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Exclusive and essential sets of implicates of Boolean functions

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RRR 10-2008,

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Exclusive and essential sets of implicates of Boolean functions

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Abstract. In this paper we study relationships between CNF representations of a given Boolean function f and certain sets of implicates of f. We introduce two definitions of sets of implicates which are both based on the properties of resolution. The first type of sets, called exclusive sets of implicates, is shown to have a functional property useful for decompositions. The second type of sets, called essential sets of implicates, is proved to posses an orthogonality property, which implies that every CNF representation and every essential set must intersect. The latter property then leads to an interesting conjecture, which we prove to be true for some special subclasses of Horn Boolean functions.

Acknowledgements: The second author gratefully acknowledges the support by the the Czech Science Foundation (grant 201/07/0205). The fourth author gratefully acknowledges the support by the Czech Science Foundation (grant 201/07/P168).

1 Introduction

One of the most commonly used representations of Boolean functions are CNFs (conjunctive normal forms). For a given function there are typically many different CNFs representing it, which may significantly vary in length. In some applications an important problem is the following: for a given function find a shortest CNF among all of its possible CNF representations. For instance, in artificial intelligence this problem is equivalent to finding a most compact representation of a given knowledge base [9, 10]. Such transformation of a knowledge base accomplishes knowledge compression, since the actual knowledge does not change, while the size of the representation can be significantly reduced. In general this problem known as Boolean minimization (BM) can be stated as follows: given a CNF ϕ find a CNF ϕ' representing the same function and such that ϕ' consists of a minimum possible number of clauses.

It is easy to see that BM is NP-hard as it contains the satisfiability problem (SAT) as its special case (an unsatisfiable CNF can be represented by an equivalent CNF consisting of only the constant "false", i.e. consisting of zero clauses or one clause depending on whether the definition of a clause admits clauses with no variables). In fact, BM is NP-hard even for some classes of Boolean functions for which SAT is solvable in polynomial time. The best known example of such a class are Horn functions (see [1, 2, 4, 9, 12] for various BM intractability results). The difficulty of BM of course raises a natural question whether for a given input CNF a nontrivial lower bound can be obtained for the number of clauses in the shortest equivalent CNF. We give a partial answer to this question in Section 6.

On the positive side, [11] introduced two subclasses of Horn functions, acyclic and quasi-acyclic functions, for which BM is solvable in polynomial time. In fact, [11] and an earlier paper [7] by the same authors served as the original motivation for the results presented in this manuscript. We generalize the key Theorems from [7, 11] used for the minimization of acyclic and quasi-acyclic functions, and derive interesting new properties.

This paper is structured as follows. In Section 2 we introduce the necessary notation and present several elementary results important for the subsequent presentation. In Section 3 we define the class of Horn functions as well as its subclasses of acyclic and quasi-acyclic functions and recall some basic properties of these classes. Section 4 contains a short collection of simple lemmas about the properties of resolution closures. The main results of this paper are presented in Sections 5 and 6. Section 5 deals with exclusive sets of implicates and yields a decomposition theorem, which is shown to be useful in Boolean minimization. Section 6 is devoted to essential sets of implicates and presents a duality relation between essential sets of clauses and CNF representations, which then leads to an interesting conjecture. Finally, Section 7 proves the validity of the conjecture for the classes of acyclic and quasi-acyclic Horn functions.

2 Basic Notation, Definitions, and Results

In this section we shall introduce the necessary notation and state several basic known results that will be needed later in the text.

A *Boolean function* f on n propositional variables $x_1, \ldots, x_n$ is a mapping $\{0,1\}^n \to \{0,1\}$. The propositional variables $x_1, \ldots, x_n$ and their negations $\overline{x}_1, \ldots, \overline{x}_n$ are called *literals* (*positive* and *negative literals*, respectively). An elementary disjunction of literals

$$C = \bigvee_{i \in I} \overline{x}_i \vee \bigvee_{j \in J} x_j \tag{1}$$

is called a *clause*, if every propositional variable appears in it at most once, i.e. if $I \cap J = \emptyset$. A *degree* of a clause is the number of literals in it. For two Boolean functions f and g we write $f \le g$ if

$$\forall (x_1, \ldots, x_n) \in \{0,1\}^n \ : \ f(x_1, \ldots, x_n) = 1 \quad \Longrightarrow \quad g(x_1, \ldots, x_n) = 1 \tag{2}$$

Since each clause is in itself a Boolean function, formula (2) also defines the meaning of inequalities $C_1 \le C_2$, $C_1 \le f$, and $f \le C_1$, where C_1, C_2 are clauses and f is a Boolean function.

We say that a clause C_1 *subsumes* another clause C_2 if $C_1 \le C_2$ (e.g. the clause $\overline{x} \vee z$ subsumes the clause $\overline{x} \vee \overline{y} \vee z$). A clause C is called an *implicate* of a function f if $f \le C$. An implicate C is called *prime* if there is no distinct implicate C' subsuming C, or in other words, an implicate of a function is prime if dropping any literal from it produces a clause which is not an implicate of that function.

