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Application of the Solution of the Univariate Discrete Moment Problem for the Multivariate Case

Gergely Mádi-Nagy

Abstract. The univariate discrete moment problem (DMP) is to find the minimum and/or maximum of the expected value of a function of a random variable which has a discrete finite support. The probability distribution is unknown, but some of the moments are given. This problem is an ill-conditioned LP, but it can be solved by the dual method presented in Prékopa (1990). The multivariate discrete moment problem (MDMP) is the generalization of the DMP where the objective function is the expected value of a function of a random vector. The MDMP has also been initiated by Prékopa and it also can be consider as an (ill-conditioned) LP. The central results of MDMP concern the structure of the dual feasible bases, provide us with bounds without any numerical difficulties. Unfortunately, in this case not all the dual feasible bases have been found, hence the multivariate counterpart of the dual method of DMP cannot be developed. However, there exists a method in Mádi-Nagy (2005), which allows us to get the basis corresponding to the best bound out of the known structures by optimizing independently on each variable. In this paper we present a method using the dual method of DMP for solving those independent subproblems. The efficiency of this new method will be illustrated by numerical examples.

Keywords: Stochastic programming, Linear programming, Moment problems

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1 Introduction

The theory of discrete moment problem (DMP) has been discussed in Prékopa (1990, 1995). Let X be a random variable with a known, finite support $Z = \{z_0, z_1, \cdots, z_n\}$, where $z_0 < \cdots < z_n$. The probability distribution of X is unknown, but some of the moments of X are known. Consider a function f with the domain Z. The objective of DMP is yielding lower and upper bounds for the expected value of $f(X)$, using the moment information.

In this paper the power moments are taken into account up to a certain order m. We introduce the following notations:

$$f_i := f(z_i),\ p_i := P(X = z_i),\ i = 0, 1, \ldots, n.$$

$$\mu_k := E(X^k),\ k = 0, 1, \ldots, m.$$

Our DMP can be represented as an LP:

$$\begin{array}{ll} & \min(\max)\ E[f(X)] = \{f_0p_0 + f_1p_1 + \cdots + f_np_n\} \\ \text{subject to} & \\ & \begin{array}{rcrcccrcl} p_0 & + & p_1 & + & \cdots & + & p_n & = & 1 \\ z_0p_0 & + & z_1p_1 & + & \cdots & + & z_np_n & = & \mu_1 \\ z_0^2p_0 & + & z_1^2p_1 & + & \cdots & + & z_n^2p_n & = & \mu_2 \\ & & & & \cdots & & & & \\ z_0^mp_0 & + & z_1^mp_1 & + & \cdots & + & z_n^mp_n & = & \mu_m \end{array} \\ & p_0 \geq 0,\ p_1 \geq 0, \ldots,\ p_n \geq 0, \end{array} \tag{1}$$

The DMP's using binomial moments can be converted into power moment problems by multiplying the coefficient matrix and the right-hand side vector by a non-singular matrix (see Prékopa 1995 p. 153). This means that our results are also valid for binomial DMP's.

The multivariate discrete moment problem (MDMP) is a generalization of the DMP for random vectors. It has been discussed in the papers by Prékopa (1992, 1998, 2000), Mádi-Nagy and Prékopa (2004). Let $\boldsymbol{X} = (X_1, \ldots, X_s)$ be a random vector with unknown distribution. We assume that the support of X_j is a known finite set $Z_j = \{z_{j0}, \ldots, z_{jn_j}\}$, consisting of distinct elements. We define

$$p_{i_1 \ldots i_s} = P(X_1 = z_{1i_1}, \ldots, X_s = z_{si_s}),\ 0 \leq i_j \leq n_j,\ j = 1, \ldots, s,$$

$$\mu_{\alpha_1 \ldots \alpha_s} = \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1 \ldots i_s},$$

where $\alpha_1, \ldots, \alpha_s$ are nonnegative integers. The number $\mu_{\alpha_1 \ldots \alpha_s}$ will be called the $(\alpha_1, \ldots, \alpha_s)$-order moment of the random vector $(X_1, \ldots, X_s)$, and the sum $\alpha_1 + \cdots + \alpha_s$ will be called the total order of the moment. Let $Z = Z_1 \times \cdots \times Z_s$ and $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ be a function on the domain Z. Let $f_{i_1\ldots i_s} = f(z_{1i_1}, \ldots, z_{si_s})$. In the paper we consider the following MDMP:

$$
\begin{array}{ll}
 & \min(\max)\ E[f(\boldsymbol{X})] = \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} f_{i_1\ldots i_s} p_{i_1\ldots i_s} \\
\text{subject to} & \\
 & \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1\ldots i_s} = \mu_{\alpha_1\ldots\alpha_s} \\
 & \alpha_j \geq 0,\ j = 1, \ldots, s;\ \alpha_1 + \cdots \alpha_s \leq m \text{ and} \\
 & \alpha_j = 0,\ j = 1, \ldots, k-1, k+1, \ldots, s, \\
 & m \leq \alpha_k \leq m_k,\ k = 1, \ldots, s; \\
 & p_{i_1\ldots i_s} \geq 0, \text{ for all } i_1, \ldots, i_s.
\end{array}
\tag{2}
$$

This means that in addition to all moments of total order at most m, the at most m_k^{th} order moments ($m_k \geq m$) of the k^{th} univariate marginal distribution are taken into account, $k = 1, \ldots, s$.

