \cap B$, and connect c_a to c_b. Let us denote the graph obtained in this way by Λ_n. It has been shown in [40] that $\mathrm{cw}(\Lambda_n) \geq n/3$. Observe that unlike Γ_n, the graph Λ_n is bipartite. We denote the class containing graphs Λ_n for each n and all their induced subgraphs by Λ.

To summarize, let us list all minimal classes of unbounded clique-width and unbounded vertex degree mentioned in the section: bipartite permutation graphs, unit interval graphs, bipartite graphs without small cycles (i.e., $(C_4, C_6, \ldots, C_{2k})$-free bipartite graphs for $k \geq 2$), the class Γ and the class Λ. By applying various complementation operations, we can obtain many other classes of this category. For instance, the class of bipartite permutation graphs can be transformed into a class of split graphs by performing subgraph complementations. Also, the class Γ can be transformed into a class of bipartite graphs by deleting the edges connecting A to B in Γ_n by means of bipartite subgraph complementation. We denote the

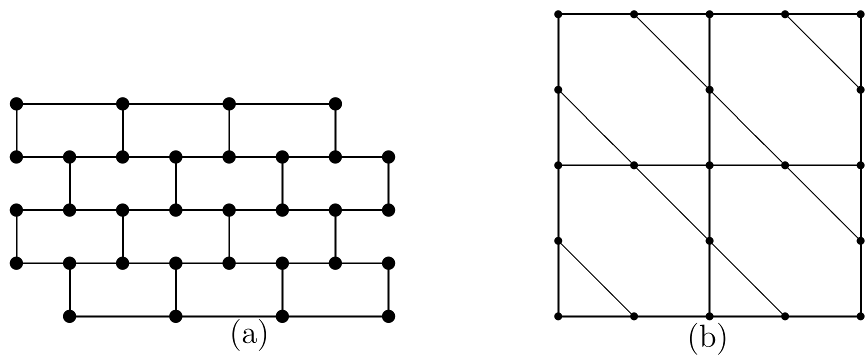

Figure 3: A wall (a) and Graph Q_3 (b)

class of bipartite graphs obtained in this way by Γ^*. To provide a uniform description of the results presented in this section, we keep only classes of bipartite graphs, since all other known classes of this category can be obtained by various complementation operations from the following graph classes:

- bipartite graphs without small cycles, i.e., $(C_4, C_6, \ldots, C_{2k})$-free bipartite graphs for $k \geq 2$;
- bipartite permutation graphs;
- the class Γ^*;
- the class Λ.

Similarly to bipartite permutation graphs, we conjecture that Γ^* and Λ are minimal hereditary classes of unbounded clique-width. This is not the case for bipartite graphs without small induced cycles. As we shall see in the next section, the clique-width in such classes remains unbounded even if we bound vertex degree.

3.2 Graphs of bounded vertex degree

All graph classes of bounded vertex degree and unbounded clique-width deal with certain types of grids. A typical example of this nature is an $n \times n$ grid G_n, i.e., the planar graph with the vertex set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ such that (i, j) and (k, l) are adjacent if and only if $|i - k| + |j - l| = 1$. A direct proof of unboundedness of the clique-width in the class of grids can be found in [27]. However, an easier way to obtain the same conclusion is to apply Proposition 2, since grids have bounded vertex degree and unbounded tree-width. Moreover, the tree-width is known to be unbounded in the class of so called walls (cf. [44]), which are planar graphs of vertex degree at most 3 (see Figure 3(a)). Again, by Proposition 2 this implies unboundedness of the clique-width in the class of walls.

Several other constructions of unbounded clique-width have been introduced in the literature. For instance, a graph Q_n can be obtained from the grid G_n as follows. First, subdivide each edge of G_n with a new vertex, and then for each original (old) vertex, connect its upper

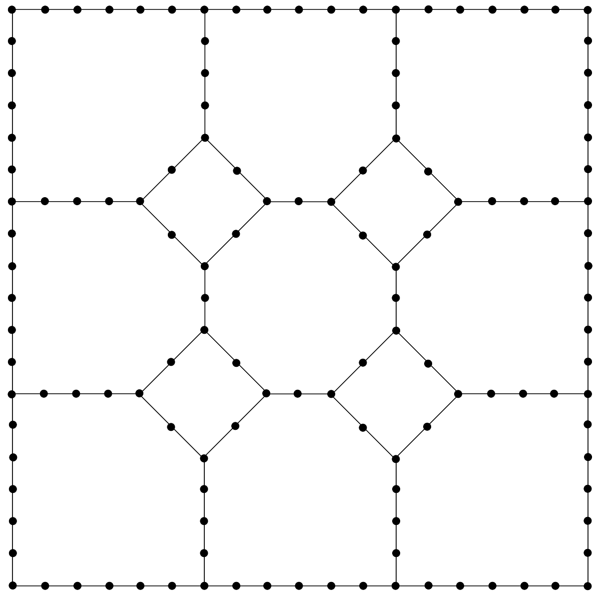

Figure 4: The grid $H_{4,2}$

new neighbor to the right one, and the left new neighbor to the lower one. An example of Q_n with $n = 3$ is represented in Figure 3(b). In [5], it has been shown that $\mathrm{cw}(Q_n) \geq n$. However, unboundedness of the clique-width in this class follows directly from Proposition 11. Indeed, it is not difficult to see that Q_n is simply the line graph of a wall.

