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Friendship two-graphs

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Abstract. A friendship graph is a graph in which every two distinct vertices have exactly one common neighbor. All finite friendship graphs are known, each of them consists of triangles having a common vertex. We extend friendship graphs to two-graphs, a two-graph being an ordered pair $G = (G_0, G_1)$ of edge-disjoint graphs G_0 and G_1 on the same vertex-set $V(G_0) = V(G_1)$. One may think that the edges of G are colored with colors 0 and 1. In a *friendship two-graph*, every unordered pair of distinct vertices u, v is connected by a unique bicolored 2-path. Friendship two-graphs are solutions to the matrix equation $AB + BA = J - I$, where A and B are $n \times n$ symmetric $0 - 1$ matrices of the same dimension, J is an $n \times n$ matrix with every entry being 1, and I is the identity $n \times n$ matrix. We show that there are no finite friendship two-graph with minimum vertex degree at most two. However, we construct an infinite such graph, and the construction can be extended to an infinite family. Also, we find a finite friendship two-graph, and conjecture that it is unique.

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1 Introduction

A friendship graph is a graph in which every two distinct vertices have exactly one common neighbor. Friendship graphs were characterized by Erdős, Rényi, and Sós [3]: a friendship graph consists of triangles incident to a common vertex. Kotzig generalized friendship graphs to graphs in which every pair of vertices is connected by λ paths of length k. His conjecture is that, for $k \geq 3$, there is no finite graph in which every pair of vertices is connected by a unique path, see also Bondy [1] and Kostochka [4].

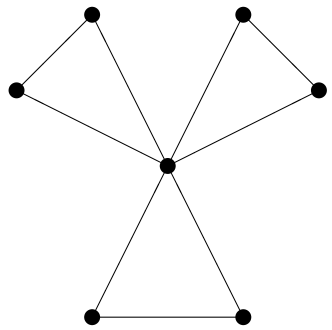

Figure 1: A friendship graph.

Here we consider another generalization. A *two-graph*, is an ordered pair (G_0, G_1) of edge-disjoint graphs G_0 and G_1 on the same vertex-set $V(G_0) = V(G_1)$. In other words, a two-graph is a graph with a partition of its edges into two color classes. The two colors will be denoted by 0 and 1. We say that vertices u and v are *i-adjacent* or they are i-neighbors of each other if the edge uv has color $i \in \{0, 1\}$. A 2-path (u, x, v) in (G, H) is called *bicolored* if either $ux \in E(G)$ and $xv \in E(H)$, or $ux \in E(H)$ and $xv \in E(G)$.

Definition 1 *A two-graph (G, H) is called a* friendship two-graph *if, for every unordered pair of distinct vertices u, v, there exists a unique bicolored 2-path connecting u and v.*

Friendship two-graphs are solutions to the matrix equation $AB + BA = J - I$, where A and B are $n \times n$ symmetric $0 - 1$ matrices of the same dimension, J is an $n \times n$ matrix with every entry being 1, and I is the identity $n \times n$ matrix. A related matrix equation was considered by Chvátal, Graham, Perold, and Whitesides [2].

2 Small minimum degree

A trivial friendship two-graph has just one vertex. The only non-trivial friendship two-graph, called F, that we know is shown in Figure 2. There are exactly 21 bicolored 2-paths in F, namely:

$(1, 7, 2)$, $(1, 2, 3)$, $(1, 7, 4)$, $(1, 6, 5)$, $(1, 7, 6)$, $(1, 6, 7)$,
$(2, 7, 3)$, $(2, 3, 4)$, $(2, 7, 5)$, $(2, 1, 6)$, $(2, 1, 7)$,
$(3, 7, 4)$, $(3, 4, 5)$, $(3, 7, 6)$, $(3, 2, 7)$,
$(4, 7, 5)$, $(4, 5, 6)$, $(4, 3, 7)$,
$(5, 7, 6)$, $(5, 4, 7)$,
$(6, 5, 7)$.

