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# Relations of Threshold and $k$-Interval Boolean Functions

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**Abstract.** Every $k$-interval Boolean function $f$ can be represented by at most $k$ intervals of integers such that vector $\boldsymbol{x}$ is a truepoint of $f$ if and only if the integer represented by $\boldsymbol{x}$ belongs to one of these $k$ (disjoint) intervals. Since the correspondence of Boolean vectors and integers depends on the order of bits an interval representation is also specified with respect to an order of variables of the represented function. Interval representation can be useful as an efficient representation for special classes of Boolean functions which can be represented by a small number of intervals. In this paper we study inclusion relations of threshold and $k$-interval Boolean functions. We show that positive 2-interval functions constitute a (proper) subclass of positive threshold functions and that such inclusion does not hold for any $k > 2$. We also prove that threshold functions do not constitute a subclass of $k$-interval functions, for any $k$.

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# 1 Introduction

Threshold functions constitute one of the most extensively studied classes of Boolean functions. The importance of this class is caused by the central role it plays in many application areas such as threshold logic ([7, 9, 10]), artificial neural networks ([1]), operational research ([6]) and game theory ([2, 11]). Moreover, threshold functions have a very simple description (since a threshold function is completely characterized by a vector of $(n + 1)$ numbers) and various other nice properties.

The concept of an interval representation has been introduced in [12] where minimal DNF representations of 1-interval functions (i.e., functions which can be represented by one interval) were studied. Using the correspondence between Boolean vectors and positive integers we can represent the truepoints of a Boolean function by a set of intervals of integers. Of course, this correspondence depends on the order of Boolean variables that determines which variable corresponds to which bit when bit vectors are viewed as numbers. Naturally, the interval representations may differ for different orders of variables not only in the boundaries of representing intervals but even in the number of intervals. Thus for special classes of functions which admit a representation using a small number of intervals, such representation can be very efficient in space (and time with respect to solving problems such as satisfiability). In [13] we have studied the problem of recognizing whether a function given by a DNF can be represented by one interval. This can be viewed as a variant of Boolean minimization (for more on Boolean minimization see e.g. [5, 3]).

In [8] we have observed that positive 1-interval functions constitute a proper subclass of threshold functions. Since threshold functions constitute such a well-studied class this property stimulated our interest in their relations with $k$-interval functions. In this paper we extend the result from [8] to a full description of the inclusion relations between positive $k$-interval and threshold functions. All results concerning positive $k$-interval functions can be easily transformed into results concerning negative $k$-interval functions (for the same value of $k$, if appropriate).

The outline of this paper is as follows: in the following section we present the notation and definitions we shall need. In Section 3 we show that threshold functions are not a subclass of positive $k$-interval functions, for any $k$. In Section 4 we extend our result from [8] by proving that positive 2-interval functions constitute a subclass of threshold functions and in Section 5 we show that this does not hold for any other $k \geq 3$ (by constructing a positive 3-interval function which is not threshold). We give a summary of the presented results and some concluding remarks in Section 6.

# 2 Basic Notation and Definitions

Let us start with some basic definitions.

A **Boolean function**, or a **function** in short, on $n$ propositional variables is a mapping $f : \{0, 1\}^n \mapsto \{0, 1\}$. A **Boolean vector of length** $n$ is an $n$-tuple of **Boolean values** 0 and 1 (usually denoted by **false** and **true**). Boolean vectors (or **vectors** for short) will be denoted by $\boldsymbol{x}, \boldsymbol{y} \ldots$. If $f(\boldsymbol{x}) = 1$ (0, resp.), then $\boldsymbol{x}$ is called a **true** (**false**, resp.) vector of $f$ (sometimes called **truepoint** resp. **falsepoint**). The set of all true vectors (false vectors) is denoted by $T(f)$ ($F(f)$). For function $f$ on $n$ variables and $v \in \{0,1\}$ we denote by $f[x_i := v]$ the function on $(n-1)$ variables, which is formed from $f$ by fixing the value of $i$-th variable to $v$.

Propositional variables $x_1, \ldots, x_n$ and their negations $\overline{x}_1, \ldots, \overline{x}_n$ are called **literals** (**positive** and **negative literals**, respectively). An elementary conjunction of literals

$$t = \bigwedge_{i \in I} x_i \wedge \bigwedge_{j \in J} \overline{x}_j \tag{1}$$

is called a **term**, if every propositional variable appears in it at most once, i.e., if $I \cap J = \emptyset$. A **disjunctive normal form** (or DNF) is a disjunction of terms. It is a well known fact (see e.g. [4].), that every Boolean function can be represented by a DNF. For DNF $\mathcal{F}$ and term $t$ we denote by $t \in \mathcal{F}$ the fact, that $t$ is contained in $\mathcal{F}$. For Boolean vector $v$ and term $t$ we say, that $t$ **satisfies** (**falsifies** resp.) $v$, if $t$ evaluates to 1 (0 resp.) on $v$. In the subsequent text the "$\wedge$" sign for conjunction will be frequently omitted.

A term $t$ defined by (1) is called **positive**, if it contains no negative literals (i.e., if $J = \emptyset$), and it is called **negative**, if it contains no positive literals (i.e., if $I = \emptyset$). A DNF is called **positive**, if it contains only positive terms. Finally, a Boolean function is called **positive** if it has at least one representation by a positive DNF. **Negative** DNFs and functions are defined similarly. The class of positive Boolean functions will be denoted by $\mathcal{C}^+$, the class of negative Boolean functions by $\mathcal{C}^-$.

Given Boolean functions $f$ and $g$ on the same set of variables, we denote by $f \leq g$ the fact, that $g$ is satisfied for any assignment of values to the variables for which $f$ is satisfied. We call a term $t$ an **implicant** of a DNF $\mathcal{F}$, if $t \leq \mathcal{F}$. We call $t$ a **prime implicant**, if $t$ is an implicant of $\mathcal{F}$ and there is no implicant $t' \neq t$ of $\mathcal{F}$, for which $t \leq t' \leq \mathcal{F}$. We call DNF $\mathcal{F}$ **prime**, if it consists of only prime implicants. We call DNF $\mathcal{F}$ **irredundant**, if for any term $t \in \mathcal{F}$, DNF $\mathcal{F}'$ produced from $\mathcal{F}$ by deleting $t$ does not represent the same function as $\mathcal{F}$.