Two other constructions have been introduced in [41]. One of them, denoted by $H_{n,q}$, can be obtained from G_n as follows (see Figure 4): Let $n \geq 4$ and $q \geq 2$.

- Replace every edge of G_n by a simple path with three edges, introducing two new vertices which are the internal vertices of the path. Let G'_n denote the resulting graph.
- For all vertices v of degree 4 in G'_n, do the following: For the four clockwise neighbors u_1, u_2, u_3, u_4 of v, omit v from G'_n and add the edges u_iu_{i+1} (index arithmetic modulo 4) such that $u_1u_2u_3u_4$ induce a C_4 in G'_n. Let G''_n denote the resulting graph.
- Replace every edge of G''_n by a simple path with q edges, introducing $q-1$ new vertices which are the internal vertices of the path. Then $H_{n,q}$ is the resulting graph.

Obviously, $H_{n,q}$ is a subdivision of a wall, and therefore, unboundedness of tree- (and hence of clique-) width follows by Proposition 10. The other construction from [41] is just the line graph of a subdivision of G_n, and hence the tree- and clique-width is unbounded in that class too.

The above discussion suggests the idea of developing a uniform tool that could be used to prove unboundedness of the clique-width for graphs of bounded degree. As we mentioned earlier, without loss of generality such a tool can be described in terms of hereditary classes. To this end, let us denote by

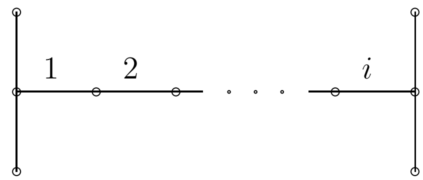

Figure 5: Graph H_i

> Φ_k the class of planar bipartite $(C_3, \ldots, C_k, H_1, \ldots, H_k)$-free graphs of vertex degree at most 3,

where H_i is the graph represented in Figure 5. It has been proved in [38] that for any positive integer k, the tree- and clique-width of graphs in Φ_k is unbounded. Translating this statement from graphs in Φ_k to line graphs of graphs in Φ_k, we obtain, by means of Proposition 11, the same conclusion for the class $L(\Phi_k)$. Combining the two statements together we conclude that

Lemma 1. *For a positive integer k, the tree- and clique-width of graphs in Φ_k and $L(\Phi_k)$ is unbounded.*

This lemma can be used to prove unboundedness of the clique-width in various classes, where "grid-like" arguments are normally applied. Moreover, it also characterizes graph classes that have the potential to be of bounded clique-width: in terms of hereditary classes, we have to exclude, as an induced subgraph, a graph from the class Φ_k and a graph from the class $L(\Phi_k)$ for each k. One possible way to do so is to forbid all long cycles as induced subgraphs. Another way is to forbid a graph from the intersection $\cap_{k\geq 3}\Phi_k$, and a graph from the intersection $\cap_{k\geq 3}L(\Phi_k)$. In the next section we shall see that both approaches lead to graphs of bounded clique-width whenever we deal with graphs of bounded vertex degree or planar graphs.

4 Graphs of bounded clique-width

We begin this section with the following remarkable result, due to Robertson and Seymour [47].

Theorem 4. *For any planar graph H there is a number N such that every graph with no minor isomorphic to H has tree-width at most N.*

According to Proposition 1, the same can be said with respect to clique-width: if a minor closed graph class does not contain some planar graph, then the clique-width of graphs in this class is bounded.

Now let us mention some other results in spirit of Theorem 4. In the course of our study, we shall use the following notation:

$\mathcal{S}$ is the class of graphs whose every connected component is of the form $S_{i,j,k}$ with $i, j, k \geq 0$ (Figure 6(a)),

$\mathcal{T}$ is the class of graphs whose every connected component is of the form $T_{i,j,k}$ with $i, j, k \geq 0$ (Figure 6(b)).