Univariate and multivariate discrete moment problems can be applied in several fields e.g., bounding expected utilities (Prékopa and Mádi-Nagy, 2008), solving geleralized s-dimensional transportation problem (Hou and Prékopa, 2007) and approximating values of multivariate generating functions (Mádi-Nagy and Prékopa, 2007). One of the main applications is to bound probabilities of Boolean functions of events. DMP's can be useful to approximate the unknown probabilities e.g., in network reliability calculation (Habib and Szántai, 2000) as well as in probabilistic constrained stochastic programming models (Prékopa, 1999, Fábián and Szőke, 2007). It could also be a good alternative of the bounding technics of Bukszár and Szántai (2002). This type of probability bounds can also be used in developing variance reduction Monte-Carlo simulation algorithms for estimating the exact probability values (Szántai, 1986, 2000).

The paper is organized as follows. In Section **??** we briefly present the solution method of the univariate DMP based on the paper by Prékopa (2001). In Section **??** we summarize the bounding techniques of MDMP and explore the structure of the coefficient matrix. The new result of the paper is in Section **??**: we introduce a bounding technique for MDMP using the method of the univariate DMP. In the last section numerical examples are presented to show the efficiency of the new method. We can see that in case of great-sized problems this method works much faster than the previous method.

2 The univariate DMP

The coefficient matrix of the univariate DMP (**??**) is an ill-conditioned Vandermonde matrix, hence the DMP usually cannot be solved by the regular methods and solvers. One way could be the usage of multiple precision arithmetic, but this leads to a very long solution time. Fortunately, under some assumptions on the function f, all the dual feasible bases of problem (**??**) can be given by the following

Theorem 2.1 (Prékopa 1995) *Suppose that all $m+1^{st}$ divided differences of the function $f(z),\ z \in \{z_0, z_1, \ldots, z_n\}$ ($z_0 < \cdots < z_n$) are positive. Then, in problem (??), all bases are dual-nondegenerate and the dual feasible bases have the following structures, presented in terms of the subscripts of the basis vectors:*

$$\begin{array}{lcc} & m+1 \textit{ even} & m+1 \textit{ odd} \\ \textit{min} & \{j, j+1, \ldots, k, k+1\} & \{0, j, j+1, \ldots, k, k+1\} \\ \textit{max} & \{0, j, j+1, \ldots, k, k+1, n\} & \{j, j+1, \ldots, k, k+1, n\} \end{array} \tag{3}$$

where in all parentheses the numbers are arranged in increasing order.

It's easy to see, if the interval $[z_0, z_n]$ is subset of the domain of the function $f(z)$ and the function has continuous, positive k^{th} derivatives in the interior of the interval, then all divided differences of order k of $f(z),\ z \in Z$ are positive.

The solution algorithm of the paper by Prékopa (1990) is the following.

The dual method of Prékopa

Step 1: Pick any dual feasible basis in agreement with Theorem **??**. Let $I_B = \{i_0, i_1, \ldots, i_m\}$ designate the set of subscripts of the basis vectors.

Step 2: Determination of the outgoing vector: Take any element $i_k \in I_B$. It can be derived (see Prékopa, 1990) that the sign of the value of the following form equals the sign of the value of the basic variable (i.e., p_{i_k}).

$$(-1)^{m-k}\left[\mu_m - \left(\sum_{J \in I \setminus \{i_k\}} z_{i_j}\right)\mu_{m-1} + \cdots + (-1)^m \prod_{J \in I \setminus \{i_k\}} z_{i_j}\right] \tag{4}$$

hence, if the value of (**??**)

- is negative, then the k^{th} vector of the basis can be the outgoing vector
- is nonnegative then seek another basis subscript.
- If the value of (**??**) is positive for all basis index, then go to **Step 4**.

Step 3: If the outgoing vector is identified, then we can choose the unique incoming vector which restores dual feasibility of the basis, by the aid of Theorem **??**. Go to **Step 2**.

Step 4: Stop, we have an optimal basis. To compute the optimum value we have to invert the optimal basis B. This can be carried out by the solution of Vandermonde equality systems. Due to numerical instability a special algorithm is needed to do the job. We use multiple precision arithmetic.