The **bits** of vector $\boldsymbol{x} \in \{0,1\}^n$ will be denoted by $x_1, \ldots, x_n$. The vector $\boldsymbol{x}$ also corresponds to an integer number $x$ with binary representation equal to $\boldsymbol{x}$. In this case $x_1$ is the most significant bit of $x$ and $x_n$ the least significant bit. Hence $x = \sum_{i=1}^{n} x_i 2^{n-i}$. Also, for any integer $x$ we denote by $\boldsymbol{x}$ the vector corresponding to the binary representation of $x$. In case integer $x$ is specified by an expression $e$, we denote the vector corresponding to its binary representation by $\overrightarrow{e}$.

**Definition 2.1.** *For vector $\boldsymbol{x} \in \{0,1\}^n$ and for permutation $\pi : \{1, \ldots, n\} \mapsto \{1, \ldots, n\}$ we denote by $\boldsymbol{x}^\pi$ the vector of $n$ bits formed by permuting bits of $\boldsymbol{x}$ by $\pi$. That means $x_i^\pi = x_{\pi(i)}$. By $x^\pi$ we denote the number with binary representation $\boldsymbol{x}^\pi$.*

Now we shall proceed with the inductive definition of $k$-interval functions.

**Definition 2.2.** *Boolean function $f : \{0,1\}^n \to \{0,1\}$ is called a* **0-interval function***, if it is identically equal to zero, i.e., $f(\boldsymbol{x}) = 0$ holds for all $\boldsymbol{x} \in \{0,1\}^n$.*

**Definition 2.3.** *Boolean function $f : \{0,1\}^n \to \{0,1\}$ is called a* $k$**-interval function***, for $k \geq 1$, if it is a $(k-1)$-interval function or there exist $k$ pairs of $n$-bit integers $a^1, b^1, \dots, a^k, b^k$, $a^1 \leq b^1 < a^2 \leq b^2 < \dots < a^k \leq b^k$, and permutation $\pi$ of $\{1, \dots, n\}$, such that for every $n$-bit vector $\boldsymbol{x} \in \{0,1\}^n$ we get $f(\boldsymbol{x}) = 1$, if and only if $x^\pi \in \cup_{i=1}^k [a^i, b^i]$. The class of $k$-interval functions will be denoted by $\mathcal{C}_{k-\text{int}}$. The class of positive (negative resp.) $k$-interval functions will be denoted by $\mathcal{C}^+_{k-\text{int}}$ ($\mathcal{C}^-_{k-\text{int}}$, resp.).*

When studying interval representations of Boolean functions it is very helpful to view the situation using a graphical information. A suitable tool for this is a **branching tree**, which is a complete binary tree with edges evaluated with values 0 and 1. Such tree displays all bit vectors of the length equal to the depth of the tree. Every path from the root to a leaf corresponds to one vector and we can identify each leaf with the integer value corresponding to this vector, where the significance of the bits decreases from the root to the leaf. An example of such tree for the case of three bits is on Figure 1.

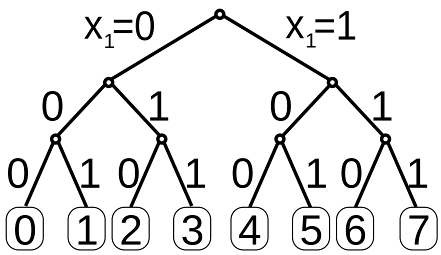


**Figure 1:** *A branching tree on 3 variables.*

When $f$ is a Boolean function on $n$ variables which can be represented by interval $[a, b]$ with respect to order $\pi$ of variables, then it means, that if we consider a branching tree of depth $n$ and evaluate the edges incident to the root by the most significant variable $x_{\pi^{-1}(1)}$, the edges adjacent to these edges by the second most significant variable $x_{\pi^{-1}(2)}$ and so on and the edges adjacent to leaves by the least significant variable $x_{\pi^{-1}(n)}$, then $f$ is 1 exactly on those bit vectors, which correspond to the leaves representing numbers from $[a, b]$.

The following lemma states a basic property of positive $k$-interval functions:

**Lemma 2.4.** *Let $f$ be a positive $k$-interval function, where $k \geq 1$, on $n$ variables, which is not a $(k-1)$-interval function and which is represented by intervals $[a^1, b^1] < \dots < [a^k, b^k]$. Then $b^k = 2^n - 1$.*

**Proof :** Clearly, if $f$ is positive (and not identically equal to 0, which is implied by the assumption, that $f$ is not a $(k-1)$-interval function), $f(11\dots1) = 1$ must hold. ■

We shall implicitly use Lemma 2.4 throughout this whole article.

For vectors $\boldsymbol{u}, \boldsymbol{v}$ of the same length $n$ we denote by $\boldsymbol{u} \geq \boldsymbol{v}$, that $\boldsymbol{u}$ is componentwise greater than or equal to $\boldsymbol{v}$, i.e., $\forall i \in \{1, \dots, n\} : u_i \geq v_i$. We use vector relations $=, \neq, <, >, \leq$ in a similar manner.

We will also use some well-known Boolean functions. The first one is the $\text{PARITY}_n$ function on $n$ variables, which is 1, if and only if the sum of all its input variables is odd. The second one is the $\text{MAJORITY}_n$ function on $n$ variables, which is 1, if and only if there are more input variables with value 1 than those with value 0.