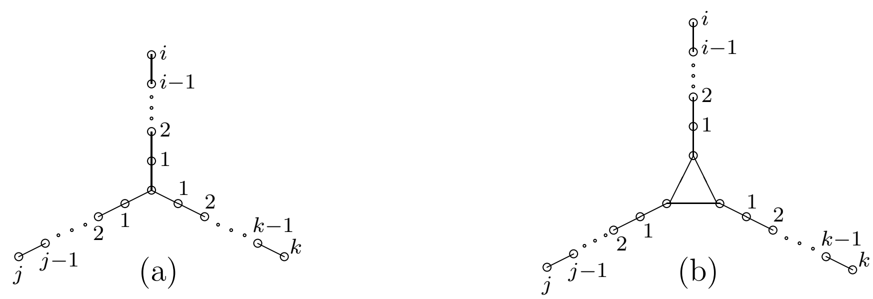

Figure 6: The graphs $S_{i,j,k}$ (a) and $T_{i,j,k}$ (b)

Notice that $\mathcal{S}$ coincides with the intersection $\cap_{k\geq 3}\Phi_k$ and $\mathcal{T}$ with the intersection $\cap_{k\geq 3}L(\Phi_k)$. The following theorem has been proved in [5].

Theorem 5. *For any graph $H \in \mathcal{S}$ there is a number N such that every graph with no subgraph (not necessarily induced) isomorphic to H has clique-width at most N.*

All other results of this section are partitioned into five major groups.

4.1 Graphs of bounded vertex degree

The chordality of a graph G is the size of a longest induced cycle in G. It has been shown by Bodlaender and Thilikos [3] that if a graph has chordality at most c and maximum degree at most k, then its tree-width is at most $k\,(k-1)^{c-3}$. Combining this result with Proposition 1 we obtain the following conclusion.

Theorem 6. *For any positive integers c and k there is a number N such that every graph of vertex degree at most k and chordality at most c has clique-width at most N.*

Another important result dealing with graphs of bounded vertex degree has been proved in [36].

Theorem 7. *For any positive integer k and any two graphs $H_1 \in \mathcal{S}$ and $H_2 \in \mathcal{T}$ there is a number N such that every graph of vertex degree at most k with no induced subgraphs isomorphic to H_1 or H_2 has clique-width at most N.*

According to Lemma 1, Theorem 7 is best possible whenever we deal with graphs of bounded vertex degree defined by finitely many forbidden induced subgraphs.

4.2 Planar and more general graphs

Surprisingly enough, many results valid for graphs of bounded degree remain true for planar graphs. Consider, for instance, the following obvious observation not mentioned in the previous section: graphs of bounded degree and bounded diameter have bounded tree- and clique-width (as there are only finitely many such connected graphs). The same is true for planar graphs, i.e., planar graphs of bounded diameter have bounded tree- and clique-width. With respect to the tree-width this has been proved by Eppstein [22]. Moreover, Eppstein showed that this property holds for any class of graphs excluding an apex graph as a minor, where an *apex graph* is a graph that contains a vertex whose deletion leaves a planar graph. Later, the proof found by Eppstein has been substantially simplified by Demaine and Hajiaghayi in [21]. The key tool for this simplification is Lemma 2 below. In this lemma, an *augmented grid* is a grid G_n augmented with additional edges (and no additional vertices). Recall that an $n \times n$ grid G_n is the graph with the vertex set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ such that (i, j) and (k, l) are adjacent if and only if $|i - k| + |j - l| = 1$. Vertices (i, j) with $\{i, j\} \cap \{1, n\} \neq \emptyset$ are *boundary vertices* of the grid; the other ones are *nonboundary.*

Lemma 2. *[21] Let H be an apex graph, let G be an H-minor-free graph, let $r = 14|V(H)| - 22$, and let $m > 2r$ be the largest integer such that $\mathrm{tw}(G) \geq m^{4|V(H)|^2(m+2)}$. Then G can be contracted into a $(m - 2r) \times (m - 2r)$ augmented grid R such that each vertex $v \in V(R)$ is adjacent to less than $(r + 1)^6$ nonboundary vertices of the grid.*

Now let us show that the same lemma can be used to obtain results analogous to Theorems 6 and 7.

With some care, one can conclude from Lemma 2 that if H is an apex graph, and X a subclass of H-minor-free graphs closed under edge contractions and vertex deletions, then the tree-width of graphs in X is bounded if and only if it is bounded for graphs of bounded vertex degree in X. More formally:

Corollary 2. *Let H be an apex graph and X a subclass of H-minor-free graphs. Denote by Y the class of all graphs which can be obtained from graphs in X by applying a sequence of (zero or more) edge contractions and vertex deletions. Suppose that for every $k \geq 1$ the tree-width of graphs in Y of degree at most k is bounded. Then, the tree-width of graphs in X is bounded.*

Proof. For $k \geq 0$, let $Y_k := \{G \in Y : \Delta(G) \leq k\}$, and suppose that there exists a function f such that the tree-width of graphs in Y_k is bounded above by $f(k)$:

$$\mathrm{tw}(G) \leq f(k), \quad \text{for all} \quad G \in Y_k\,. \tag{1}$$

Let $r = 14|V(H)| - 22$, and consider a graph $G \in X$ with $\mathrm{tw}(G) \geq (2r + 1)^{4|V(H)|^2(2r+3)}$. (If there is no such graph, then the tree-width of graphs in X is bounded and we are done.)