It is a well-known fact that every Boolean function f can be represented by a conjunction of clauses (see e.g. [6]). Such an expression is called a *conjunctive normal form* (or CNF) of the Boolean function f. It should be noted that a given Boolean function may have many CNF representations (typically exponentially many in the number of propositional variables). If two distinct CNFs, say ϕ_1 and ϕ_2, represent the same function, we say that they are *equivalent*, and denote this fact by $\phi_1 \equiv \phi_2$. A CNF ϕ representing a function f is called *prime* if each clause of ϕ is a prime implicate of the function f. A CNF ϕ representing a function f is called *irredundant* if dropping any clause from ϕ produces a CNF that does not represent f.

Example 2.1 *Consider the CNF*

$$(\overline{x}_1 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee x_4) \wedge (\overline{x}_2 \vee \overline{x}_3 \vee x_4) \wedge (\overline{x}_3 \vee \overline{x}_4).$$

The 2nd clause can be dropped (although it is prime), and the 4th clause can be shortened (i.e. it is not prime). In fact, the same function can be represented by the CNF

$$(\overline{x}_1 \vee x_2) \wedge (\overline{x}_1 \vee x_4) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_3 \vee \overline{x}_4)$$

which is both prime and irredundant.

Two clauses C_1 and C_2 are said to be *resolvable* if they contain exactly one complementary pair of literals, i.e. if there exists exactly one propositional variable that appears uncomplemented in one of the clauses and complemented in the other. That means that we can write $C_1 = \tilde{C}_1 \vee x$ and $C_2 = \tilde{C}_2 \vee \overline{x}$ for some propositional variable x and clauses $\tilde{C}_1$ and $\tilde{C}_2$ which contain no complementary pair of literals. The clauses C_1 and C_2 are called *parent clauses* and the disjunction $R(C_1, C_2) = \tilde{C}_1 \vee \tilde{C}_2$ is called the *resolvent* of the parent clauses C_1 and C_2. Note that the resolvent is a clause (does not contain a propositional variable and its negation). The following is an easy lemma.

Lemma 2.2 *Let C_1 and C_2 be two resolvable implicates of a Boolean function f. Then $R(C_1, C_2)$ is also an implicate of f.*

Resolutions have a very important property usually called the *completeness of resolution*. Sometimes this property is also referred to as the Quine theorem after the author of one of the first papers in which this property was proved [13, 14].

Theorem 2.3 *Let ϕ be a CNF representation of a Boolean function f and let C be a prime implicate of f. Then C can be derived from ϕ by a series of resolutions.*

Throughout this paper we shall also use the following notation. For an arbitrary set of clauses $\mathcal{C}$ the *resolution closure* of $\mathcal{C}$ denoted by $\mathcal{R}(\mathcal{C})$ is the set of all clauses obtainable through series of resolutions from the set $\mathcal{C}$ (allowing the resolvents to become parent clauses in subsequent resolutions).

The following two notational conventions will allow us to switch back and forth between sets of clauses and CNFs. For an arbitrary set of clauses $\mathcal{C}$ the symbol $\phi(\mathcal{C})$ denotes the CNF obtained by taking a conjunction of all clauses in $\mathcal{C}$. On the other hand, for an arbitrary CNF ϕ the symbol $\mathcal{C}(\phi)$ denotes the set of all clauses present in ϕ. We shall use the notion of "representing a given function" interchangeably for both CNFs and sets of clauses, i.e. if a CNF ϕ represents a function f we shall also say that the set of clauses $\mathcal{C}(\phi)$ represents f.

For a Boolean function f let us denote by $\mathcal{I}^p(f)$ the set of its prime implicates, and let $\mathcal{I}(f) = \mathcal{R}(\mathcal{I}^p(f))$. Note that not all implicates of f may belong to $\mathcal{I}(f)$. For instance, if f is defined by the CNF $\phi = (x_1 \vee x_2) \wedge (x_2 \vee x_3)$, then we have $\mathcal{I}(f) = \mathcal{I}^p(f) = \{(x_1 \vee x_2), (x_2 \vee x_3)\}$, however the clause $(x_1 \vee x_2 \vee x_3)$ is also a implicate of f.

Let us now turn our attention to a subclass of Boolean functions which will be frequently used in this paper, namely the class of Horn functions.

3 Horn functions and their subclasses

In this section we will describe a class of Horn functions, the properties of which served as the original motivation for deriving the more general results presented in this paper. We will also define two subclasses of Horn functions which are the main subject of the last section.

A clause C defined by (1) is called *negative* if it contains no positive literals (i.e. if $J = \emptyset$). It is called *pure Horn* (or in some literature *definite Horn*) if it contains exactly one positive literal (i.e. if $|J| = 1$). To simplify notation, we shall sometimes write a pure Horn clause $C = \bigvee_{x \in S} \overline{x} \vee y$ simply as $C = S \vee y$. Each propositional variable $x \in S$ is called a *subgoal of* C and the propositional variable y is called the *head of* C.[1] We shall denote $Head(C) = y$, $Subg(C) = S$, and $Vars(C) = S \cup \{y\}$.