**Theorem 2.5.** *Any Boolean function on n variables has at most $2^{n-1}$ intervals. Moreover, this bound is tight.*

**Proof :** It is obvious that we get the most intervals by alternating zeros and ones on every neighbouring vectors in given order. In $[0, 2^n - 1]$ this means we can create $2^{n-1}$ intervals. We can easily observe, that the $\text{PARITY}_n$ function does exactly this kind of alternating, starting with 0 in $00\ldots0$. Also, PARITY is symmetrical with respect to the order of its variables, hence $\text{PARITY}_n \in \mathcal{C}_{2^{n-1}-int} \setminus \mathcal{C}_{(2^{n-1}-1)-int}$. ■

Boolean function $f : \{0,1\}^n \mapsto \{0,1\}$ on variables $x_1, \ldots x_n$ is called a **threshold function**, if there exist weight function $\omega : \{x_1, \ldots, x_n\} \to \mathbf{R}$ and real number $t$, such that for any vector $\boldsymbol{x} \in \{0,1\}^n$ it holds that $f(\boldsymbol{x}) = 1$, if and only if $\omega(\boldsymbol{x}) \geq t$, where $\omega(\boldsymbol{x}) = \sum_{i=1}^{n} x_i \omega(x_i)$. A pair $(\omega, t)$ satisfying such equivalence for a threshold function $f$ is then called a **threshold structure** for $f$. A threshold structure $(\omega, t)$ is positive, if $\omega$ assigns only non-negative weights. The class of threshold functions will be denoted by $\mathcal{C}_{\text{th}}$, the class of positive threshold functions by $\mathcal{C}_{\text{th}}^{+}$. It is easy to observe, that threshold function $f$ is positive, if and only if there is a positive threshold structure for $f$.

**Definition 2.6.** *Let $f$ be a positive Boolean function on $n$ variables $x_1, \ldots x_n$ and let $x_1$ and $x_2$ be two of its variables. We say that $x_1$ has* **equal or greater strength** *than $x_2$ in $f$, if for any evaluation $e$ of variables $x_3, \ldots x_n$ it holds, that $f(0, 1, e(x_3), \ldots, e(x_n)) \leq f(1, 0, e(x_3), \ldots, e(x_n))$. We denote it by $x_1 \succeq x_2$. If $x_1 \succeq x_2$ and $x_2 \succeq x_1$ we say, that $x_1$ and $x_2$ have* **equal strength** *in $f$ and denote it by $x_1 \sim x_2$. If $x_1 \succeq x_2$ and not $x_2 \succeq x_1$ we say, that $x_1$ is* **stronger** *than $x_2$ in $f$ and denote it $x_1 \succ x_2$.*

Positive Boolean function $f$ on $n$ variables is called a **regular function**, if it holds that $x_1 \succeq x_2 \succeq \ldots \succeq x_n$ in $f$. Positive Boolean function $f$ on $n$ variables is called a **2-monotonic function**, if there exists a linear order of its variables which respects their strength, i.e., it is possible to rename variables in $f$ to get a regular function. The class of 2-monotonic functions will be denoted by $\mathcal{C}_{\text{2m}}$.

# 3 Relations of Threshold and $k$-Interval Functions

In this section we shall show that threshold functions do not constitute a subclass of $k$-interval functions for any $k$. We prove it using the following theorem:

**Theorem 3.1.** *The $\text{MAJORITY}_n$ function on $n$ variables is a threshold function and cannot be represented by $2^{\lfloor n/2 \rfloor} - 2$ or less intervals in any order of its variables.*

**Proof :** A threshold structure with all weights of variables set to 1 and the threshold set to $\lfloor n/2 \rfloor + 1$ proves that MAJORITY$_n$ is a (positive) threshold function.

Now we will prove the second part of the theorem. First of all, the MAJORITY$_n$ function is symmetrical with respect to its variables hence we only need to consider one (arbitrary) order of its variables. Let us take the natural order $x_1, x_2, \ldots, x_n$.

To determine the number of intervals needed to represent MAJORITY$_n$ we shall count the right boundaries of these intervals. One of them is, naturally, the vector of all ones. The other ones are identified by $n$-bit numbers $x$ binary representations of which have at least $\lfloor n/2 \rfloor + 1$ bits with value 1 and their successors, $x+1$, have binary binary representation with at most $\lfloor n/2 \rfloor$ bits with value 1. The set

$$X_n(l) = \{x \mid 0 < x < 2^n - 1 \ \& \ |ON(\boldsymbol{x})| = l \ \& \ |ON(\overrightarrow{x+1})| \leq \lfloor n/2 \rfloor\}$$

contains all such numbers having exactly $l$ bits with value 1 in their binary representation. Let us denote $|X_n(l)|$ by $x_n(l)$.

Now we shall count $x_n(l)$. Adding 1 to some $n$-bit number $x < 2^n - 1$ replaces the last (least significant) bit with value 0 in its binary representation by 1 and all subsequent (less significant) bits are switched from 1 to 0. Hence the numbers $x$ that we count in $x_n(l)$ must have binary representations such that the last $l - \lfloor n/2 \rfloor + 1$ bits have value 1 and the other bits can be an arbitrary combination of $\lfloor n/2 \rfloor - 1$ ones and $n - l$ zeros. This implies that for any such $x$ integer $x+1$ has at most $l - (l - \lfloor n/2 \rfloor + 1) + 1 = \lfloor n/2 \rfloor$ bits with value 1. Therefore

$$x_n(l) = \binom{n - (l - \lfloor n/2 \rfloor + 1)}{\lfloor n/2 \rfloor - 1} = \binom{n - l + \lfloor n/2 \rfloor - 1}{\lfloor n/2 \rfloor - 1}$$

To count the number of right boundaries of intervals representing MAJORITY$_n$ we need to sum $x_n(l)$ for $l$ ranging from $\lfloor n/2 \rfloor + 1$ (because there have to be enough bits with value 1) to $n-1$ (because we are not counting $2^n - 1$). Therefore $m_n$, the number of the right boundaries of inner intervals (i.e., not having their right boundary in $2^n - 1$), is

$$m_n = \sum_{l=\lfloor n/2 \rfloor + 1}^{n-1} x_n(l) = \sum_{l=\lfloor n/2 \rfloor + 1}^{n-1} \binom{n - l + \lfloor n/2 \rfloor - 1}{\lfloor n/2 \rfloor - 1}$$

This can be simplified using a substitution $i = n - 1 - l$ as follows (note, that we sum in the reverse order):

$$m_n = \sum_{i=0}^{\lceil n/2 \rceil - 2} \binom{\lfloor n/2 \rfloor + i}{\lfloor n/2 \rfloor - 1} = \sum_{i=0}^{\lceil n/2 \rceil - 2} \binom{\lfloor n/2 \rfloor + i}{i + 1}$$

For our purpose we shall use a very simple approximation as follows:

$$m_n \geq \sum_{i=0}^{\lfloor n/2 \rfloor - 2} \binom{\lfloor n/2 \rfloor}{i+1} = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{\lfloor n/2 \rfloor}{j} - \binom{\lfloor n/2 \rfloor}{0} - \binom{\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} = 2^{\lfloor n/2 \rfloor} - 2 \qquad (2)$$

The first inequality follows from the obvious fact, that $\binom{p+q}{r} \geq \binom{p}{r}$ for any non-negative integers, and the last equality is derived using the binomical theorem.