According to Lemma 2, G can be contracted into an $(m-2r) \times (m-2r)$ augmented grid R such that each vertex $v \in V(R)$ is adjacent to less than $(r + 1)^6$ nonboundary vertices of the grid, where $m > 2r$ is the largest integer such that $\mathrm{tw}(G) \geq m^{4|V(H)|^2(m+2)}$. Let R_0

denote the subgraph of R induced by the nonboundary vertices of R. Since $R_0 \in Y_{(r+1)^6}$, it follows from (1) that the tree-width of R_0 is at most $f((r+1)^6)$. As the tree-width of a minor of a graph never exceeds the tree-width of the graph, and since the tree-width of an $n \times n$ grid is n, we conclude that $m - 2r - 1 \leq \mathrm{tw}(R_0) \leq f((r+1)^6)$, implying $m + 1 \leq f((r+1)^6) + 2r + 2 =: C$. This inequality and the choice of m then imply that the tree-width of G is bounded above by the constant $C^{4|V(H)|^2(C+2)}$. □

Trivially, any class of bounded chordality is closed under edge contractions and vertex deletions. Together with the preceding results this implies the following analog of Theorem 6.

Theorem 8. *For any apex graph H and any positive integer c, there is a number N such that every H-minor-free graph of chordality at most c has clique-width at most N.*

In order to obtain a statement analogous to Theorem 7, we have to prove that for any $H_1 \in \mathcal{S}$ and $H_2 \in \mathcal{T}$, the class of (H_1, H_2)-free graphs is closed under edge contractions (closedness under vertex deletion is obvious). Unfortunately, this is not generally true. However, one can prove the following lemma, where by nT we denote the graph consisting of n disjoint copies of a graph T.

Lemma 3. *Let G be an $(nS_{k,k,k}, nT_{k,k,k})$-free graph, and let G' be a graph obtained from G by a sequence of edge contractions. Then, G' is $((2n-1)S_{k+1,k+1,k+1}, (2n-1)T_{k+1,k+1,k+1})$-free.*

Sketch of Proof. Assume by contradiction that G' contains an induced subgraph H isomorphic either to $(2n-1)S_{k+1,k+1,k+1}$ or to $(2n-1)T_{k+1,k+1,k+1}$.

Let $n = 1$. Denote by U the vertices of degree 3 in H and by R the remaining vertices of H. Also, let U_G, R_G be the vertices of G "contracted" to U and R, respectively. It is not difficult to see that, without loss of generality, we may assume that $R_G = R$. Finally, let a, b, c denote the vertices of R that have a neighbor of degree 3 in U. Obviously, b must be connected to c in G by a path P whose every internal vertex (other than b and c) belongs to U_G. Similarly, there must exist a path Q connecting a to P with all internal vertices belonging to U_G. If P and Q are chordless, the reader can easily find in G an induced $S_{k,k,k}$ or $T_{k,k,k}$, contradicting the assumption.

For $n > 1$, we apply the above arguments to each connected component of H and conclude by analogy that each of them corresponds to an induced copy of $S_{k,k,k}$ or $T_{k,k,k}$ in G. Since the number of components is $2n-1$, we conclude that G contains either $nS_{k,k,k}$ or $nT_{k,k,k}$ as an induced subgraph. □

Obviously, every graph $H_1 \in \mathcal{S}$ is an induced subgraph of $nS_{k,k,k}$ for some n and k, and every graph $H_2 \in \mathcal{T}$ is an induced subgraph of $nT_{k,k,k}$ for some n and k. Therefore, summarizing the above discussion we conclude that

Theorem 9. *For any apex graph H and any two graphs $H_1 \in \mathcal{S}$ and $H_2 \in \mathcal{T}$ there is a number N such that every H-minor-free graph with no induced subgraphs isomorphic to H_1 or H_2 has clique-width at most N.*

4.3 Bipartite graphs

Clique-width of bipartite graphs is of special interest. Most classes of unbounded clique-width mentioned in Section 3 are bipartite. Many important examples of graphs of bounded clique-width are also bipartite. We partition these examples into two general groups: subclasses of chordal bipartite graphs and graphs defined by finitely many forbidden induced bipartite subgraphs.