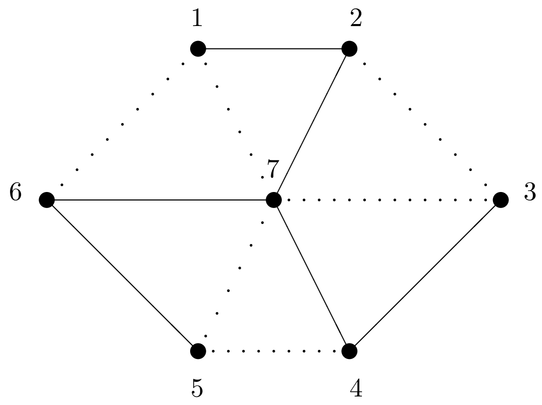

Figure 2: The friendship two-graph F.

We conjecture that F is a unique non-trivial friendship two-graph.

Theorem 1 *Every non-trivial friendship two-graph has minimum degree at least three.*

Let $G = (G_0, G_1)$ be a non-trivial friendship two-graph having a vertex v of degree at most two. Clearly, v cannot be an isolated vertex, so we may assume that v is 1-adjacent to a vertex w. Consider the unique bicolored 2-path (v, u, w) connecting v and w. If uv is a 0-edge then G does not have a bicolored 2-path connecting u and v, since v has degree at most two (in fact, exactly two). Thus, uv is a 1-edge and uw is a 0-edge, see Figure 3.

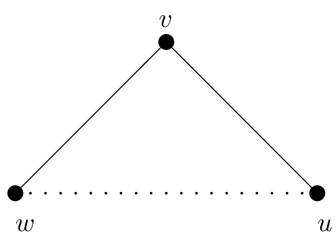

Figure 3: The subgraph induced by the set $\{u, v, w\}$.

A $(2+2)$-*cycle* is a 4-cycle that contains exactly two 0-edges and exactly two 1-edges.

Property 1 *A friendship two-graph does not have* $(2+2)$-*cycles.*

Consider a $(2+2)$-cycle (a, b, c, d), see Figure 4. If ab and cd are 0-edges then there are two bicolored 2-paths connecting a and c, a contradiction. If ab and bc are 0-edges then there are two bicolored 2-paths connecting b and d, a contradiction.

Figure 4: Two $(2 + 2)$-cycles.

Now consider a bicolored 2-path (u, x, w) connecting u and v. By symmetry, we may assume that wx is a 1-edge, and ux a 0-edge. Clearly, x is non-adjacent to v.

Property 2 *For every vertex $z \neq v$, exactly one of uz or wz is a 0-edge.*

Indeed, either (v, u, z) or (v, w, z) is a bicolored 2-path, but not both.

Property 3 (i) *The only 1-edge incident to u is uv.*

(ii) *The only 1-edges incident to w are vw and wx.*

(i) Suppose that uz is a 1-edge with $z \neq v$. By Property 2 w and z are 0-adjacent. We obtain a $(2+2)$-cycle (u, z, w, x), a contradiction to Property 1.

(ii) Now let wz be a 1-edge with $z \neq v, x$. By Property 2 u and z are 0-adjacent, and (w, z, u, x) is a $(2 + 2)$-cycle, a contradiction to Property 1.

There must be a bicolored 2-path (w, y, x) connecting w and x. By Property 3, xy is a 1-edge, and therefore wy is a 0-edge. Property 2 and Property 3 show that y is non-adjacent to u.

Property 4 *The only 1-edges incident to x are wx and xy.*

Suppose that xz is a 1-edge with $z \neq w, y$. If uz is a 0-edge then (u, w, x, z) is a $(2+2)$-cycle, a contradiction to Property 1. By Property 2 w and z are 0-adjacent. But then (w, y, x, z) is a $(2+2)$-cycle, a contradiction to Property 1.

The current subgraph H induced by the set $\{u, v, w, x, y\}$ is shown in Figure 5. It can be viewed as a particular snake two-graph $S(5)$ defined below.

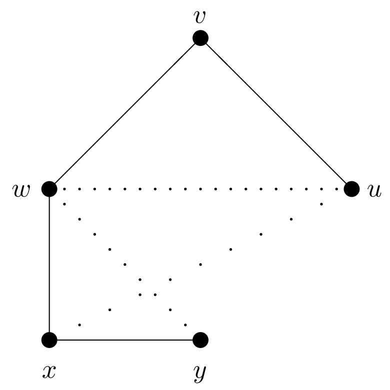

Figure 5: The subgraph H induced by the set $\{u, v, w, x, y\}$.