3 The multivariate DMP

As we have seen in (??), the MDMP serves for bounding

$$E[f(X_1,\dots,X_s)] \tag{5}$$

where all moments of total order at most m and the at most m_k^{th} order moments ($m_k \geq m$) of the k^{th} univariate marginal distribution are known, $k=1,\dots,s$. Unfortunately, in the multivariate case not all the dual feasible bases are known, hence we cannot construct a robust dual simplex method to solve it. However, some dual feasible basis structures can be given and by the aid of them bounds can be derived for (??). Furthermore, if the cardinality of the known dual feasible bases is great enough then the best corresponding bounds are close to the optimal values (min and max) of the MDMP (??). In this section we summarize the former results related to dual feasible bases and then in the following section we introduce the new method to find the best among the corresponding bounds.

We will use the notations of the compact matrix form of problem (??) (compatible with the notation of Mádi-Nagy and Prékopa, 2004 and Mádi-Nagy, 2005):

$$\begin{array}{rrcl} & \min(\max) & \boldsymbol{f}^T\boldsymbol{p} & \\ \text{subject to} & & & \\ & \hat{A}\boldsymbol{p} & = & \hat{\boldsymbol{b}} \\ & \boldsymbol{p} & \geq & \boldsymbol{0}. \end{array} \tag{6}$$

Consider the set of subscripts

$$I = I_0 \cup \left(\cup_{j=1}^s I_j\right), \tag{7}$$

where

$$I_0 = \{(i_1,\dots,i_s) |\ 0 \leq i_j \leq m-1,\ \text{integers},\ j=1,\dots,s,\ i_1+\ldots+i_s \leq m\} \tag{8}$$

and

$$\begin{array}{l} I_j = \{(i_1,\dots,i_s) |\ i_j \in K_j,\ i_l = 0\ l \neq j\} \\ K_j = \{k_j^{(1)},\dots,k_j^{(|K_j|)}\} \subset \{m,m+1,\dots,n_j\},\ j=1,\dots,s. \end{array} \tag{9}$$

Let us consider four different structures for K_j:

$$\begin{array}{lcc} & |K_j|\ \textit{even} & |K_j|\ \textit{odd} \\ \textit{min} & u^{(j)},u^{(j)}+1,\dots,v^{(j)},v^{(j)}+1 & m,u^{(j)},u^{(j)}+1,\dots,v^{(j)},v^{(j)}+1 \\ \textit{max} & m,u^{(j)},u^{(j)}+1,\dots,v^{(j)},v^{(j)}+1,n_j & u^{(j)},u^{(j)}+1,\dots,v^{(j)},v^{(j)}+1,n_j. \end{array} \tag{10}$$

We consider the following

Assumption 3.1 *The function $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ has nonnegative divided differences of total order $m+1$, and in addition, in each variable z_j it has positive divided differences of order $m_j+1 := m+|K_j|$.*

We remark that for the results of this section it is enough to assume nonnegativity of the univariate moments. Positivity is needed in the following section for the new algorithm.

Theorem 3.1 *(Mádi-Nagy and Prékopa (2004)) Let* $z_{j0} < z_{j1} < \cdots < z_{jn_j}$, $j = 1, \ldots, s$. *Suppose that function* f *fulfils Assumption* **??**, *where the set* K_j *has one of the min structures in (??).*

Under these conditions the set of columns $\widehat{B}$ *of* $\widehat{A}$ *in problem (??), with the subscript set* I*, is a dual feasible basis in the minimization problem (??), and*

$$E[f(X_1, \ldots, X_s)] \geq \boldsymbol{f}_{\widehat{B}}^T \boldsymbol{p}_{\widehat{B}}. \tag{11}$$

If $\widehat{B}$ *is also a primal feasible basis in problem (??), then the inequality (??) is sharp.*

Theorem 3.2 *(Mádi-Nagy and Prékopa (2004)) Let* $z_{j0} > z_{j1} > \cdots > z_{jn_j}$, $j = 1, \ldots, s$. *Suppose that function* f *fulfils Assumption* **??**, *where* K_j *has one of the structures in (??) that we specify below.*

(a) If $m + 1$ *is even,* $|K_j|$ *is even and* K_j *has the max structure in (??) or* $m + 1$ *is even,* $|K_j|$ *is odd and* K_j *has the min structure in (??), then the set of columns* $\widehat{B}$ *in* $\widehat{A}$*, corresponding to the subscripts* I*, is a dual feasible basis in the minimization problem (??). We also have the inequality*

$$E[f(X_1, \ldots, X_s)] \geq \boldsymbol{f}_{\widehat{B}}^T \boldsymbol{p}_{\widehat{B}}. \tag{12}$$

(b) If $m + 1$ *is odd,* $|K_j|$ *is even and* K_j *has the max structure in (??) or* $m + 1$ *is odd,* $|K_j|$ *is odd and* K_j *has the min structure in (??), then the basis* $\widehat{B}$ *is dual feasible in the maximization problem (??). We also have the inequality*

$$E[f(X_1, \ldots, X_s)] \leq \boldsymbol{f}_{\widehat{B}}^T \boldsymbol{p}_{\widehat{B}}. \tag{13}$$

The above two theorems yield dual feasible basis structures by the aid of the subscript set I defined in (**??**), ordering the elements of Z_j's increasingly or decreasingly. In the bivariate case ($s = 2$), (still at Assumption **??**) we can give much more dual feasible bases corresponding to I, by suitable (not necessary increasing or decreasing) ordering of the variables. In the following, we sketch these methods. Detailed discussion with illustrative figures and examples can be found in Mádi-Nagy and Prékopa (2004).