A CNF is called *Horn* if it contains only negative and pure Horn clauses. A CNF is called *pure Horn* if it contains only pure Horn clauses. Finally, a Boolean function is called *Horn* if it has at least one representation by a Horn CNF, and similarly a Boolean function is called *pure Horn* if it has at least one representation by a pure Horn CNF.

It is known (see [7]) that each prime implicate of a Horn function is either negative or pure Horn, and each prime implicate of a pure Horn function is pure Horn. Thus, in particular, any prime CNF representing a Horn function is Horn, and any prime CNF representing a pure Horn function is pure Horn.

Let us now recall some very useful definitions from [11], associating directed graphs to Horn CNFs and Horn functions. Let us start by reviewing several standard notions from graph theory.

A *directed graph* (or a *digraph*) is an ordered pair $\mathbf{D} = (\mathbf{N}, \mathbf{A})$ where $\mathbf{N}$ is the set of *nodes* and $\mathbf{A}$ is the set of *arcs*, where an arc is an ordered pair of nodes. A *directed path* is a sequence of arcs a_1, a_2,...,a_p such that $a_i = (x_i, x_{i+1})$ for some nodes x_1, x_2, ..., x_{p+1}. A *directed cycle* is a directed path such that $x_1 = x_{p+1}$. A directed graph is called *strongly connected* if for any two nodes x and y there exist both a directed path from x to y and a directed path from y to x. If a graph $\mathbf{D}$ is not strongly connected then its node set can be decomposed in a unique way into maximal strongly connected subsets, called the *strong components* of $\mathbf{D}$. A directed graph is called *acyclic* if it contains no directed cycle. Note that in such a case every strong component consists of a single node. If $\mathbf{D}$ is an acyclic directed graph with a node set $\mathbf{N} = \{x_1, \ldots, x_n\}$, then an ordering of the nodes $(x_{i_1}, \ldots, x_{i_n})$ is called a *topological order* on $\mathbf{N}$ if for every arc $(x_{i_j}, x_{i_k}) \in \mathbf{A}$ we have $i_j < i_k$.

If $\mathbf{D} = (\mathbf{N}, \mathbf{A})$ is a directed graph, then the *transitive closure* of $\mathbf{D}$ is a directed graph $\overline{\mathbf{D}} = (\mathbf{N}, \overline{\mathbf{A}})$ where $(x, y) \in \overline{\mathbf{A}}$, whenever there is a directed path from x to y in the digraph $\mathbf{D}$. Obviously, for a given $\mathbf{D}$ the transitive closure $\overline{\mathbf{D}}$ is uniquely defined. Finally, if $\mathbf{D} = (\mathbf{N}, \mathbf{A})$ is a directed graph with strong components $C_1, \ldots, C_s$, then the directed graph $\mathbf{D}' = (\mathbf{N}', \mathbf{A}')$ on the set of nodes $\mathbf{N}' = \{C_1, \ldots, C_s\}$ with arcs

$$(C_i, C_j) \in \mathbf{A}' \text{ iff } \exists x \in C_i \ \exists y \in C_j \text{ such that } (x, y) \in \mathbf{A}$$

is called the *acyclic condensation* of the digraph $\mathbf{D}$.

Definition 3.1 *For a Horn CNF ϕ let $\mathbf{G}_\phi = (\mathbf{N}, \mathbf{A}_\phi)$ be the digraph defined by*

$$\mathbf{N} = \{x |\ x \text{ is a propositional variable in } \phi\}$$

$$\mathbf{A}_\phi = \{(x, y) \mid \exists \text{ a clause } C \in \mathcal{C}(\phi) \text{ such that } C \geq \overline{x} \vee y\}.$$

[1]This terminology comes from the area of artificial intelligence, where the clause C is thought of as a "rule" $S \longrightarrow y$.

In other words, for each pure Horn clause C in ϕ, the graph $\mathbf{G}_\phi$ contains as many arcs as is the number of subgoals in C, with each arc going from the corresponding subgoal to the head of C. Since a Horn function can be represented by several different Horn CNFs, seemingly we can associate in this way several different graphs to a Horn function. However, as it was shown in [3], all these graphs share several important features.

Theorem 3.2 *Let ϕ_1 and ϕ_2 be two distinct prime CNFs representing the same Horn function f and let x, y be arbitrary propositional variables from f. Then there is a directed path from x to y in $\mathbf{G}_{\phi_1}$ if and only if there is a directed path from x to y in $\mathbf{G}_{\phi_2}$. Moreover, it then follows that $\mathbf{G}_{\phi_1}$ and $\mathbf{G}_{\phi_2}$ have identical transitive closures, identical strong components, and identical acyclic condensations.*

Theorem 3.2 allows us to associate a graph directly to a Horn function rather than to its particular CNF representations.

Definition 3.3 *Let f be a Horn function and ϕ its arbitrary prime CNF representation. Then we define $\mathbf{G}_f$ as the transitive closure of $\mathbf{G}_\phi$.*

The graph $\mathbf{G}_f$ moreover has the following useful property with respect to the set $\mathcal{I}(f)$.