For an integer $n \geq 1$, the *snake two-graph* of order n, $S(n)$, is defined by the following:

- $V(S(n)) = \{s_1, s_2, \ldots, s_n\}$, and we also use alternative names of the vertices: $s_{4k-3} = p_k$, $s_{4k-2} = p'_k$, $s_{4k-1} = q_k$, $s_{4k} = q'_k$ for $k \geq 1$,
- the set of 1-edges is $\{s_1s_2, s_2s_3, \ldots, s_{n-1}s_n\}$, and
- the set of 0-edges is generated by the following two rules:
  - every vertex $p_i \in V(S(n))$ is 0-adjacent to all q_j and q'_j with $j \geq i$,
  - every vertex $q_i \in V(S(n))$ is 0-adjacent to all p_j and p'_j with $j \geq i$.

Figure 6 shows an example of a snake two-graph.

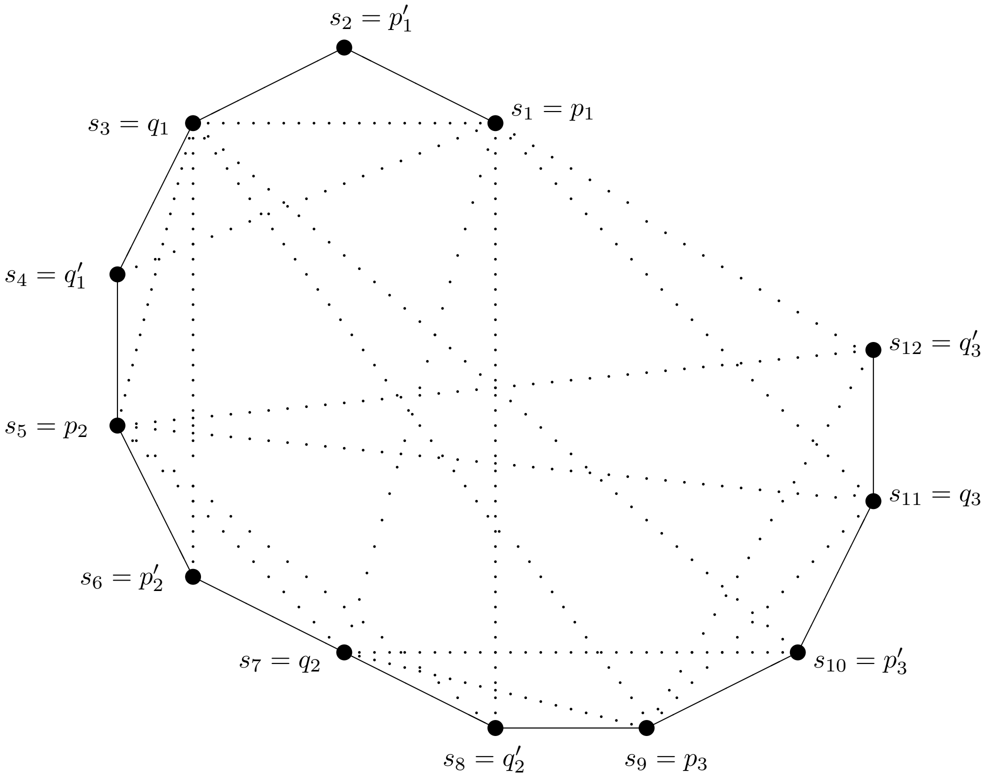

Figure 6: The snake two-graph $S(12)$.

Now we extend the induced subgraph H to an inclusion-wise maximal induced subgraph $S = S(n)$ of G [with $V(S(n)) = \{s_1 = u, s_2 = v, s_3 = w, s_4 = x, s_5 = y, \dots, s_n\}$] satisfying the following condition.

Condition 1 (i) *The only* 1*-edge of* G *incident to* s_1 *is* s_1s_2.

(ii) *The only* 1*-edges of* G *incident to* s_i, $2 \le i \le n-1$, *are* $s_{i-1}s_i$ *and* s_is_{i+1}.

Note that the subgraph H satisfies Condition 1 according to Property 3 and Property 4. The vertex s_n may be incident to a 1-edge distinct from $s_{n-1}s_n$.