Due to (2), the MAJORITY$_n$ function cannot be represented by $2^{\lfloor n/2 \rfloor} - 2$ intervals or less. ■

**Corollary 3.2.**

*(a)* $\mathcal{C}_{\text{th}} \not\subseteq \mathcal{C}_{k-\text{int}}$ *for any* $k$.

*(b)* $\mathcal{C}^+_{\text{th}} \not\subseteq \mathcal{C}_{k-\text{int}}$ *for any* $k$.

**Proof :** The second proposition follows from the fact that majority is a positive function (in every variable). ■

# 4 Relation of Threshold and Positive 2-Interval Functions

In this section we shall gradually construct the proof that positive 2-interval functions constitute a subclass of threshold functions. Due to Corollary 3.2 this inclusion is proper. First we show that the classes of (positive) $k$-interval functions are closed under partial assignment.

**Lemma 4.1.** *Let* $f$ *be a Boolean function on* $n$ *variables and let* $k$ *be the minimum number of intervals that it can be represented by (in a suitable order of variables). Then for any* $i \in \{1, \ldots, n\}$ *and* $c \in \{0, 1\}$ *the function* $f' = f[x_i := c]$ *can be represented by* $k$ *intervals or less.*

**Proof :** Let $\pi$ be the order of variables for which $f$ can be represented by $k$ intervals. The high-level idea of this lemma is that when we fix the value of variable $x_i$ to $c$ then the branching tree corresponding to $f$ and order $\pi$ loses one level and the intervals of *true* and *false* values "shrink" and maybe merge because some of these values are left out. But in this process no new interval can be introduced.

Formally, we will proceed by contradiction. Suppose that $f'$ cannot be represented by $k$ or less intervals with respect to order $\pi'$ which is $\pi$ projected on $1, \ldots, i-1, i+1, \ldots, n$. Then there are $2k+1$ vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{k+1}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ of $(n-1)$ bits with the following properties:

1. $\forall j \in \{1, \ldots, k+1\} : f'(\boldsymbol{a}_j) = true$
2. $\forall j \in \{1, \ldots, k\} : f'(\boldsymbol{b}_j) = false$
3. $\forall j \in \{1, \ldots, k\} : a_j^{\pi'} < b_j^{\pi'} < a_{j+1}^{\pi'}$

These vectors prove that $f'$ cannot be represented by $k$ intervals with respect to order $\pi'$. But if we insert the value $c$ as the $i$-th bit in every vector $\boldsymbol{a}_j$ and $\boldsymbol{b}_l$ we will get $2k+1$ vectors of $n$-bits proving that $f$ cannot be represented by $k$ intervals with respect to the order $\pi$. This is the desired contradiction. ■

In the following part we show several rather technical lemmas which describe basic properties of positive $k$-interval functions. We formulate these lemmas for general $k$ although we will need them later only for the case $k = 2$. A corollary of these properties will show that in order to prove the desired inclusion between threshold and positive 2-interval functions we can consider only those positive 2-interval functions for which there is no index $i$ satisfying at least one of the following conditions:

$$\forall \boldsymbol{x}(f(\boldsymbol{x}) = 1 \quad \Rightarrow \quad x_i = 1) \tag{3}$$

$$\forall \boldsymbol{x}(f(\boldsymbol{x}) = 0 \quad \Rightarrow \quad x_i = 0) \tag{4}$$

i.e., all truepoints or all falsepoints have the same value of $i$-th bit. We note that for any positive function $f$ it is not possible for all truepoints to contain 0 in $i$-th bit, for any $i$, because when $\boldsymbol{x}$ is a truepoint of $f$ such that $x_i = 0$, then $\boldsymbol{x}'$ formed from $\boldsymbol{x}$ by switching bit $x_i$ to 1 is also a truepoint of $f$ because $f$ is positive. Similarly, it is not possible for all falsepoints of $f$ to contain 1 in $i$-th bit, for any $i$.

**Lemma 4.2.** *Let $f$ be a positive $k$-interval function on $n$ variables. Let $i$ be an index $1 \le i \le n$ such that at least one of the conditions (3), (4) is satisfied. Then there is order $\pi$ with respect to which $f$ can be represented by $l$ intervals $[c^1, d^1] < \ldots < [c^l, 2^n - 1]$, where $l \le k$, and such that $\pi(i) = 1$, i.e., $x_i$ is the most significant variable with respect to $\pi$.*

**Proof :** It is best to view the situation on a branching tree. Figure 2 captures the branching trees of a function in which index $i$ satisfies the condition (3) or (4). We can see that if we

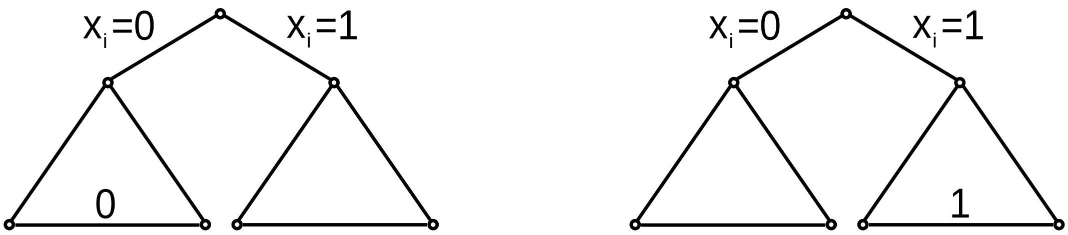


**Figure 2:** *Condition (3) on the left and condition (4) on the right in a branching tree.*

take $x_i$ as the most significant variable then one of the root subtrees contains either only truepoints or only falsepoints. Moreover, since the function considered is positive the other subtree of the root has the nearest vector also truepoint or falsepoint (same as all the vectors in the other subtree) or it is identically equal to zero or one. Hence the interval representation of this possibly non-constant subtree can be easily extended to an interval representation of the whole tree which has the same number of intervals.