Lemma 3.4 *Let $C = \overline{x}_1 \vee \overline{x}_2 \vee \cdots \vee \overline{x}_k \vee y \in \mathcal{I}(f)$ be arbitrary. Then (x_i, y) is an arc in $\mathbf{G}_f$ for every $1 \leq i \leq k$.*

Proof. If C is prime then it suffices to pick ϕ containing C in Definition 3.3 and the claim for C follows. Now observe that resolving two clauses in which each subgoal is connected to the head by an arc in $\mathbf{G}_f$ produces a resolvent in which each subgoal is connected to the head by a directed path in $\mathbf{G}_f$. However, $\mathbf{G}_f$ is transitively closed and thus the claim follows for any $C \in \mathcal{I}(f)$. ∎

Theorem 3.2 suggests that for a given Horn function f the strong components of $\mathbf{G}_f$ play an important role in how the set of all prime CNF representations of f is structured. In what follows we shall call $\mathbf{G}_\phi$ and $\mathbf{G}_f$ the *implication graph*s of ϕ and f, respectively. The notion of implication graph of a Horn function f carries a lot of information about the CNF representations of f, allowing the characterization of two important subclasses of Horn functions.

A Horn CNF ϕ is said to be *acyclic* if its associated implication graph $\mathbf{G}_\phi$ is acyclic. A Horn function f is called *acyclic* if it admits at least one acyclic CNF representation. It was shown in [11] that every pure Horn acyclic function has a unique irredundant and prime representation. This fact has an important consequence: obviously, this unique CNF is also the shortest possible CNF representing the given pure Horn function. Moreover, as transforming an arbitrary Horn CNF into an equivalent prime and irredundant CNF can be done in polynomial time [7], this unique minimal CNF can be found efficiently. It was also shown in [7] that in fact this procedure can be used to minimize any acyclic Horn CNF (not necessarily pure Horn).

Let us call two propositional variables x and y *logically equivalent* in a Horn function f if the clauses $\overline{x} \vee y$ and $\overline{y} \vee x$ are implicates of f. A Horn CNF ϕ is then said to be *quasi-acyclic* (see [11]) if every strong component of its associated implication graph $\mathbf{G}_\phi$ consists of a set of logically equivalent propositional variables. A Horn function f is called *quasi-acyclic* if it admits at least one quasi-acyclic CNF representation.

Note that every acyclic CNF ϕ is quasi-acyclic since each strong component of $\mathbf{G}_\phi$ is a singleton. Also note that every *quadratic* Horn CNF ϕ (i.e. a CNF consisting of clauses of degree at most two) is quasi-acyclic as every strong component of $\mathbf{G}_\phi$ in this case consists of logically equivalent variables.

The name "quasi-acyclic" comes from the property that picking a representative in each set of logically equivalent propositional variables and substituting this representative for all the other logically equivalent variables in the set results in an acyclic CNF (i.e. the CNF is essentially acyclic except for the fact that each variable can have several "names"). In order to understand the structure of quasi-acyclic functions it is important to realize that if f is a quasi-acyclic function and x, y are propositional variables from the same strong component of $\mathbf{G}_f$ then both $\overline{x} \vee y$ and $\overline{y} \vee x$ are implicates of f. Hence no prime pure Horn implicate of f with degree three or more may contain a subgoal from the same strong component of $\mathbf{G}_f$ as the head. This means that the pure Horn clauses in any prime CNF representation of f can be partitioned into two groups. The first group (let us call it group A) contains clauses where all the subgoals are in different strong component(s) of $\mathbf{G}_f$ than the head, while the second group (group B) contains quadratic clauses with both the subgoal and the head belonging to the same strong component of $\mathbf{G}_f$. Loosely speaking, the clauses in group B "generate" the strong components of $\mathbf{G}_f$ while the clauses in group A "generate" its acyclic condensation. It was proved in [11] that similarly to the acyclic functions, a shortest CNF representation (i.e a representation with the minimum possible number of clauses) of a given quasi-acyclic function can be found in polynomial time.

Let us now leave the class of Horn functions and return (for the next three sections) to general Boolean functions.

4 Properties of resolution closures

In this section we prove several simple properties of resolution closures of sets of clauses and of CNF representations of Boolean functions. In the remainder of this section let us consider an arbitrary but fixed Boolean function f, the set $\mathcal{I}^p(f)$ of all prime implicates of f, and the set $\mathcal{I}(f) = \mathcal{R}(\mathcal{I}^p(f))$ of all implicates of f that can be generated from the prime implicates of f by series of resolutions. Note that $\mathcal{I}(f) = \mathcal{R}(\mathcal{I}(f))$, i.e. the set $\mathcal{I}(f)$ is closed under resolution.