One can directly check that there exist a unique bicolored 2-path P connecting distinct vertices $s_i \ne s_n$ and $s_j \ne s_n$.

1) If $s_i = p_k$, $s_j = p_l$ and $i < j$, then $P = (p_k, q'_{l-1}, p_l)$.
2) If $s_i = p_k$, $s_j = p'_l$ and $i \le j$, then $P = (p_k, q_l, p'_l)$.
3) If $s_i = p_k$, $s_j = q_l$ and $i \le j$, then $P = (p_k, q'_l, q_l)$.
4) If $s_i = p_k$, $s_j = q'_l$ and $i \le j$, then $P = (p_k, q_l, q'_l)$.
5) If $s_i = p'_k$, $s_j = p'_l$ and $i < j$, then $P = (p'_k, q_k, p'_l)$.
6) If $s_i = p'_k$, $s_j = q_l$ and $i \le j$, then $P = (p'_k, p_k, q_l)$.

7) If $s_i = p'_k$, $s_j = q'_l$ and $i \leq j$, then $P = (p'_k, p_k, q'_l)$.
8) If $s_i = q_k$, $s_j = q_l$ and $i < j$, then $P = (q_k, p_k, q_l)$.
9) If $s_i = q_k$, $s_j = q'_l$ and $i \leq j$, then $P = (q_k, p_k, q'_l)$.
10) If $s_i = q'_k$, $s_j = q'_l$ and $i < j$, then $P = (q'_k, p_k, q'_l)$.

Case 1. $s_n \in \{p_k, q'_k\}$.

In this case (q_i, s_n), $i = 1, 2, \dots, k$ are the only pairs of S that are not connected by a bicolored 2-path. In particular, there exists a vertex $s_{n+1} \notin V(S)$ such that (q_1, s_{n+1}, s_n) is a bicolored 2-path. Condition 1 shows that $q_1 s_{n+1}$ is a 0-edge and therefore $s_{n+1} s_n$ is a 1-edge.

Property 5 *There is no vertex $z \notin \{s_{n-1}, s_{n+1}\}$ which is 1-adjacent to the vertex s_n.*

Clearly, $z \notin V(S)$. By Property 2 exactly one of $p_1 z$ or $q_1 z$ is a 0-edge. Then either (p_1, s_{n-1}, s_n, z) or (q_1, s_{n+1}, s_n, z) a $(2+2)$-cycle, a contradiction to Property 1.

Condition 1 shows that s_n is the only vertex of S which is 1-adjacent to s_{n+1}. We claim that s_{n+1} is 0-adjacent to all $q_i \in V(S)$. Indeed, otherwise s_{n+1} is non-adjacent to some q_i, and there must be a bicolored 2-path (q_i, z, s_n) with $z \neq s_{n+1}$. It is impossible by Condition 1 and Property 5.

Finally, we note that s_{n+1} is non-adjacent to all vertices p_i and q'_i in $V(S) \setminus \{z_n\}$. Indeed, if s_{n+1} is adjacent to some p_i, then $s_{n+1}p_i$ a 0-edge. We obtain a second bicolored 2-path (p_i, s_{n+1}, s_n) connecting p_i and s_n, a contradiction. A similar contradiction arises with a 0-edge $s_{n+1}q'_i$.

Thus, the set $\{s_1, s_2, \dots, s_{n+1}\}$ induces the snake two-graph $S(n+1)$, contradiction to maximality of n.

Case 2. $s_n \in \{p'_k, q_k\}$.

The only pairs of S that are not connected by a bicolored 2-path are (p_i, s_n), $i = 1, 2, \dots, k$. As in Case 1, one can extend S to the snake two-graph $S(n+1)$, obtaining a contradiction to maximality of n.

3 Balls of snakes

If we continue the construction in the proof of Theorem 1, we obtain an infinite two-graph $S(\infty)$ on vertex-set $\{s_1, s_2, \dots, s_n, \dots\}$. It is easy to see that $S(\infty)$ is a friendship two-graph with minimum vertex degree $\delta = 2$. We distinguish two-graphs up to renaming of the two colors, that is (G_0, G_1) and (G_1, G_0) are considered as the same two-graph. We are going to show that $S(\infty)$ is not unique infinite friendship two-graph with minimum vertex degree $\delta \leq 2$.