Consider first the case, where we want to construct a lower bound. We may assume, without loss of generality, that the sets Z_1 and Z_2 are the following: $Z_1 = \{0, 1, \ldots, n_1\}$, $Z_2 = \{0, 1, \ldots, n_2\}$.

Min Algorithm (Mádi-Nagy and Prékopa (2004))

Algorithm to find $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$**.**

Step 0. Initialize $t = 0$, $-1 \le q_1 \le m-1$, $L = (0, 1, \ldots, q_1)$, $U = (n_1, n_1 - 1, \ldots, n_1 - (m - q_1 - 2))$. Let $(z_{10}, \ldots, z_{1(m-1)})$ = (arbitrary merger of the sets L, U). If $|U|$ is even, then $z_{20} = 0$, $l^0 = 1$, $u^0 = n_2$, and if $|U|$ is odd, then $z_{20} = n_2$, $l^0 = 0$, $u^0 = n_2 - 1$. If $t = m - 1$, then go to Step 2. Otherwise go to Step 1.

Step 1. If $z_{1(m-1-t)} \in L$, then let $z_{2(t+1)} = l^t$, $l^{t+1} = l^t + 1$, $u^{t+1} = u^t$, and if $z_{1(m-1-t)} \in U$, then let $z_{2(t+1)} = u^t$, $u^{t+1} = u^t - 1$, $l^{t+1} = l^t$. Set $t \leftarrow t + 1$. If $t = m - 1$, then go to Step 2. Otherwise repeat Step 1.

Step 2. Stop, $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$ have been created.

Let $0, 1, \ldots, q_2, n_2, \ldots, n_2 - (m - q_2 - 2)$ be the numbers used to construct $z_{20}, z_{21}, \ldots, z_{2(m-1)}$. Then let $(z_{jm}, z_{j(m+1)}, \ldots, z_{jn_j}) = (q_j + 1, q_j + 2, \ldots, n_j - (m - q_j - 1))$, $j = 1, 2$. If $m - 1 - q_j$ even, then K_j should follow a minimum structure in (**??**), and if $m - 1 - q_j$ odd, then K_j should follow a maximum structure, $j = 1, 2$.

We have completed the construction of the dual feasible basis related to the subscript set I.

If we want to construct an upper bound, then only slight modification is needed in the above algorithm to find $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$. We only have to rewrite Step 0 and keep the other steps unchanged, and then give the appropriate K_j structures.

Max Algorithm (Mádi-Nagy and Prékopa (2004))

Algorithm to find $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$.

Step 0. Initialize $t = 0$, $-1 \le q_1 \le m-1$, $L = (0, 1, \ldots, q_1)$, $U = (n_1, n_1 - 1, \ldots, n_1 - (m - q_1 - 2))$. Let $(z_{10}, \ldots, z_{1(m-1)})$ = (arbitrary merger of the sets L, U). If $|U|$ is odd, then $z_{20} = 0$, $l^0 = 1$, $u^0 = n_2$, and if $|U|$ is even, then $z_{20} = n_2$, $l^0 = 0$, $u^0 = n_2 - 1$. If $t = m - 1$, then go to Step 2. Otherwise go to Step 1, etc.

In case of the upper bound we have to choose K_j the other way around as in case of the Min Algorithm. If $m - 1 - q_j$ even, then K_j should follow a maximum structure, otherwise a minimum structure.

We have completed the construction of the dual feasible basis related to the subscript set I.

The multivariate generalization of these algorithms can be found in Mádi-Nagy (2007) where the MDMP is slightly different to (**??**) but the algorithm remains nearly the same. Hence, the further results of our paper can be applied for the MDMP of Mádi-Nagy (2007), as well.

In the theorems above and also in the algorithms we dealt with bases correspond to the subscript set I. Let us call them Z_I-type bases. Our aim is to find the basis among them which gives the maximum (minimum) objective value function in case of the minimum (maximum) problem of (**??**). The diversity of Z_I-type bases is given by the order of Z_j's and the choices of the subscript sets K_j's. If the order of Z_j's are given then by the method of Mádi-Nagy (2005) the best K_j's, in sense of the objective function value, can be found independently. The sketch of the method is the following.