Lemma 4.1 *If $\mathcal{A} \subseteq \mathcal{B}$ are arbitrary sets of clauses, then we have $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}(\mathcal{B})$ and $\mathcal{R}(\mathcal{R}(\mathcal{A})) = \mathcal{R}(\mathcal{A})$.*

Proof. Immediate by the definition of the resolution closure. ∎

Lemma 4.2 *A set of clauses $\mathcal{C} \subseteq \mathcal{I}(f)$ represents the function f if and only if $\mathcal{I}(f) = \mathcal{R}(\mathcal{C})$.*

Proof. Assume first that $\mathcal{C} \subseteq \mathcal{I}(f)$ represents f. Then by Theorem 2.3 we have $\mathcal{I}^p(f) \subseteq \mathcal{R}(\mathcal{C})$, and thus $\mathcal{I}(f) = \mathcal{R}(\mathcal{I}^p(f)) \subseteq \mathcal{R}(\mathcal{R}(\mathcal{C})) = \mathcal{R}(\mathcal{C}) \subseteq \mathcal{R}(\mathcal{I}(f)) = \mathcal{I}(f)$ follows by Lemma 4.1.

For the reverse direction let us assume that $\mathcal{I}(f) = \mathcal{R}(\mathcal{C})$. That means that starting with the set $\mathcal{C}$, resolution generates all (prime) implicates of f, which implies $\phi(\mathcal{C}) \leq f$. However, $\mathcal{C} \subseteq \mathcal{R}(\mathcal{C}) = \mathcal{I}(f)$ also implies $\phi(\mathcal{C}) \geq f$ which concludes the proof. ∎

Lemma 4.3 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two sets of clauses. Then $\mathcal{R}(\mathcal{C}_1) = \mathcal{R}(\mathcal{C}_2)$ implies that $\phi(\mathcal{C}_1) \equiv \phi(\mathcal{C}_2)$, i.e. if the sets have the same resolution closure then they represent the same function.*

Proof. By Lemma 2.2, adding a resolvent of two clauses present in a given CNF to this CNF does not change the function being represented. Hence $\phi(\mathcal{C}) \equiv \phi(\mathcal{R}(\mathcal{C}))$ holds for any set $\mathcal{C}$ of clauses. Therefore $\mathcal{R}(\mathcal{C}_1) = \mathcal{R}(\mathcal{C}_2)$ implies $\phi(\mathcal{C}_1) \equiv \phi(\mathcal{R}(\mathcal{C}_1)) = \phi(\mathcal{R}(\mathcal{C}_2)) \equiv \phi(\mathcal{C}_2)$ which completes the proof. ∎

Let us note that despite the similarity to Lemma 4.2 in the above statement, it is not an equivalence. This is due to the fact that if we do not assume $\mathcal{C} \subseteq \mathcal{I}(f)$, then we can still have $\phi(\mathcal{C}) = f$ while $\mathcal{R}(\mathcal{C}) \neq \mathcal{I}(f)$. For instance, we have $(x \vee y) \wedge x \equiv x$, but the resolution closures of these two CNF-s are not the same, obviously.

The following two simple technical lemmas will be useful in our proofs later.

Lemma 4.4 *Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{I}(f)$ be two arbitrary sets of clauses. Then $\mathcal{R}(\mathcal{V} \cup \mathcal{W}) = \mathcal{R}(\mathcal{V} \cup \mathcal{R}(\mathcal{W}))$.*

Proof. The inclusion $\mathcal{R}(\mathcal{V} \cup \mathcal{W}) \subseteq \mathcal{R}(\mathcal{V} \cup \mathcal{R}(\mathcal{W}))$ follows by Lemma 4.1, since $\mathcal{V} \cup \mathcal{W} \subseteq \mathcal{V} \cup \mathcal{R}(\mathcal{W})$ by the definition of the resolution closure. To prove the opposite inclusion, let us assume that $C \in \mathcal{R}(\mathcal{V} \cup \mathcal{R}(\mathcal{W}))$. Then, any $C' \in \mathcal{R}(\mathcal{W})$ used in the derivation of C from the set $\mathcal{V} \cup \mathcal{R}(\mathcal{W})$ by a series of resolutions is, by assumption, itself derivable from the set $\mathcal{W}$, and hence $C \in \mathcal{R}(\mathcal{V} \cup \mathcal{W})$ holds. ∎

Let us close out this section with a lemma which roughly says the following: replacing a set of clauses $\mathcal{E}$ by its subset $\mathcal{Q}$ does not change the function if the resolution closures of $\mathcal{E}$ and $\mathcal{Q}$ are the same.

Lemma 4.5 *Let $\mathcal{C} \subseteq \mathcal{I}(f)$ and $\mathcal{Q} \subseteq \mathcal{E} \subseteq \mathcal{R}(\mathcal{C})$ be sets of clauses such that $\mathcal{R}(\mathcal{Q}) = \mathcal{R}(\mathcal{E})$. Then $\mathcal{R}((\mathcal{C} \setminus \mathcal{E}) \cup \mathcal{Q}) = \mathcal{R}(\mathcal{C})$.*

Proof. Using Lemma 4.4 with $\mathcal{V} = \mathcal{C} \setminus \mathcal{E}$ twice (first with $\mathcal{W} = \mathcal{Q}$ and then with $\mathcal{W} = \mathcal{E}$) we get

$$\mathcal{R}((\mathcal{C} \setminus \mathcal{E}) \cup \mathcal{Q}) = \mathcal{R}((\mathcal{C} \setminus \mathcal{E}) \cup \mathcal{R}(\mathcal{Q})) = \mathcal{R}((\mathcal{C} \setminus \mathcal{E}) \cup \mathcal{R}(\mathcal{E})) = \mathcal{R}((\mathcal{C} \setminus \mathcal{E}) \cup \mathcal{E}) = \mathcal{R}(\mathcal{C})$$

which is the stated result. ∎

5 Exclusive sets and exclusive components of functions

As in the previous section let us consider throughout this section an arbitrary but fixed Boolean function f and the sets of clauses $\mathcal{I}^p(f)$ and $\mathcal{I}(f)$ associated with it. Let us now define the first of the two key concepts of this paper.