Consider and arbitrary infinite friendship two-graph G with minimum vertex degree $\delta \leq 2$. The proof of Theorem 1 shows that G must contain $H = S(\infty)$ as an induced subgraph. As before, we denote $V(H) = \{s_1, s_2, \dots, s_n, \dots\}$, see Figure 6.

First note that there are no 1-edges connecting a vertex of H with a vertex of X, see Condition 1. Therefore X induces a friendship two-graph H' (finite or infinite). Using Property 2, we subdivide X into disjoint subsets A and B such that every vertex of A (respectively, B) is 0-adjacent to the vertex s_1 (respectively, s_3) of H.

The set of all 1-edges within A constitutes a perfect matching M_A to guarantee the existence of a bicolored 2-path connecting s_1 and an arbitrary vertex of A and to avoid $(2+2)$-cycles (s_1, a_1, a_2, a_3), where $a_1, a_2, a_3 \in A$. The set of all 1-edges between A and B is a disjoint union of stars $S(1), S(2), \dots, S(k)$ centered at some vertices of A and such that every vertex of B is a pendant vertex of a unique star $S(i)$. The stars provide bicolored 2-paths from s_1 to an arbitrary vertex of B. In fact, every star $S(i)$ is just a 1-edge $a_i b_i$, $a_i \in A$ and $b_i \in B$, otherwise there is a $(2+2)$-cycle of the form (s_3, b, a, b'), where $b, b' \in B$ are pendant vertices of a star centered at $a \in A$. Thus, we have a matching $M_{AB} = \{a_1b_1, a_2b_2, \dots, a_kb_k\}$ of 1-edges which covers B, that is $B = \{b_1, b_2, \dots, b_k\}$.

The set of all 1-edges within B constitutes a matching M_B, not necessarily perfect and possibly empty. Indeed, 1-edges b_1b_2 and b_2b_3, $b_i \in B$, produce a $(2+2)$-cycle (s_3, b_1, b_2, b_3), which is impossible. Let

$H' = (H'_0, H'_1)$. The matchings M_A, M_{AB} and M_B constitute edge-set of H'_1, and H'_1 is disjoint union of paths (finite or infinite) and/or even cycles. Every component K of H'_1 by itself induces a friendship two-graph.

Claim 1 *If K is a cycle C_n, then $n = 4k$ and K does not induce a friendship two-graph.*

The fact $n = 4k$ is easy. We show that it is impossible to add 0-edges to $K = C_{4k}$ to obtain a friendship two-graph. Suppose it is possible. For $t \geq 2$, define a *t-chord* as a 0-edge connecting two vertices at distance t along the cycle C_{4k}. Let $D(l)$ be the set of all unordered pairs of vertices at distance l along the cycle C_{4k}. Clearly, $|D(1)| = |D(2)| = \cdots = |D(2k-1)| = 4k$, and $|D(2k)| = 2k$, and $|D(l)| = 0$ for all $l \geq 2k+1$. Every t-chord produces bicolored 2-paths connecting two pairs in $D(t-1)$ and bicolored 2-paths connecting two pairs in $D(t+1)$. To create $4k$ bicolored 2-paths for pairs in $D(2)$ we must add $2k$ 3-chord. These 2-paths automatically satisfy all pairs in $D(4)$. Then we must add $2k$ 7-chord to create $4k$ bicolored 2-paths for pairs in $D(6)$. These 2-paths automatically satisfy all pairs in $D(8)$, and so on. Finally, we obtain a contradiction to the fact $|D(2k)| = 2k$, $2k$ $(2k-1)$-chord will create $4k$ bicolored 2-paths for pairs in $D(2k)$.

Thus, K must be a path. We show that it is possible. For that we define an infinite *bi-snake*, denoted by $B(\infty)$, on vertex set

$$\{\ldots, a_{-1}, a'_{-1}, b_{-1}, b'_{-1}, a_0, a'_0, b_0, b'_0, a_1, a'_1, b_1, b'_1, \ldots\}.$$

The set of 1-edges form the path $(\ldots, a_{-1}, a'_{-1}, b_{-1}, b'_{-1}, a_0, a'_0, b_0, b'_0, a_1, a'_1, b_1, b'_1, \ldots)$. Every vertex a_i is 0-adjacent to all b_j and b'_j with $j \geq i$. Every vertex b_i is 0-adjacent to all a_j and a'_j with $j \geq i$.