4.3.1 Subclasses of chordal bipartite graphs

The most remarkable class of this category is the class of trees (or more generally, forests). An important extension of trees, which is of bounded clique-width, is the class of distance-hereditary graphs (see Section 4.5). Bipartite distance-hereditary graphs can be characterized in terms of forbidden induced subgraphs as *domino*-free chordal bipartite graphs (where *domino* is the graph obtained from a cycle C_6 by connecting two vertices at distance 3). Besides trees, bipartite distance-hereditary graphs generalize another interesting subclass mentioned before, the class of chain graphs.

It is worth mentioning that chordal bipartite graphs of bounded vertex degree have bounded clique-width, as they have bounded chordality. A more general conclusion obtained in [37] is that k-fork-free chordal bipartite graphs have bounded clique-width for any k, where a k-fork is the graph obtained by a single subdivision of exactly one edge of a $K_{1,k+1}$. Observe that if we subdivide at least three edges of a $K_{1,k+1}$, then we obtain a graph that contains $S_{2,2,2}$ as an induced subgraph. The class of $S_{2,2,2}$-free chordal bipartite graphs contains all bipartite permutation graphs and hence is not of bounded clique-width. This leads to the following open problem. Denote by E_k the graph obtained by single subdivisions of exactly two edges of a $K_{1,k+1}$.

Open Problem 2. *For a fixed $k \geq 3$, determine whether the clique-width of E_k-free chordal bipartite graphs is bounded or unbounded.*

As we shall see in the next section, the clique-width is bounded for general E_2-free bipartite graphs (not necessarily chordal bipartite), since $E_2 = S_{1,2,2}$. This conclusion cannot be extended to larger values of k, because the class of E_3-free bipartite graphs contains all bipartite graphs of vertex degree at most three. However, for E_3-free *chordal* bipartite graphs the problem is open.

4.3.2 Subclasses of bipartite graphs defined by finitely many forbidden induced bipartite subgraphs

A systematic investigation of the clique-width of bipartite graphs defined by finitely many forbidden induced bipartite subgraphs has been initiated in [38]. According to Lemma 1, a necessary condition for boundedness of the clique-width in such classes is that the set of forbidden graphs contains a graph from the class $\mathcal{S}$. We also know that the set of forbidden graphs must contain a bipartite permutation graph, a graph from Γ^* and a graph from Λ

(see Section 3.1). Moreover, by Proposition 7, we also must forbid bipartite complements of graphs from each of the above four classes $\mathcal{S}$, bipartite permutation graphs, Γ^* and Λ. Whether this set of requirements is sufficient to bound the clique-width is an open problem.

Open Problem 3. *Let X be a class of bipartite graphs defined by a finite set M of forbidden induced bipartite subgraphs, and let M contain a bipartite permutation graph, a graph from $\mathcal{S}$, a graph from Γ^*, a graph from Λ, the bipartite complement of a bipartite permutation graph, of a graph from $\mathcal{S}$, of a graph from Γ^*, and of a graph from Λ. Determine whether the clique-width of graphs in X is bounded or not.*

A natural first step to approach this general problem would be to investigate it for small values of $|M|$. When $|M| = 1$, we call the class of M-free bipartite graphs *monogenic*. Solving Problem 3 for monogenic classes is equivalent to enumerating all graphs in the intersection of the above eight classes (bipartite permutation graphs, $\mathcal{S}$, Γ^*, Λ, and their bipartite complements) and analyzing the clique-width of H-free bipartite graphs for each graph H in the intersection. This work was started in [35] and completed in [40]. We summarize the results in Theorem 10. For definitions of graphs mentioned in this theorem, we refer the reader to Figure 7.

Theorem 10. *If H is an induced subgraph of one of the graphs $S_{1,2,3}$, $K_{1,3} + 3K_1$, $K_{1,3} + e$, $S_{1,1,3} + v$, then the clique-width of H-free graphs is bounded. Otherwise, it is unbounded.*

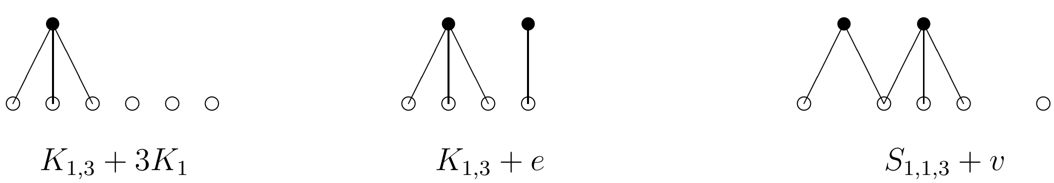

Figure 7: Three critical bipartite graphs

For classes defined by two forbidden induced subgraphs, Problem 3 is open. We formulate it in the following way.