Definition 5.1 *Given a set $\mathcal{C}$ of clauses, a subset $\mathcal{X} \subseteq \mathcal{C}$ is called an* exclusive subset of $\mathcal{C}$ *if for every pair of resolvable clauses $C_1, C_2 \in \mathcal{C}$ the following implication holds:*

$$R(C_1, C_2) \in \mathcal{X} \Longrightarrow C_1 \in \mathcal{X} \text{ and } C_2 \in \mathcal{X},$$

i.e. the resolvent belongs to $\mathcal{X}$ only if both parent clauses are in $\mathcal{X}$. In particular, if $\mathcal{C} = \mathcal{I}(f)$ for a Boolean function f, we call such a subset $\mathcal{X}$ an exclusive set of clauses of f (or simply an exclusive set, if f or $\mathcal{C}$ is clear from the context).

Note that the above definition trivially implies that $\mathcal{C}$ itself is the largest (with respect to inclusion) exclusive subset of $\mathcal{C}$ (for any $\mathcal{C}$), and in particular $\mathcal{I}(f)$ is by definition the largest exclusive set of clauses of f. Let us first claim in the next lemma some simple properties possessed by exclusive sets. Since all these properties follow directly from Definition 5.1 we shall omit the proofs.

Lemma 5.2 *Let $\mathcal{C}$ be an arbitrary set of clauses. Then,*

(a) if $\mathcal{A}$ is an exclusive subset of $\mathcal{B}$ and $\mathcal{B}$ is an exclusive subset of $\mathcal{C}$, then $\mathcal{A}$ is an exclusive subset of $\mathcal{C}$;

(b) if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, and $\mathcal{A}$ is an exclusive subset of $\mathcal{C}$, then it is also an exclusive subset of $\mathcal{B}$;

(c) if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ are both exclusive subsets of $\mathcal{C}$, then $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B}$ are also exclusive (and hence all exclusive subsets of $\mathcal{C}$ form a lattice). ■

To see an interesting example of exclusive sets of clauses, let us for a moment return to Horn functions. Let h be a Horn function and let us partition the set $\mathcal{I}(h)$ into two subsets $\mathcal{I}(h) = \mathcal{H} \cup \mathcal{N}$ where $\mathcal{H}$ is the set of all pure Horn clauses in $\mathcal{I}(h)$ and $\mathcal{N}$ is the set of all negative clauses in $\mathcal{I}(h)$[2]. Then it is not hard to see that $\mathcal{H}$ is an exclusive set of h (the resolvent is in $\mathcal{H}$ only if both parent clauses are in $\mathcal{H}$).

The partition $\mathcal{I}(h) = \mathcal{H} \cup \mathcal{N}$ has some important properties, shown in [7]). The first such property states that if ϕ_1 and ϕ_2 are two distinct prime CNFs representing h, then the pure Horn parts of ϕ_1 and ϕ_2 (i.e. the conjunctions of all pure Horn clauses in the given CNFs) also represent the same pure Horn function, called in [7] the *pure Horn component* of h.

[2]It is left to the reader to verify the easy fact that $\mathcal{H}$ and $\mathcal{N}$ indeed constitute a partition of $\mathcal{I}(h)$, i.e. that no clause which is neither pure Horn nor negative can appear in $\mathcal{I}(h)$ (recall that each prime implicate of a Horn function is either pure Horn or negative.)

Proposition 5.3 ([7]) *Let ϕ_1 and ϕ_2 be two distinct prime CNFs of a Horn function h. Then $\phi(\mathcal{C}(\phi_1) \cap \mathcal{H}) \equiv \phi(\mathcal{C}(\phi_2) \cap \mathcal{H})$.*

The proof of this proposition is based on the above mentioned fact that the only way how to generate a pure Horn clause by resolution is to use two pure Horn clauses as the parent clauses. Using the just defined terminology, the proof rests on the fact that pure Horn clauses constitute an excluive set of h. Thus it is obviously very tempting to generalize the result to all exclusive sets. However, first we need to state a simple lemma.