Claim 2 *$B(\infty)$ is an infinite friendship two-graph.*

Straightforward.

The *A-set* (respectively, *B-set*) of $B(\infty)$ consists of all vertices a_j and a'_j (respectively, a_j and a'_j).

Theorem 2 *There are infinitely many infinite friendship two-graphs with minimum vertex degree $\delta = 2$, and all of them contain $S(\infty)$ as an induced subgraph.*

For an integer $n \geq 0$, we define a *ball of snakes* as an infinite friendship two-graph G_n consisting of one copy H of $S(\infty)$, n pairwise vertex-disjoint copies H_n of $B(\infty)$ and an additional set S of 0-edges. Every vertex p_i (respectively, q_i) of H is 0-adjacent to all vertices in the A-set (respectively, B-set) of H_n. For H_m and H_n with $m < n$, the set S has following 0-edges connecting H_m and H_n: every vertex a_i (respectively, b_i) of H_m is 0-adjacent to all vertices in the A-set (respectively, B-set) of H_n.

It is easy to see that G_n is a friendship two-graph for every $n \geq 0$.

4 Augmenting infinite paths

We use the proof of Claim 1 to solve the following problem: Given in infinite path

$$P = (\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots)$$

consisting of 1-edges $u_i u_{i+1}$, add 0-edges to P to obtain a friendship two-graph. We show that there are uncountably many solutions. Using the terminology in the proof of Claim 1, we first introduce a set of 2-chords to create bicolored 2-paths between vertices at distance 1 along P. Consider u_0 and u_1. For them, there are two variants: either

(V1) u_0 is 0-adjacent to u_2, or

(V2) u_{-1} is 0-adjacent to u_1.

These variants are inconsistent, since we have a $(2+2)$-cycle (u_0, u_2, u_1, u_{-1}). Let us consider (V1). It creates a bicolored 2-paths between the vertices u_1 and u_2, and therefore the 2-chord u_1u_3 should be rejected. To have a bicolored 2-paths between the vertices u_2 and u_3, we must introduce the 2-chord u_2u_4. In turn, u_3u_5 is forbidden. Now it is clear that we must choose exactly one of the two sets of 2-chords, namely

$$S_2 = \{u_{2i}u_{2i+2} : i \in Z\}$$

and

$$S'_2 = \{u_{2i+1}u_{2i+3} : i \in Z\}.$$

Each of the two sets produces bicolored 2-paths between all pairs of vertices at distance 1 and 3. It implies that there are no 4-chord at all.

A similar situation takes place for pairs of vertices at distance 2. For u_0 and u_2, we should introduce a 3-chord, and there are two inconsistent variants: u_0u_3 and $u_{-1}u_2$. The variant u_0u_3 creates also a bicolored 2-path connecting u_1 and u_3. Hence the 3-chord u_1u_4 is forbidden. It implies the existence of the 3-chord u_2u_5 to satisfy the pair u_2, u_4. As before, we must choose exactly one of the two sets of 3-chords, namely

$$S_3 = \{u_{2i}u_{2i+3} : i \in Z\}$$

and

$$S'_3 = \{u_{2i+1}u_{2i+4} : i \in Z\}.$$

Either of them produces bicolored 2-paths for all pairs at distance 2 and 4. It implies that there are no 5-chord at all.

In general, we always have two choices, $S_{4k-2} = \{u_{2i}u_{2i+4k-2} : i \in Z\}$ and $S'_{4k-2} = \{u_{2i+1}u_{2i+4k-1} : i \in Z\}$, for $(4k-2)$-chords, $k \geq 1$. Each of them creates all required paths between pairs of vertices at distance $4k-3$ and $4k-1$, implying that there are no $4k$-chords. Similarly, there are exactly two choices S_{4k-1} and S'_{4k-1}, for $(4k-1)$-chords, $k \geq 1$, and there are no $(4k+1)$-chords for all $k \geq 1$.

Theorem 3 *There are uncountably many infinite friendship two-graphs in which the* 1*-edges constitute an infinite path* $(\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots)$.