Now we proceed with the formal proof. First, if both conditions (3) and (4) are satisfied by index $i$ then $f$ does not depend on values of other input variables but $x_i$ because $f(\boldsymbol{x}) = 1$

for any vector $\boldsymbol{x}$ such that $x_i = 1$ and $f(\boldsymbol{x}) = 0$ for any vector $\boldsymbol{x}$ such that $x_i = 0$. In this case $l = 1$ because $f$ can be represented by interval $[2^{n-1}, 2^n - 1]$ for any order having $x_i$ as the most significant variable.

In the rest of the proof we assume that $i$ satisfies exactly one of condiditions (3), (4). We shall define function $g$ on $n-1$ variables. We set $g = f[x_i := 1]$ if $i$ satisfies the condition (3) or $g = f[x_i := 0]$ otherwise. In either case $g$ is a $k$-interval function according to Lemma 4.1. Let $\pi'$ be an order of variables $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$ with respect to which $g$ can be represented by $l \leq k$ intervals. We construct the desired order $\pi$ from $\pi'$ in the following way. We set $\pi(i) = 1$ and for all $j \neq i$ we set $\pi(i) = \pi'(i) + 1$, i.e., we add $x_i$ as the most significant variable to $\pi'$. We claim that $f$ is represented by $l$ intervals with respect to $\pi'$. This is true because if the condition (3) is satisfied then all vectors $\boldsymbol{x}$ with $x_i = 0$ are falsepoints, every truepoint $\boldsymbol{u}$ has $u_i = 1$, hence $u^\pi > x^\pi$, and $f(\boldsymbol{y}) = 1$ holds for any vector $\boldsymbol{y}$ with $y_i = 1$ if and only if $g(\boldsymbol{y}') = 1$, where $\boldsymbol{y}'$ is formed from $\boldsymbol{y}$ by removing the $i$-th bit. If the condition (4) is satisfied then all vectors $\boldsymbol{x}$ with $x_i = 1$ are truepoints, every falsepoint $\boldsymbol{v}$ has $v_i = 0$, hence $v^\pi < x^\pi$, and $f(\boldsymbol{y}) = 0$ holds for any vector $\boldsymbol{y}$ with $y_i = 0$ if and only if $g(\boldsymbol{y}') = 0$, where $\boldsymbol{y}'$ is again formed from $\boldsymbol{y}$ by removing the $i$-th bit. Thus if $[c^1, d^1] < \ldots < [c^l, 2^{n-1} - 1]$ are the intervals representing $g$ with respect to $\pi'$ then we can form from them an interval representation of $f$ with respect to $\pi$ by adding 1 (in case the condition (3) is satified) or 0 (in case the condition (4) is satisfied) as the most significant bit to all interval boundaries except $2^{n-1} - 1$ which is always transformed into $2^n - 1$. This completes the proof. ■

**Lemma 4.3.** *Let $f$ be a positive $k$-interval function on $n$ variables and let $i$ be an index $1 \leq i \leq n$, which satisfies exactly one of the conditions (3), (4). Let us set $c = 1$, if $i$ satisfies the condition (3), or $c = 0$, if $i$ satisfies the condition (4). Then $f$ is a positive threshold function, if and only if $f' = f[x_i := c]$ is a positive threshold function.*

**Proof :** To explain the idea of the proof we shall again refer to Figure 2. In case of the condition (3) function $f$ has positive threshold structure $(\omega, t)$ if and only if function $f'$, which corresponds to the right subtree of the root, has positive threshold structure $(\omega, t - \omega(x_i))$. Similarly, in case of the condition (4) function $f$ has positive threshold structure $(\omega, t)$ if and only if $f'$, which corresponds to the left subtree of the root, has positive threshold structure $(\omega, t)$.

Now we show the formal proof. Let $f$ be a positive threshold function and let $(\omega, t)$ be a positive threshold structure for $f$. Let $\omega'$ be a restriction of $\omega$ on $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ and let $t' = t - (c \cdot \omega(x_i))$. Then $(\omega', t')$ is a positive threshold structure for $f'$, because for any vector $\boldsymbol{x}'$ of length $n - 1$ it holds, that $f'(\boldsymbol{x}') = 1$, if and only if $f(\boldsymbol{x}) = 1$, where the vector $\boldsymbol{x}$ is formed from $\boldsymbol{x}'$ by inserting $c$ as the $i$-th bit. Moreover, $f(\boldsymbol{x}) = 1$ holds, if and only if $\omega(\boldsymbol{x}) \geq t$, and this holds, if and only if $\omega'(\boldsymbol{x}') \geq t'$. Hence $f'$ is a positive threshold function.

Conversely, let $f'$ be a positive threshold function on $n - 1$ variables and $(\omega', t')$ be a positive threshold structure for $f'$. Let $\Omega$ be the sum of the weights of all variables in $f'$ (let $\Omega = 0$ if $f'$ has no variables). We set $\omega(x_i) = \Omega + 1$ and $\omega(x_j) = \omega'(x_j)$ for any $x_j \neq x_i$.