Lemma 5.4 *Let $\mathcal{X} \subseteq \mathcal{I}(f)$ be an exclusive set of clauses (of f) and $\mathcal{C} \subseteq \mathcal{I}(f)$ be a set of clauses such that $\mathcal{X} \subseteq \mathcal{R}(\mathcal{C})$. Then $\mathcal{R}(\mathcal{X}) = \mathcal{R}(\mathcal{C} \cap \mathcal{X})$.*

Proof. The fact that $\mathcal{X}$ is exclusive means that no clause in $\mathcal{C} \setminus \mathcal{X}$ can appear as a parent clause in a resolution leading to a resolvent in $\mathcal{X}$. That, together with the inclusion $\mathcal{X} \subseteq \mathcal{R}(\mathcal{C})$, implies that $\mathcal{X} \subseteq \mathcal{R}(\mathcal{C} \cap \mathcal{X})$ must hold, which in turn implies $\mathcal{R}(\mathcal{X}) \subseteq \mathcal{R}(\mathcal{R}(\mathcal{C} \cap \mathcal{X})) = \mathcal{R}(\mathcal{C} \cap \mathcal{X})$ by Lemma 4.1. The reverse inclusion $\mathcal{R}(\mathcal{C} \cap \mathcal{X}) \subseteq \mathcal{R}(\mathcal{X})$ follows also from Lemma 4.1, since $\mathcal{C} \cap \mathcal{X} \subseteq \mathcal{X}$ holds trivially. ■

We are now ready to generalize Proposition 5.3 to all exclusive sets.

Theorem 5.5 *Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{I}(f)$ be two distinct sets of clauses such that $\phi(\mathcal{C}_1) \equiv \phi(\mathcal{C}_2) \equiv f$, i.e. such that both sets represent f, and let $\mathcal{X} \subseteq \mathcal{I}(f)$ be an exclusive set of clauses. Then $\phi(\mathcal{C}_1 \cap \mathcal{X}) \equiv \phi(\mathcal{C}_2 \cap \mathcal{X})$.*

Proof. Since both $\mathcal{C}_1$ and $\mathcal{C}_2$ represent f, we have $\mathcal{R}(\mathcal{C}_1) = \mathcal{R}(\mathcal{C}_2) = \mathcal{I}(f)$ by Theorem 2.3, and hence $\mathcal{X} \subseteq \mathcal{R}(\mathcal{C}_1) = \mathcal{R}(\mathcal{C}_2)$. Now by Lemma 5.4 we get $\mathcal{R}(\mathcal{X}) = \mathcal{R}(\mathcal{C}_1 \cap \mathcal{X}) = \mathcal{R}(\mathcal{C}_2 \cap \mathcal{X})$ which then implies $\phi(\mathcal{C}_1 \cap \mathcal{X}) \equiv \phi(\mathcal{C}_2 \cap \mathcal{X})$ by Lemma 4.3. ■

It is immediate to see that Proposition 5.3 is just a special case of Theorem 5.5. We can also generalize the notion of a "pure Horn component".

Definition 5.6 *Let f be an arbitrary Boolean function, $\mathcal{X} \subseteq \mathcal{I}(f)$ be an exclusive set of clauses of f, and $\mathcal{C} \subseteq \mathcal{I}(f)$ be a set of clauses which represents f (i.e. $\phi(\mathcal{C}) \equiv f$). The Boolean function $f_{\mathcal{X}}$ represented by the set $\mathcal{C} \cap \mathcal{X}$ is called the $\mathcal{X}$-*component *of the function f. We shall simply call a function g an* exclusive component of f*, if $g = f_{\mathcal{X}}$ for some exclusive subset $\mathcal{X} \subseteq \mathcal{I}(f)$.*

Theorem 5.5 guarantees that the $\mathcal{X}$-component $f_{\mathcal{X}}$ is well defined for every exclusive set $\mathcal{X} \subseteq \mathcal{I}(f)$. Let us now briefly return to Proposition 5.3. It has the following consequence: given a Horn CNF one can extract the pure Horn sub-CNF which represents the pure Horn component, find its shortest CNF representation, and then insert this new sub-CNF back into the input CNF. This is exactly how the minimization procedure for acyclic and quasi-acyclic functions works. A similar (but more general) consequence can be drawn from Theorem 5.5.

Corollary 5.7 *Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{I}(f)$ be two distinct sets of clauses such that $\phi(\mathcal{C}_1) \equiv \phi(\mathcal{C}_2) \equiv f$, i.e. such that both sets represent f, and let $\mathcal{X} \subseteq \mathcal{I}(f)$ be an exclusive set of clauses. Then $\phi((\mathcal{C}_1 \setminus \mathcal{X}) \cup (\mathcal{C}_2 \cap \mathcal{X})) \equiv f$.*

Proof. Similarly as in the proof of Theorem 5.5 we get $\mathcal{X} \subseteq \mathcal{R}(\mathcal{C}_1) = \mathcal{R}(\mathcal{C}_2)$ and also $\mathcal{R}(\mathcal{X}) = \mathcal{R}(\mathcal{C}_1 \cap \mathcal{X}) = \mathcal{R}(\mathcal{C}_2 \cap \mathcal{X})$. Now using Lemma 4.5 with $\mathcal{Q} = \mathcal{C}_2 \cap \mathcal{X}$, $\mathcal{E} = \mathcal{X}$, and $\mathcal{C} = \mathcal{C}_1$ (note that $\mathcal{C}_2 \cap \mathcal{X} \subseteq \mathcal{X} \subseteq \mathcal{R}(\mathcal{C}_1)$ and so the assumptions of Lemma 4.5 are satisfied) we obtain

$$\mathcal{R}((\mathcal{C}_1 \setminus \mathcal{X}) \cup (\mathcal{C}_2 \cap \mathcal{X})) = \mathcal{R}(\mathcal{C}_1)$$

which together with the fact $\mathcal{R}(\mathcal{C}_1) = \mathcal{R}(\mathcal{C}_2) = \mathcal{I}(f)$ completes the proof by Lemma 4.3. ■