5 Two-graphs having dominating vertices

A *dominating vertex* in a two-graph G is a vertex which is 0- or 1-adjacent to all other vertices of G.

Theorem 4 *The only friendship two-graph having a dominating vertex is the two-graph* F *of Figure 2.*

Let $G = (G_0, G_1)$ be a friendship two-graph with a dominating vertex u. Denote by N_0 (respectively, N_1) the set of all 0-neighbors (respectively, 1-neighbors) of u. Since u is a dominating vertex, $V(G) = \{u\} \cup N_0 \cup N_1$.

Fact 1 *No two vertices in* N_0 *are* 1*-adjacent, and no two vertices in* N_1 *are* 0*-adjacent.*

Suppose that vertices $v, w \in N_0$ are 1-adjacent, and consider a bicolored 2-path (v, x, w). By symmetry, we may assume that vx is a 1-edge, and xw is a 0-edge. Clearly $x \neq q$, and therefore either $x \in N_0$ or $x \in N_1$. If $x \in N_0$ then (u, x, v, w) a $(2+2)$-cycle, a contradiction. Thus, $x \in N_1$, and (u, x, w, v) a $(2+2)$-cycle, a contradiction.

The second statement is similar.

A *star* (x, P) consists of a *central vertex* x and a set of *pendant vertices* P, each vertex of P being adjacent to u only. Note that the set P may be empty, in which case (x, P) has just one vertex x. Let X and Y be disjoint subsets of vertices. A *multi-star* (X, Y) consists of $|X|$ vertex-disjoint stars (x_i, P_i) centered at the vertices of X, all P_i are subsets of Y, and they constitute a partition of Y. An example of a multi-star (X, Y) is shown in Figure 7 for $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$.

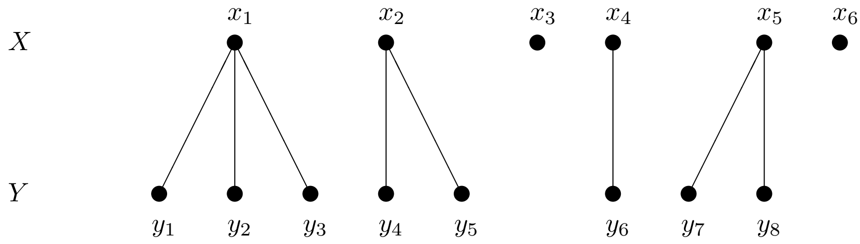

Figure 7: An example of a multi-star (X, Y).

Fact 2 (i) *The subgraph of G_0 induced by $N_0 \cup N_1$ is a multi-star (N_1, N_0).*

(ii) *The subgraph of G_1 induced by $N_0 \cup N_1$ is a multi-star (N_0, N_1).*

(i) Let $S(i)$ be the maximal star of 0-edges centered at an arbitrary vertex $x_i \in N_1$. By Fact 1, all pendant vertices of each $S(i)$ are in N_0. The stars $S(i)$ are pairwise vertex-disjoint. Indeed, if $S(i)$ and $S(j)$, $i \neq j$, have a common vertex $v \in N_0$, then (u, x_i, v, x_j) is a forbidden $(2+2)$-cycle. It remains to show that N_0 is covered by the pendant vertices of all $S(i)$. For every vertex $v \in N_0$, there must be a bicolored 2-path (u, x, v). Clearly, ux is a 1-edge and therefore xv is a 0-edge. Thus, v is covered by the star centered at x.

(ii) follows by symmetry.

Now consider all bicolored 2-paths connecting a fixed vertex $v \in N_0$ with all other vertices of N_0. By Fact 1, every such 2-path (v, x, v') has $x \in N_1$. If vx is a 0-edge then Fact 2(i) shows that v' is unique. Hence all but two vertices in N_0 are connected with v by a bicolored 2-path (v, x, v') such that vx is a 1-edge. Let $M(v)$ be the set of the end-vertices $v' \in N_0$. Thus, $|M(v)| = |N_0| - 2$. Fact 2 implies that $M(v) \cup M(w) = emptyset$ whenever $v \neq w$. We obtain

$$|M(v)| \cdot |N_0| = |N_0|.$$

Since $N_0 \neq \emptyset$, we have $|M(v)| = |N_0| - 2 = 1$, or $|N_0| = 3$. By symmetry, $|N_1| = 3$. Note that the conclusion $|N_0| = |N_1| = 3$ is valid even for infinite two graph G. It shows that all stars in the multi-stars (N_0, N_1) and (N_1, N_0) are just edges. There is just one variant (up to isomorphism) for the subgraph induced by $N_0 \cup N_1$, see Figure 8, where $N_0 = \{v_1, v_2, v_3\}$ and $N_1 = \{w_1, w_2, w_3\}$.