Open Problem 4. *Classify the family of bipartite graph classes defined by two forbidden induced bipartite subgraphs with respect to bounded/unbounded clique-width.*

The only available result of this type that is not covered by any monogenic class, deals with $(S_{2,2,2}, A)$-free bipartite graphs [4], where A is the graph obtained from a *domino* by disconnecting two adjacent vertices of degree 2. Observe that $S_{2,2,2}$ is not a bipartite permutation graph and $A \notin \mathcal{S}$, which means that the clique-width is bounded neither in the class of $S_{2,2,2}$-free bipartite graphs nor in the class of A-free bipartite graphs. On the other hand, it is not hard to verify that $S_{2,2,2} \in \mathcal{S}$, while A belongs to the other seven critical classes. As a result, it is of no surprise that the clique-width of $(S_{2,2,2}, A)$-free bipartite graphs is bounded [5]. Taking into account the critical role of the graph A in this topic, we distinguish the following subproblem of Problem 4.

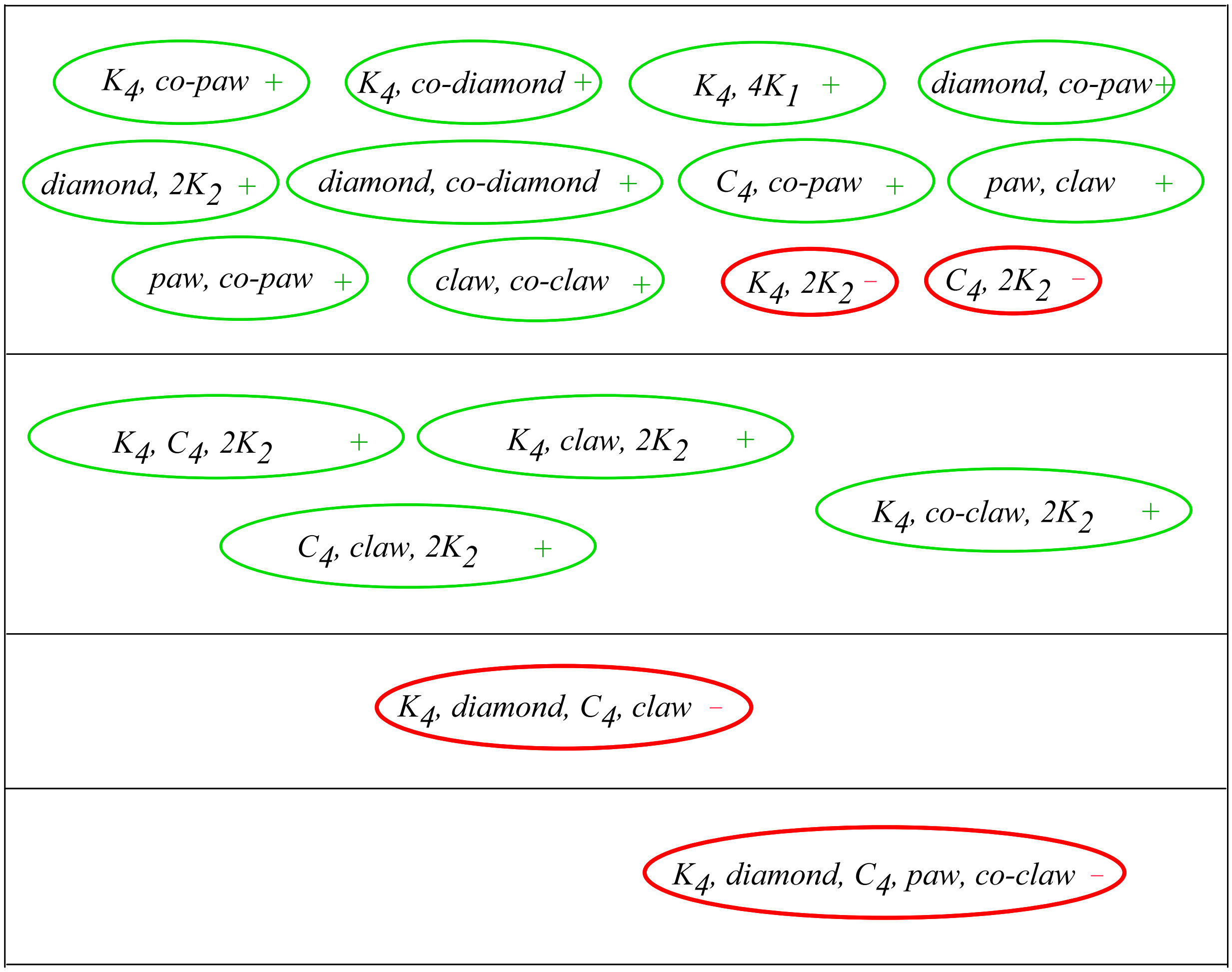

Figure 8: Essential classes for all combinations of forbidden 4-vertex graphs; + (−) denotes bounded (unbounded) clique-width