Also, we set $t = t' + (c \cdot \omega(x_i))$. Then $(\omega, t)$ is a positive threshold structure for $f$, because

- for any vector $\boldsymbol{x}$, such that $x_i > c$, we have $\omega(\boldsymbol{x}) > t$, and such $\boldsymbol{x}$ is a truepoint, because $c = 0$ and hence all falsepoints have 0 in the $i$-th bit,
- for any vector $\boldsymbol{x}$, such that $x_i < c$, we have $\omega(\boldsymbol{x}) < t$, and such $\boldsymbol{x}$ is a falsepoint, because $c = 1$ and hence all truepoints have 1 in the $i$-th bit, and
- for any vector $\boldsymbol{x}$, such that $x_i = c$, we have $\omega(\boldsymbol{x}) = (c \cdot \omega(x_i)) + \omega'(\boldsymbol{x}')$, where $\boldsymbol{x}'$ is formed from $\boldsymbol{x}$ by removing the $i$-th bit, and thus $\omega(\boldsymbol{x}) \geq t$, if and only if $\omega'(\boldsymbol{x}') \geq t'$.

Therefore $f$ is a positive threshold function. ■

In the proof of Lemma 4.3 it is also shown how to construct a threshold structure for a positive $k$-interval function $f$ for which there is an index satisfying one of the conditions (3), (4) provided that we have a threshold structure for $f'$.

**Corollary 4.4.** *Let $k \geq 2$ be given. Then $\mathcal{C}^+_{k-\mathrm{int}} \subseteq \mathcal{C}_{\mathrm{th}}$ if and only if every positive $k$-interval function, for which there is no index $i$ satisfying one of the conditions (3) and (4), is a threshold function.*

**Proof :** While there is some index satisfying one of the conditions (3) or (4) we can use Lemma 4.2 (and also its proof) to fix its value and we again get a positive $k$-interval function which, according to Lemma 4.3, is a positive threshold functions if and only if $f$ is a positive threshold function. ■

Using this corollary we shall show that positive 2-interval functions constitute a subclass of threshold functions. We shall proceed with some more properties of $k$-interval functions that we need to prove this inclusion.

**Lemma 4.5.** *Let $f$ be a positive function. Let $i$ be an index not satisfying either of the conditions (3),(4). Then $f(\boldsymbol{x}) = 0$ holds for the vector $\boldsymbol{x}$ with bit $x_i$ equal to 1 and with all other bits equal to 0, and $f(\boldsymbol{y}) = 1$ holds for the vector $\boldsymbol{y}$ with bit $y_i$ equal to 0 and with all other bits equal to 1.*

**Proof :** We proceed by contradiction. If $f(\boldsymbol{y})$ is 0, then all vectors with the $i$-th bit equal to 0 must be mapped to 0 by $f$ as well (because $f$ is a positive function). In that case all truepoints would have 1 as the $i$-th bit and hence $i$ would satisfy the condition (3), contradiction. Therefore $f(\boldsymbol{y}) = 1$.

In case $f(\boldsymbol{x})$ is 1, then all vectors with the $i$-th bit equal to 1 must be truepoints and hence all falsepoints have the $i$-th bit equal to 0, i.e., $i$ would satisfy the condition (4), contradiction. Therefore $f(\boldsymbol{x}) = 0$. ■

**Corollary 4.6.** *Let $f$ be a positive 2-interval function which is not a 1-interval function and let $\pi$ be an order with respect to which $f$ can be represented by intervals $[a^1, b^1] < [a^2, 2^n - 1]$ and such that $i$, for which $\pi(i) = 1$ (i.e., the index of the most significant variable with* *respect to $\pi$), does not satisfy either of the conditions (3), (4). Then $b^1 = y^\pi$, where $\boldsymbol{y}$ is the vector with $y_i = 0$ and all other bits 1.*

This corollary shows that if there is no index satisfying either of the conditions (3), (4) then each of the two intervals representing a positive function spans the numbers corresponding to truepoints with the same value of the most significant bit. It should be also clear that by fixing the value (either to 1 or 0) of the variable corresponding to the most significant bit we get a (positive) 1-interval function.

The following lemma establishes a relation between the conditions (3), (4) and properties of prime DNF representations of positive functions.

**Lemma 4.7.** *Let $\mathcal{F}$ be a prime DNF representing positive function $f$. Then*

*(a) every term of $\mathcal{F}$ contains variable $x_i$, if and only if $i$ satisfies the condition (3),*

*(b) variable $x_i$ forms a linear term in $\mathcal{F}$, if and only if $i$ satisfies the condition (4).*

**Proof :** We shall start with the first implications of the equivalences. In (a), if every term of $\mathcal{F}$ contains $x_i$, then any vector $\boldsymbol{x}$ having $x_i = 0$ is a falsepoint, hence every truepoint must have the $i$-th bit equal to 1. Similarly, if in (b) variable $x_i$ forms a linear term in $\mathcal{F}$, then any vector $\boldsymbol{x}$ having $x_i = 1$ is a truepoint, hence every falsepoint must have the $i$-th bit equal to 0.

Now we shall prove the second implications. If $i$ satisfies the condition (3), then by fixing $x_i$ to 0 in $\mathcal{F}$ we must get an empty DNF, because it represents the function identically equal to 0. Hence $x_i$ must occur in every term of $\mathcal{F}$. Similarly, if $i$ satisfies the condition (4), then by fixing $x_i$ to 1 in $\mathcal{F}$, we must get a DNF representing the function, which is identically equal to 1. Hence $x_i$ must form a linear term in $\mathcal{F}$. ■

**Corollary 4.8.** *Let $f$ be a positive 1-interval function on $n$ variables which is not identically equal to zero. Let $\pi$ be an order with respect to which $f$ can be represented by one interval and let $x_i$ be the most significant variable, i.e., $\pi(i) = 1$, and let $\mathcal{F}$ be the prime DNF representing $f$. Then $\mathcal{F}$ contains the linear term $x_i$ or the variable $x_i$ is contained in every term of $\mathcal{F}$.*

**Proof :** Follows from Lemma 4.7 because $i$ must satisfy at least one of the conditions (3), (4) otherwise $f$ is not a 1-interval function due to Corollary 4.6. ■