All MDMP's with Assumption **??** can be converted into an equivalent problem, such that Assumption **??** remains valid and

$$z_{j0} = 0,\ j = 1, \ldots, s \tag{14}$$

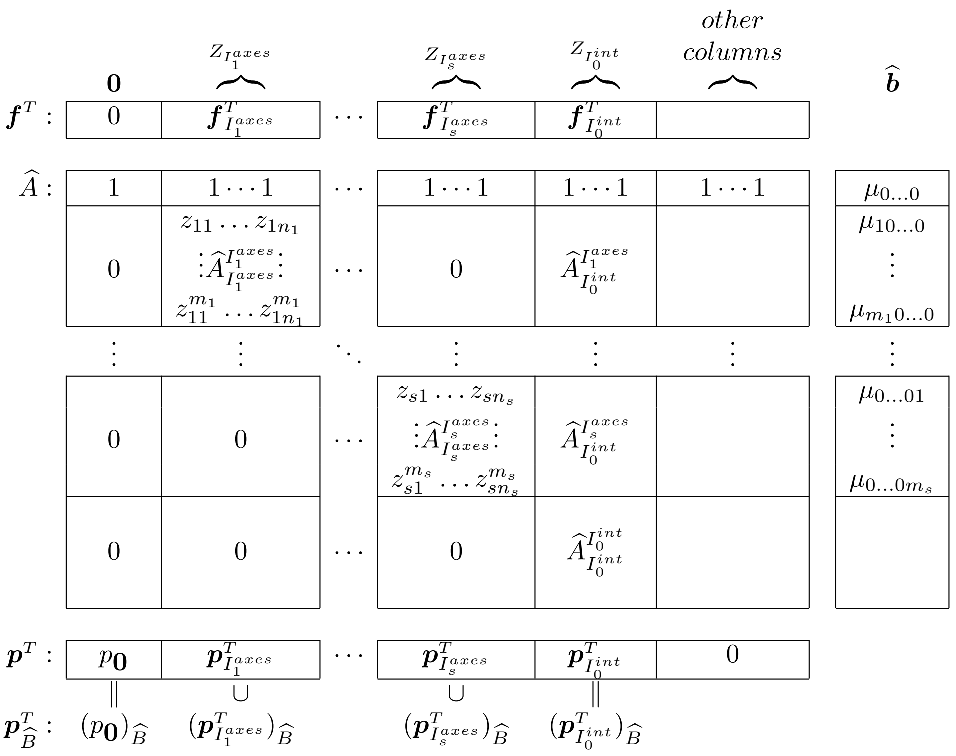
and

$$f(z_{10}, \ldots, z_{s0}) = 0. \tag{15}$$

Consider the following subscript sets:

$$\begin{aligned} I_0^{int} &= \{(i_1, \ldots, i_s) |\ 1 \le i_j \le m-1,\ \text{integer},\ j = 1, \ldots, s,\ i_1 + \cdots + i_s \le m\}, \\ I_j^{axes} &= \{(i_1, \ldots, i_s) |\ 1 \le i_j \le n_j,\ \text{integer};\ i_l = 0 \text{ for } l \ne j\}. \end{aligned} \tag{16}$$

If we reorder the columns and rows of the constraint matrix of the converted problem according to the subscript sets above, we get a more perspicuous structure:



(17)

At problem (**??**) we have introduced some new notations, which help us in the following arguments.

- The subscripts denote the columns of the matrix, while the superscripts refer to the rows.

- $\boldsymbol{p}$ denotes an appropriate basic solution, while the $\boldsymbol{p}_{\widehat{B}}$ vector consists of the components of the basic variables. From the structure of the Z_I bases follows that there are no basic variables in the last block of $\boldsymbol{p}^T$, hence these components have the value zero.

- Between the rows $\boldsymbol{p}^T_{\widehat{B}}$ and $\boldsymbol{p}^T$ we referred by equality signs to the fact, that all variables of $p_{0\ldots0}$ and $\boldsymbol{p}_{I_0^{int}}$ are also basic variables for each Z_I-type basic solution.

At first, let us consider the rows of the mixed moments. Since the variables of the last block are zero and the coefficients of the variables of Z_I are zero except $Z_{I_0^{int}}$ and the matrix $\widehat{A}^{I_0^{int}}_{I_0^{int}}$ is a non-singular square matrix:

$$\boldsymbol{p}_{I_0^{int}} = \left(\widehat{A}^{I_0^{int}}_{I_0^{int}}\right)^{-1} \boldsymbol{\mu}_{I_0^{int}}. \tag{18}$$

Let $\widehat{\boldsymbol{b}}_1 = \widehat{\boldsymbol{b}} - A_{I_0{}^{int}}\boldsymbol{p}_{I_0{}^{int}}$. Then the problem is broken into the following type of smaller subproblems:

$$\begin{array}{ll} & \max(\min) \quad \boldsymbol{f}^T_{I_j{}^{axes}}\boldsymbol{p}_{I_j{}^{axes}} \\ \text{subject to} & \\ & \begin{array}{rcl} \widehat{A}^{I_j{}^{axes}}_{I_j{}^{axes}}\boldsymbol{p}_{I_j{}^{axes}} & = & \widehat{\boldsymbol{b}}_1^{I_j^{axes}} \\ \boldsymbol{p}_{I_j{}^{axes}} & = & \end{array} \\ & \text{the corresponding part of a } Z_I\text{-type basic solution,} \end{array} \tag{19}$$

$j = 1, \ldots, s$. The last constraint above means that the subscript set of the basic variables of $\boldsymbol{p}_{I_j{}^{axes}}$ is the union of the part of I_0 which contains the related axis and the set I_j that is characterized by K_j.