Open Problem 5. *Determine those graphs $H \in \mathcal{S}$ for which the clique-width of (H, A)-free bipartite graphs is bounded.*

4.4 Classes defined by small forbidden induced subgraphs

Several classes of graphs where the clique-width has been shown to be bounded are defined by forbidding certain small induced subgraphs. For instance, (P_6, K_3)-free graphs, generalizing P_6-free bipartite graphs, provide such an example [12]. Another remarkable example of this type is the class of P_4-free graphs. Graphs from this class admit a recursive decomposition into connected components or co-components (i.e., components of the complement) [13]. This nice decomposability property has led to efficient solutions for many algorithmic graph problems that are NP-hard in general [14]. In retrospect, this also follows from the fact

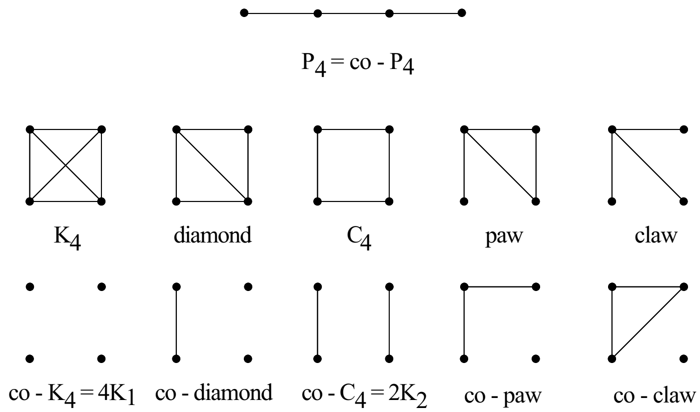

Figure 9: All four-vertex graphs

that P_4-free graphs are exactly the graphs of clique-width at most 2. Boundedness of the clique-width in the class of P_4-free graphs motivated several lines of research.

4.4.1 Graphs defined by forbidden induced subgraphs with at most four vertices

A natural question to ask is, besides P_4-free graphs, are there other classes defined by a single forbidden induced subgraph H with at most four vertices for which the clique-width is bounded? By a simple inspection one can conclude that if H has at most 4 vertices and is not an induced subgraph of a P_4, then either H or its complement contains a cycle. Therefore, according to Lemma 1, the clique-width of H-free graphs is unbounded. However, if we forbid two or more induced subgraphs with at most 4 vertices, we may obtain a class of bounded clique-width. A complete classification of all such classes with respect to bounded/unbounded clique-width can be found in [11]. All essential classes of this type (i.e., minimal classes of unbounded clique-width and maximal classes of bounded clique-width) are presented in Figure 8. For the names of 4-vertex graphs we refer the reader to Figure 9.

4.4.2 Graphs defined by forbidding one-vertex extensions of P_4

Another line of research is devoted to classes defined by forbidding one-vertex extensions of P_4. There are ten such graphs (see Figure 10). A complete classification of all graph classes defined by forbidding one-vertex extensions of P_4 with respect to bounded/unbounded clique-width was obtained in [8]. The results from this paper are summarized in Table 1. As before, we list only maximal classes of bounded clique-width and minimal classes of unbounded clique-width. Also, out of each pair of classes containing graphs and their complements, we keep only one representative.