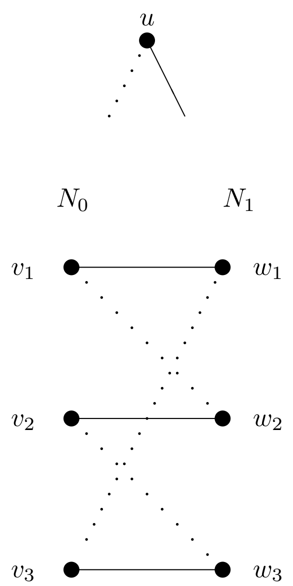

Figure 8: The subgraph induced by the set $N_0 \cup N_1$.

It is clear that the sets N_0 and N_1 induce edgeless graphs. Thus, G is the two-graph F of Figure 2.

6 A criterion

For $i \in \{0, 1\}$, let $\deg_i(u)$ denote the *i-degree* of a vertex u in a two-graph $G = (G_0, G_1)$, that is the total number of i-edge incident to u. The ordinary degree of u is $\deg(u) = \deg_0(u) + \deg_1(u)$.

Theorem 5 *$G = (G_0, G_1)$ is a friendship two-graph if and only if*

$$\sum_{u \in V(G)} \deg_0(u)\deg_1(u) = n(n-1)/2, \tag{1}$$

and there are no $(2 + 2)$-cycles in G.

The number of bicolored 2-paths centered at a fixed vertex u is exactly $\deg_0(u)\deg_1(u)$, so the left-hand side in (1) must be equal to the number of unordered pairs of distinct vertices, that is $n(n-1)/2$. Thus, (1) is equivalent to the statement that there are exactly $n(n-1)/2$ bicolored 2-paths. Finally, the existence of a $(2 + 2)$-cycle is equivalent to the statement that some unordered pairs of distinct vertices is connected by two bicolored 2-paths.

Theorem 5 implies a lower bound on the maximum vertex degree $\Delta(G)$ of a friendship two-graph G.

Corollary 1 *If G is a friendship two-graph then*

$$\Delta(G) \geq \sqrt{2n-2}, \tag{2}$$

where $n = |V(G)|$.

An arbitrary term $\deg_0(u)\deg_1(u)$ in (1) does not exceed $\Delta^2(G)/4$. Therefore Theorem 5 gives $n\Delta^2(G)/4 \geq n(n-1)/2$, which is equivalent to (2).

For an integer $k \geq 0$, let $\mathcal{DELTA}(k)$ denote the class of all two-graphs G with $\Delta(G) \leq k$.

Corollary 2 *For every k, the class $\mathcal{DELTA}(k)$ contains finitely many friendship two-graphs.*

Indeed, (2) implies that $(k^2+2)/2 \geq n$, that is all friendship two-graph in $\mathcal{DELTA}(k)$ have a bounded number of vertices.

References

[1] J. A. Bondy, Kotzig's conjecture on generalized friendship graphs – a survey, in: *Cycles in graphs (Burnaby, B.C., 1982)*, North-Holland Math. Stud. **115** (North-Holland, Amsterdam, 1985) 351–366

[2] V. Chvátal, R. L. Graham, A. F. Perold, and S. H. Whitesides, Combinatorial designs related to the strong perfect graph conjecture, Discrete Math. **26** (2) (1979) 83–92

[3] P. Erdős, A. Rényi, and V. T. Sós, On a problem of graph theory, Studia Sci. Math. Hungar. **1** (1966) 215–235

[4] A. V. Kostochka, The nonexistence of certain generalized friendship graphs, in: *Combinatorics (Eger, 1987)*, Colloq. Math. Soc. János Bolyai **52** (North-Holland, Amsterdam, 1988) 341–356