Finally, if we solved problems (**??**) then

$$p^{opt}_{\mathbf{0}} = (\widehat{\boldsymbol{b}}_1)^{\mathbf{0}} - (1, \ldots, 1)^T \boldsymbol{p}^{opt}_{(I_1{}^{axes},\ldots,I_s{}^{axes})}.$$

By the aid of this method the closest Z_I-type bounds can be found in a much shorter way than by calculating the objective function value for all possible Z_I-type bases. Using the method of the next section the bounding procedure can be further shortened.

4 Application of the univariate method in the solution of MDMP

We focus on subproblems (**??**) that give the subscript sets K_j's corresponding to the Z_I-type basis yielding the best bound.

First we prove

Theorem 4.1 *The corresponding parts of Z_I-type bases, $\widehat{B}^{I_j^{axes}}_{I_j^{axes}}$'s, are dual feasible in problem*

$$\begin{array}{ll} & \min(\max) \quad \boldsymbol{f}^T_{I_j{}^{axes}}\boldsymbol{p}_{I_j{}^{axes}} \\ \textit{subject to} & \\ & \begin{array}{rcl} \widehat{A}^{I_j{}^{axes}}_{I_j{}^{axes}}\boldsymbol{p}_{I_j{}^{axes}} & = & \widehat{\boldsymbol{b}}_1^{I_j^{axes}} \\ \boldsymbol{p}_{I_j{}^{axes}} & \geq & \mathbf{0} \end{array} \end{array} \tag{20}$$

Proof. Let us consider the following problem:

$$\begin{aligned} &\min(\max) \quad (f_{\mathbf{0}}, \boldsymbol{f}^T_{I_j{}^{axes}}) \cdot \begin{pmatrix} p_{\mathbf{0}} \\ \boldsymbol{p}_{I_j{}^{axes}} \end{pmatrix} \\ \text{subject to} \quad & \\ &\begin{pmatrix} 1 & \mathbf{1}^T \\ \mathbf{0} & \widehat{A}^{I_j{}^{axes}}_{I_j{}^{axes}} \end{pmatrix} \cdot \begin{pmatrix} p_{\mathbf{0}} \\ \boldsymbol{p}_{I_j{}^{axes}} \end{pmatrix} = \begin{pmatrix} b_0 \\ \widehat{\boldsymbol{b}}_1^{I_j^{axes}} \end{pmatrix} \\ &\begin{pmatrix} p_{\mathbf{0}} \\ \boldsymbol{p}_{I_j{}^{axes}} \end{pmatrix} \geq \mathbf{0}, \end{aligned} \tag{21}$$

where $f_{\mathbf{0}} = 0$. The coefficient matrix, in fact, a Vandermonde matrix, i.e.,

$$\begin{pmatrix} 1 & \mathbf{1}^T \\ \mathbf{0} & \widehat{A}^{I_j{}^{axes}}_{I_j{}^{axes}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_{j0} & z_{j1} & \cdots & z_{jn_j} \\ \vdots & \vdots & \ddots & \vdots \\ z_{j0}^{m_j} & z_{j1}^{m_j} & \cdots & z_{jn_j}^{m_j} \end{pmatrix},$$

where $z_{j0} = 0$. The function $f(\boldsymbol{z})$ fulfils Assumption **??**, hence the m_j+1^{st} divided differences of $f(0,\dots,0,z_j,0,\dots,0)$ in $z_j \in Z_j$ are positive.

If we look at the function $f(0,\dots,0,z_j,0,\dots,0)$ as a univariate function of $z_j \in Z_j$ then from Theorem **??** follows that the corresponding part of any Z_I-type basis is dual feasible. We know that the first column (the corresponding part of the column of the variable $(z_{10},\dots,z_{s0}) = (0,\dots,0)$) is always in the corresponding part of a Z_I-type basis. Hence, the basis B can be written in the following form:

$$B = \begin{pmatrix} 1 & \mathbf{1}^T \\ \mathbf{0} & \widehat{B}^{I_j^{axes}}_{I_j^{axes}} \end{pmatrix},$$

where $\widehat{B}^{I_j^{axes}}_{I_j^{axes}}$ are the corresponding part of the same Z_I-type basis regarding problem (**??**).