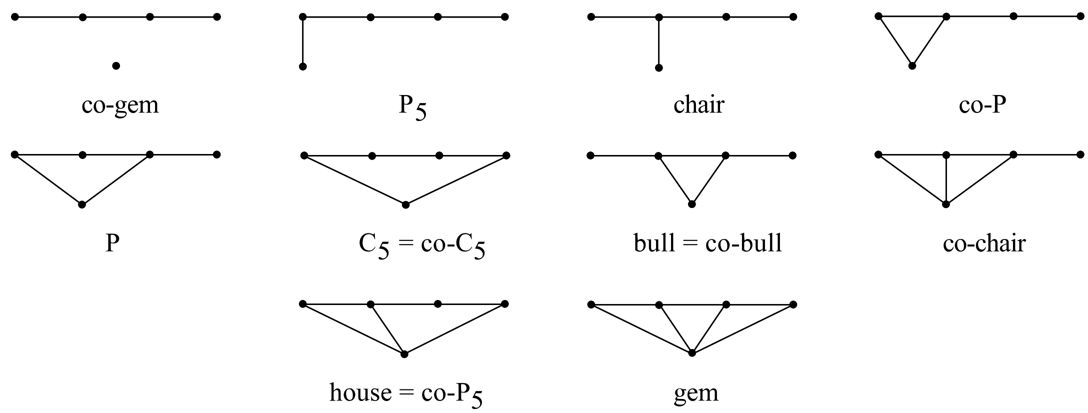

Figure 10: All one-vertex extensions of a P_4

bounded clique-width	unbounded clique-width
(co-gem,house)-free graphs	(co-gem,P_5,chair,co-P,C_5,co-chair)-free graphs
(co-gem, gem)-free graphs	(co-gem,P_5,chair,co-P,P,C_5,bull)-free graphs
(co-gem,P,co-chair)-free graphs	(P_5,co-P,P,C_5,house)-free graphs
(co-gem,bull,co-chair)-free graphs	
(P_5,chair,house)-free graphs	
(P_5,P,co-chair)-free graphs	
(P_5,bull,co-chair)-free graphs	
(P_5,bull,house)-free graphs	
(chair,bull,co-chair)-free graphs	
(chair,co-P,P,co-chair)-free graphs	

Table 1: Classes of graphs defined by forbidding one-vertex extensions of P_4

	$q=4$	$q=5$	$q=6$	$q=7$	$q=8$	$q=9$	$q=10$	...
$t=0$	+	+	+	+	+	+	+	+
$t=1$	-	+	+	+	+	+	+	+
$t=2$	-	-	+	+	+	+	+	+
$t=3$	-	-	-	+	+	+	+	+
$t=4$	-	-	-	+	+	+	+	+
$t=5$	-	-	-	-	+	+	+	+
$t=6$	-	-	-	-	?	+	+	+
$t=7$	-	-	-	-	-	?	+	+
$t=8$	-	-	-	-	-	-	?	+
$t=9$	-	-	-	-	-	-	-	?
...	-	-	-	-	-	-	-	-

Table 2: Classes of (q,t)-graphs with bounded (+) and unbounded (-) clique-width.

4.4.3 (q,t)-graphs

One more line of research was initiated by Hoàng who introduced in [31] the class of P_4-*sparse graphs*, defined as graphs in which every set of 5 vertices contains at most one P_4. This notion was later generalized to (q,t)-graphs. A (q,t)-graph is a graph in which every induced subgraph on q vertices contains at most t distinct induced P_4's. Clearly, P_4-free graphs are exactly $(4,0)$-graphs, and $(5,1)$-graphs are exactly P_4-sparse graphs. Every class of (q,t)-graphs can be equivalently defined by forbidding finitely many induced subgraphs, i.e., all those graphs on q vertices that contain more than t P_4's. Notice that $(q,t-1)$-graphs and $(q+1,t)$-graphs are subclasses of (q,t)-graphs.

A systematic study of (q,t)-graphs appears in [41]. The authors want to determine for what values of parameters q and t the class of (q,t)-graphs is of bounded/unbounded clique-width. The results of this paper are summarized in Table 2. The question mark in this table corresponds to unsolved cases, which naturally leads to the following open problem

Open Problem 6. *For a fixed integer $q \geq 8$, determine whether the clique-width of graphs in the class $(q,q-2)$ is bounded or unbounded.*

4.4.4 Partner-limited graphs

A *partner* of a P_4 (induced by vertices a,b,c,d) is a vertex v (different from a,b,c,d) such that the set $\{a,b,c,d,v\}$ induces at least two P_4's. A graph G is called *partner-limited* if no induced P_4 in G has more than two partners. The class of partner-limited graphs obviously has a characterization in terms of forbidden seven-vertex subgraphs.

In [52], Vanherpe showed that the clique-width of partner-limited graphs is at most four.

4.5 Various classes

A graph G is called *distance-hereditary* if for every connected induced subgraph H of G the distance between every pair of vertices in H is the same as in G. The clique-width of distance-hereditary graphs has been shown to be bounded by three [28]. Interestingly, this result also follows directly from Proposition 5 and the fact that distance-hereditary graphs are exactly graphs of rank-width at most one [42].

An interesting subclass of distance-hereditary graphs is the class of so-called *ptolemaic graphs*. They are exactly the intersection of chordal graphs and gem-free graphs. Interestingly enough, the clique-width is also bounded for chordal graphs that are co-gem-free [10]. Moreover, the gem and the co-gem are the only one-vertex P_4-extensions H such that chordal H-free graphs have bounded clique-width.

Finally, let us mentioned a couple of examples of graphs of bounded clique-width defined in terms of existence of specific induced subgraphs. For instance, we know from Section 4.4.2 that P_5- and chair-free graphs have unbounded clique-width. Moreover, it is unbounded even if we additionally forbid a co-gem. However, if a (P_5,chair)-free graph G contains a co-gem as an induced subgraph, then the clique-width of G is at most three [7]. Similarly, the clique-width of (chair,bull)-free graphs containing a co-gem is at most one [7].

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