**Lemma 4.9.** *Let $\mathcal{F}$ be the prime DNF representing positive 2-interval function $f$ which is not a 1-interval function and such that there is no index $i$ satisfying either of the conditions (3), (4). Let $\pi$ be an order of variables with respect to which $f$ can be represented by 2 intervals. Then $\mathcal{F}$ has the form*

$$\mathcal{F} = x_i x_j \vee x_i \mathcal{G} \vee x_j \mathcal{H},$$

*where $x_i, x_j$ are the first and the second most significant variables in the order $\pi$, and $\mathcal{G}$ and $\mathcal{H}$ are positive DNFs not containing variables $x_i$ and $x_j$. Moreover, $\mathcal{G}$ and $\mathcal{H}$ represent positive functions, which can be represented by one interval with respect to the same order of variables.*

**Proof :** Let $\pi$ be an order, with respect to which $f$ can be represented by two intervals $[a^1, b^1] < [a^2, 2^n - 1]$. According to Corollary 4.6, $b = 2^{n-1} - 1$. Therefore, both $f_0 = f[x_i := 0]$ and $f_1 = f[x_i := 1]$ are positive functions, which can be represented by one interval, moreover, with respect to the same order $\pi'$, formed from $\pi$ by restriction on variables $\{x_1, \ldots, x_{i-1}, x_{i+1}, x_n\}$.

Then $x_j$ is the most significant variable with respect to $\pi'$. According to Corollary 4.8, the prime DNF representations $\mathcal{F}_0$ and $\mathcal{F}_1$ of $f_0$ and $f_1$ contain a linear term formed by $x_j$, or contain $x_j$ in every term. But $x_j$ cannot form a linear term in $\mathcal{F}_0$, because otherwise this term is also present in $\mathcal{F}$ and that is not possible according to Lemma 4.7 and the assumptions of the lemma, that we are proving. Therefore $x_j$ has to be contained in every term of $\mathcal{F}_0$. If $x_j$ was also contained in every term of $\mathcal{F}_1$, then $x_j$ would be present in every term of $\mathcal{F}$, which is again not possible due to our assumptions. Therefore, $x_j$ must form a linear term in $\mathcal{F}_1$. However, since $\mathcal{F}$ does not have linear terms, it is only possible for this term to occur in $\mathcal{F}_1$, if there is the term $x_i x_j$ in $\mathcal{F}$.

Putting this together with the fact that $\mathcal{F}$ is prime we get that

1. $x_i x_j$ is a term in $\mathcal{F}$ and this implies that no other term contains both varibles $x_i$ and $x_j$
2. in every term $t$ of $\mathcal{F}$ not containing $x_j$ there is variable $x_i$ (because none of these terms is in $\mathcal{F}_0$)

These conditions imply the desired form of $\mathcal{F}$. ∎

**Lemma 4.10.** *Let $f$ be a positive 2-interval function on $n$ variables which is not a 1-interval function. Moreover, let us suppose that there is no index satisfying either of the conditions (3), (4). Let $\pi$ be an order of its variables with respect to which $f$ can be represented by intervals $[a, 2^n - 1], [c, 2^n - 1]$. Then*

*(a) the most significant bit of $a$ is 0 and the second most significant bit of $a$ is 1,*

*(b) the most significant bit of $c$ is 1 and the second most significant bit of $c$ is 0.*

**Proof :** Let us start with (a). Using Lemma 4.9 we get, that any vector, which has the first two most significant bits 0, is a falsepoint of $f$. This, together with Corollary 4.6, proves the first part of the lemma.

We proceed with (b). First, according to Corollary 4.6, the most significant bit of $c$ is 1. Using Lemma 4.9, we know that $\mathcal{F} = x_i x_j \vee x_i \mathcal{G} \vee x_j \mathcal{H}$, where $\mathcal{F}$ is the prime DNF representing $f$, $x_i$ is the most significant variable and $x_j$ is the second most significant variable with respect to $\pi$. It follows, that any vector, which has the most significant bit 1 and the second most significant bit 0, is a truepoint of $f$, if and only if $\mathcal{G}$ is satisfied by the values of the remaining bits. The second most significant bit of $c$ is 1, if and only if all vectors with the first two most significant bits $1, 0$ are falsepoints and this holds, if and only if $\mathcal{G}$ represents the function, which is identically equal to zero. However, this is not possible, because it means, that the prime representation of $f$ contains the second most significant variable in every term, which contradicts the assumptions of this lemma according to Lemma 4.7. ■

Lemma 4.10 together with Corollary 4.6 imply that the branching tree of a positive 2-interval function in which no index of a variable satisfies either of the conditions (3), (4) has the form visualized on Figure 3. Now we are ready to prove the main result of this section.

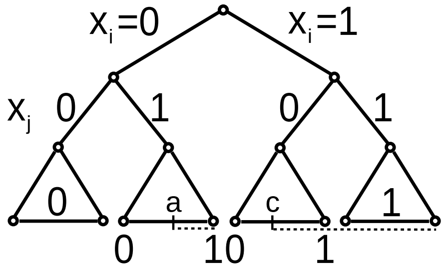


**Figure 3:** *A branching tree of a positive 2-interval function, which is represented by intervals* $[a, 2^{n-1} - 1], [c, 2^n - 1]$ *and in which no index of a variable satisfies conditions (3) and (4). The pair of variables* $x_i, x_j$ *satisfies Lemma 4.9.*

**Theorem 4.11.** *Let* $f$ *be a positive 2-interval function which is not a 1-interval function and for which there is no index satisfying either of the conditions (3), (4). Then* $f$ *is a threshold function.*

**Proof :** Again, we first explain the idea of the proof. We refer to Figure 3. We will assign the weights to variables $x_i$ and $x_j$ in such way that the vectors corresponding to the interval boundaries $a$ and $c$ will exactly reach the threshold. This is possible due to Lemma 4.10.

Now let us prove the lemma formally. Let $f$ be represented by $[a, 2^{n-1} - 1] < [c, 2^n - 1]$ with respect to some order $\pi$. First we note, that according to Lemma 4.10, the two most significant bits of $a$ are $0, 1$ and the two most significant bits of $c$ are $1, 0$. Let $\boldsymbol{b}$ be the vector of $n$ bits, such that $b^\pi = a$, i.e., $\boldsymbol{b}$ is the vector corresponding to the boundary $a$ (with respect to $\pi$). Hence $f(\boldsymbol{b}) = 1$ and

$$f(\overrightarrow{(b^\pi - 1)^{\pi^{-1}}}) = 0 \ .$$

Similarly, let $\boldsymbol{d}$ be the vector, such that $d^\pi = c$.