Finally, we prove that from the dual feasibility of the basis B in problem (**??**) follows the dual feasibility the basis $\widehat{B}^{I_j^{axes}}_{I_j^{axes}}$ in problem (**??**). In case of min (max) problem dual feasibility of B means that

$$(0, \boldsymbol{f}^T_{I_j{}^{axes}}) \cdot B^{-1} \cdot \begin{pmatrix} 1 \\ z_{ji} \\ \vdots \\ z_{ji}^{m_j} \end{pmatrix} - f(0,\dots,0,z_{ji},0,\dots 0) \leq (\geq) 0,$$

for all $i = 0,\dots,n_j$. It's easy to see that

$$B^{-1} = \begin{pmatrix} 1 & -\mathbf{1}^T \left(\widehat{B}^{I_j^{axes}}_{I_j^{axes}}\right)^{-1} \\ \mathbf{0} & \left(\widehat{B}^{I_j^{axes}}_{I_j^{axes}}\right)^{-1} \end{pmatrix}. \tag{22}$$

From this the left-hand side of (**??**) equals:

$$
\begin{aligned}
&(0, \boldsymbol{f}^T_{I_j^{axes}}) \cdot \begin{pmatrix} 1 & -\mathbf{1}^T \left(\widehat{B}^{I_j^{axes}}_{I_j^{axes}}\right)^{-1} \\ \mathbf{0} & \left(\widehat{B}^{I_j^{axes}}_{I_j^{axes}}\right)^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ z_{ji} \\ \vdots \\ z_{ji}^{m_j} \end{pmatrix} - f(0,\dots,0,z_{ji},0,\dots 0) \\
&= \boldsymbol{f}^T_{I_j^{axes}} \cdot \left(\widehat{B}^{I_j^{axes}}_{I_j^{axes}}\right)^{-1} \cdot \begin{pmatrix} z_{ji} \\ \vdots \\ z_{ji}^{m_j} \end{pmatrix} - f(0,\dots,0,z_{ji},0,\dots 0).
\end{aligned} \tag{23}
$$

This means that basis $\widehat{B}^{I_j^{axes}}_{I_j^{axes}}$ is dual feasible in problem (**??**) □

Remark 4.1 *As regards the choice of K_j, it's easy to see, that the corresponding parts of Z_I-type bases are the only dual feasible bases of problem (**??**) within the case where $z_{j0},\dots,z_{j(m-1)}$ are basic variables. Considering (**??**) the same is true for problem (**??**) within the case where $z_{j1},\dots,z_{j(m-1)}$ are basic variables.*

From the construction of Z_I follows that either all z_{ji} are positive or all z_{ji} are negative for $1 \le i \le n_j$.

If we substitute $x_i = |z_{ji_j}| p_{0\dots0i_j0\dots0}$ into (**??**) then we have the following equivalent problem:

$$
\begin{array}{l}
\min(\max)\ \dfrac{f_{0\dots010\dots0}}{|z_{j1}|}x_1 + \dfrac{f_{0\dots020\dots0}}{|z_{j2}|}x_2 + \dots + \dfrac{f_{0\dots0n_j0\dots0}}{|z_{jn_j}|}x_{n_j} \\
\text{subject to} \\
\begin{array}{rcrcccrcl}
x_1 & + & x_2 & + & \cdots & + & x_{n_j} & = & b_1^{I_j^{axes}} \cdot arg(z_{j1}) \\
z_{j1}x_1 & + & z_{j2}x_2 & + & \cdots & + & z_{jn_j}x_{n_j} & = & b_2^{I_j^{axes}} \cdot arg(z_{j1}) \\
 & & & & \cdots & & & & \\
z_{j1}^{m_j-1}x_1 & + & z_{j2}^{m_j-1}x_2 & + & \cdots & + & z_{jn_j}^{m_j-1}x_{n_j} & = & b_{m_j}^{I_j^{axes}} \cdot arg(z_{j1}) \\
 & & & & \boldsymbol{x} & \ge & \mathbf{0} & &
\end{array}
\end{array} \tag{24}
$$

Considering the results above we can elaborate the following method in order to solve (**??**). We look for the best corresponding Z_I-type basis of the equivalent problem (**??**) similarly as in the univariate dual method of Prékopa. The new method is based on the result of Theorem **??** and Remark **??** and the coefficient matrix of (**??**).

Partial dual method for finding the solution of the subproblem (??), i.e., for finding the best K_j

Step 1: Pick the corresponding part of any Z_I-type basis in agreement with Theorem **??** (**??**) or the Min (Max) Algorithm. Let $I_B = \{1,\dots,m-1,i_0,i_1,\dots,i_{m_j-m}\}$ designate the set of subscripts of the basis vectors, where $m \le i_0, i_1, \dots, i_{m_j-m} \le n_j$. Let $K = 0,\dots,m_j - m$.

Step 2: Determination of the outgoing vector: Take any element $i_k, k \in K$. It can be derived (based on Prékopa, 1990) that the sign of the value of the following form equals the sign of the value of the basic variable (i.e., x_{i_k} as well as p_{i_k}).

$$(-1)^{m_j-(q_j+k+1)}$$
$$\times \left[b_{m_j}^{I_j^{axes}} - \left(\sum_{J \in I_j^{axes} \setminus \{i_k\}} z_{ji_j} \right) b_{m_j-1}^{I_j^{axes}} + \cdots + (-1)^{m_j-1} \left(\prod_{J \in I_j^{axes} \setminus \{i_k\}} z_{i_j} \right) b_1^{I_j^{axes}} \right], \tag{25}$$

where q_j is the parameter of the Min (Max) Algorithm, in case of Theorem **??** $q_j = m - 1$, in case of Theorem **??** $q_j = -1$. Hence, if the value of (**??**)

- is negative, then the i_k^{th} vector of the basis can be the outgoing vector
- is nonnegative then seek another basis subscript.
- If the value of (**??**) is positive for all basis subscripts $i_k, k \in K$, then go to **Step 4**.

Step 3: If the outgoing vector is identified, then we can choose at most one incoming vector which restores the Z_I-type structure of the basis. If we found the incoming vector then consider the new Z_I-type basis and $K := 0, \dots, m_j - m$ else $K := K \setminus \{k\}$. Go to **Step 2**.

Step 4: Stop, we have found the solution of (**??**), i.e., the corresponding part of Z_I-type solution that gives the best bound.

The advantage of this new method is that we find the best basis of the subproblem (**??**) through bases having greater (smaller) objective function values in case on min (max) problem in each step, i.e., we don't have to examine all of the possible bases. In addition we don't have to calculate either the objective function value or the inverse of the basis matrix, just the value of (**??**).

5 Numerical examples

In this section we present the efficiency of the above method. For the sake of simplicity we restrict ourselves to the bivariate case. Two examples of Mádi-Nagy (2005), extended with greater sized problems, are recalculated. These problems cannot be solved by CPLEX and most of them cannot be solved by any numerically stable solvers in reasonable time. The best lower and upper bounds of the Min and Max Algorithms are given. These bounds are good approximations of the minimum and maximum of the objective functions. The method of Mádi-Nagy (2005) as well as our new method are used. The related CPU times (denoted by CPU_n and CPU_u, respectively) are also shown. The algorithms are implemented in Wolfram's Mathematica 5.1 (www.wolfram.com). Comparing the running times we can see how much faster and more effective the method of our paper is. This also means that by the aid of the new method greater sized problems can be solved.

Example 5.1 *Let $m = 4$, $m_1 = m_2 = 6$, and generate the moments by the discrete uniform distribution on $\{0, \ldots, 14\} \times \{0, \ldots, 14\}$. We obtain:*

μ_{ij}	0	1	2	3	4	5	6
0	1	7	203/3	735	127687/15	102655	3818459/3
1	7	49	1421/3	5145			
2	203/3	1421/3	41209/9				
3	735	5145					
4	127687/15						
5	102655						
6	3818459/3						

Consider the function

$$f(z_1, z_2) = e^{z_1/25 + z_1 z_2/400 + z_2/15}.$$

Let the support be $Z = Z_1 \times Z_2$, where $Z_j = \{0, \Delta z, 2\Delta z, \ldots, 14\}$ $j = 1, 2$. The results, depending on the value of Δz, are given in the following tableau:

Δz	*Lower*	CPU_n	CPU_u	*Upper*	CPU_n	CPU_u
1	2.61201563	1.16	0.52	2.67123415	1.14	0.5
0.5	2.60896245	0.97	0.73	2.67474361	0.99	0.70
0.2	2.60707222	3.00	1.70	2.67686786	2.97	1.73
0.1	2.60643222	6.84	3.42	2.67757838	6.83	3.38
0.05	2.60611033	15.77	7.19	2.67793372	16.14	7.11
0.01	2.60585192	143.78	49.81	2.67821877	142.97	51.31

Example 5.2 *Let $Z_1 = Z_2 = \{0, \ldots, 200\}$, $m = 6$, $m_1 = m_2 = 4$. First, generate the moments by the discrete uniform distribution on Z.*

We also consider the moments of the random vector

$$(min(X + Y_1, 200), min(X + Y_2, 200)),$$

where X, Y_1, Y_2 are random variables having Poisson distributions with λ parameters $30, 40, 50$, respectively.

The results, corresponding to the above moments at case of some functions, are shown below.

- *First we consider*

$$f(z_1, z_2) = log[(e^{0.75 z_1 + 2} - 1)(e^{1.25 z_2 + 3} - 1) - 1].$$

This function is a member of the function class of the paper by Prékopa and Mádi-Nagy (2008). It has the property that its odd (even) order derivatives are nonnegative (nonpositive) for all $z_1 \geq 0, z_2 \geq 0$.

Moments	*Lower*	CPU_n	CPU_u	*Upper*	CPU_n	CPU_u
Uniform	204.9826	28.73	5.61	205.0000	28.72	5.54
Poisson	157.4995	23.78	4.61	157.5000	23.61	4.56

- *The second function is*

$$f(z_1, z_2) = e^{z_1/200 + z_1 z_2/50000 + z_2/400},$$

which has positive derivatives for all $z_1 \geq 0, z_2 \geq 0$.

Moments	*Lower*	CPU_n	CPU_u	*Upper*	CPU_n	CPU_u
Uniform	2.82204	25.16	4.97	3.03179	25.06	4.84
Poisson	1.93597	17.79	4.00	1.97805	17.82	4.00

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