We shall construct a positive threshold structure $(\omega, t)$ for $f$. We set weight $\omega(x_i)$ of variable $x_i$ to $2^{n-\pi(i)}$ with the only exception of variable $x_j$, such that $\pi(j) = 2$, which will have weight $\omega(x_j) = \omega(\boldsymbol{d}) - \omega(\boldsymbol{b}')$, where $\boldsymbol{b}'$ is the vector of length $n - 2$ formed from $\boldsymbol{b}$ by removing the two most significant bits with respect to $\pi$. This definition of $\omega(x_j)$ is correct, because $c$ (and consequently also $\boldsymbol{d}$) has 0 in the bit corresponding to variable $x_j$.

Threshold $t$ will be set to $\omega(\boldsymbol{d}) = \omega(\boldsymbol{b}') + \omega(x_j) = \omega(\boldsymbol{b})$. The result is, that the weights are assigned in such way, that $\omega(\boldsymbol{b}) = \omega(\boldsymbol{d})$ and the threshold is set to this value as well. Hence, for any vector $\boldsymbol{x}$, such that $x^\pi < a$, we have $f(\boldsymbol{x}) = 0$ and $\omega(\boldsymbol{x}) < \omega(\boldsymbol{b}) = t$, because $\omega(x_j) > \omega(x_{\pi^{-1}(1)}) - \sum_{i=0}^{n-3} 2^i = 2^{n-1} - 2^{n-2} - 1 = 2^{n-2} - 1$ and hence no vector having two zeros in the two most significant bits can reach the threshold. Also, for any vector $\boldsymbol{y}$, such that $\boldsymbol{y}^\pi \geq 100\dots0$ and $y^\pi < c$, we have $f(\boldsymbol{y}) = 0$ and $\omega(\boldsymbol{y}) < \omega(\boldsymbol{d}) = t$. This completes our proof. ■

**Corollary 4.12.** $\mathcal{C}^+_{2-\mathrm{int}} \subset \mathcal{C}_{\mathrm{th}}$.

**Proof :** Follows from Corollary 4.4 and Theorem 4.11. ■

# 5 Relation of Positive Threshold and Positive 3-Interval Functions

In previous sections we proved that positive 2-interval functions constitute a proper subclass of (positive) threshold functions. Also, some of the lemmas leading to this result were formulated for general $k$ hence it would be quite natural to expect that also positive 3-interval functions constitute a subclass of threshold functions. However, in this section we prove the opposite result.

**Theorem 5.1.** $\mathcal{C}^+_{3-\mathrm{int}} \not\subseteq \mathcal{C}_{2\mathrm{m}}$.

**Proof :** We shall show an example of a positive 3-interval function and prove that it is not a 2-monotonic function. Let $f$ be the function on 4 variables represented by prime positive DNF

$$x_2x_4 \vee x_1x_3 \vee x_1x_2.$$

Function $f$ can be represented by three intervals $[5,5], [7,7], [10,15]$ with respect to the natural order $x_1, x_2, x_3, x_4$. Hence it is a positive 3-interval function. However, $f$ is not a 2-monotonic function because the truepoint 1010 and the falsepoint 1001 require $x_3 \succ x_4$ and the truepoint 0101 and the falsepoint 0110 require $x_3 \prec x_4$. Hence $x_3$ and $x_4$ are not of comparable strength in $f$. ■

**Corollary 5.2.** $\mathcal{C}^+_{3-\mathrm{int}} \not\subseteq \mathcal{C}_{\mathrm{th}}$.

**Proof :** Follows from Theorem 5.1 because every positive threshold function is a 2-monotonic function as well (see [10]). ■

**Corollary 5.3.** $\mathcal{C}^+_{k-\mathrm{int}} \not\subseteq \mathcal{C}_{\mathrm{th}}$ *for any* $k \geq 4$.

**Proof :** Follows from Corollary 5.2 and the fact that $\mathcal{C}^+_{3-\mathrm{int}} \subset \mathcal{C}^+_{k-\mathrm{int}}$ for any $k \geq 4$. ■

# 6 Conclusions

In this paper we studied the relations of the classes of $k$-interval Boolean functions and threshold functions. First, we have shown that unless we take $k \geq 2^{n-1}$ the class of threshold functions on $n$ variables does not constitute a subclass of $k$-interval functions. To prove this result, we have shown that for a given value of $k$ there is a positive threshold function on $O(\log k)$ variables which cannot be represented by $k$ intervals or less. This bound is probably not tight as we used very relaxed approximations.

The second main result shows that the class of positive 2-interval functions is a proper subclass of threshold functions. We closed the paper by showing that for any $k \geq 3$ that class of positive $k$-interval functions does not constitute a subclass of threshold functions nor of 2-monotonic functions. This concludes a complete description of the inclusion relations among the classes of 2-monotonic, threshold and positive $k$-interval functions.

Regarding general (i.e., non-monotone) $k$-interval functions it is easy to observe that even the class of 1-interval functions cannot be a subclass of threshold functions. The reason is that threshold functions are monotone in each variable while for every $k \geq 1$ there is a $k$-interval function with at least one variable in which the function is not monotone. For example if $k$ is 1 we can take a function $f$ which is represented by interval $[a, b]$ where the least significant bit of $a$ is 1 and the least significant bit of $b$ is 0. Then $f$ is not negative in variable $x_i$ corresponding to this least significant bit because the truepoint corresponding to $a$ (in the order with respect to which $[a, b]$ represents $f$) can be changed to a falsepoint only by switching $x_i$ from 1 to 0. Similarly, $f$ is not positive in $x_i$ either because by switching its value from 0 to 1 in the vector corresponding to $b$ we change a truepoint to a falsepoint. Hence $f$ is an example of a 1-interval function which is not threshold. Thus the description of inclusion relations between the classes of $k$-interval and threshold functions is complete.

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