The algorithmic meaning of Corollary 5.7 can be stated as follows. If $\mathcal{C}_1$ is the input CNF, one can extract the sub-CNF $\mathcal{C}_1 \cap \mathcal{X}$ which represents the $\mathcal{X}$-component $f_{\mathcal{X}}$, find its shortest CNF representation (say $\mathcal{C}_2 \cap \mathcal{X}$), and then insert this new sub-CNF back into the input CNF. That suggests a decomposition approach for minimization algorithms. Whenever one can find an exclusive subset of clauses of a given function or several pairwise disjoint exclusive subsets of clauses of a given function, it is possible to decompose the minimization problem, solve the subproblems separately, and then compose the obtained solutions back together.

Let us close this section by a simple corollary about certain redundant sets of clauses. A clause $C \in \mathcal{I}(f)$ is called *redundant* with respect to a function f if C does not appear in any irredundant CNF representation $\mathcal{C} \subseteq \mathcal{I}(f)$ of f (in other words, $C \notin \mathcal{C}$ for any minimal $\mathcal{C} \subseteq \mathcal{I}(f)$ such that $\mathcal{R}(\mathcal{C}) = \mathcal{I}(f)$). A set $\mathcal{S} \subseteq \mathcal{I}(f)$ of clauses is called *redundant* with respect to a function f if every clause in $\mathcal{S}$ is redundant with respect to f, i.e., if $\mathcal{S} \cap \mathcal{C} = \emptyset$ for every irredundant representation $\mathcal{C} \subseteq \mathcal{I}(f)$ of f.

Corollary 5.8 *For every exclusive set $\mathcal{X} \subseteq \mathcal{I}(f)$ we have $\mathcal{R}(\mathcal{X}) = \mathcal{I}(f_{\mathcal{X}})$, furthermore the set $\mathcal{R}(\mathcal{X}) \setminus \mathcal{X}$ is redundant with respect to $f_{\mathcal{X}}$, as well as with respect to f.*

Proof. By Lemma 5.4 we have $\mathcal{R}(\mathcal{X}) = \mathcal{R}(\mathcal{X} \cap \mathcal{C})$ for every irredundant representation $\mathcal{C} \subseteq \mathcal{I}(f)$ of f, and hence $\mathcal{C} \cap (\mathcal{R}(\mathcal{X}) \setminus \mathcal{X}) = \emptyset$ follows for all such representations by their irredundancy. Furthermore, by Corollary 5.7 and Lemma 4.5, we can choose $\mathcal{C}$ such that $\mathcal{C} \cap \mathcal{X}$ is a prime irredundant representation of $f_{\mathcal{X}}$, from which $\mathcal{I}(f_{\mathcal{X}}) = \mathcal{R}(\mathcal{X} \cap \mathcal{C}) = \mathcal{R}(\mathcal{X})$ follows (again by Lemma 5.4). ■

6 Essential sets and a min-max relation

Let us now introduce the second key concept of this paper, which leads to an interesting conjecture.

Definition 6.1 *Given a set $\mathcal{C}$ of clauses, a subset $\mathcal{E} \subseteq \mathcal{C}$ is called an* essential subset of $\mathcal{C}$ *if for every pair of resolvable clauses $C_1, C_2 \in \mathcal{C}$ the following implication holds:*

$$R(C_1, C_2) \in \mathcal{E} \Longrightarrow C_1 \in \mathcal{E} \text{ or } C_2 \in \mathcal{E},$$

i.e. the resolvent belongs to $\mathcal{E}$ only if at least one of the parent clauses is from $\mathcal{E}$. In particular, if $\mathcal{C} = \mathcal{I}(f)$ for a Boolean function f, we call $\mathcal{E}$ an essential set of clauses of f (or simply an essential set, if f or $\mathcal{C}$ is clear from the context).

It is easy to see that every exclusive set of clauses (and the set $\mathcal{I}(f)$ in particular) is also essential. We summarize in the following lemma a few simple properties of essential sets. Since all these properties follow directly from Definitions 5.1 and 6.1 we shall omit the proofs.

Lemma 6.2 *Let $\mathcal{C}$ be an arbitrary set of clauses. Then,*

(a) if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ are both essential subsets of $\mathcal{C}$, then $\mathcal{A} \cup \mathcal{B}$ is also essential;

(b) if $\mathcal{R}(\mathcal{C}) = \mathcal{C}$ and $\mathcal{A}$ is an essential subset of $\mathcal{C}$, then $\mathcal{C} \setminus \mathcal{A}$ is closed under resolution, i.e. $\mathcal{C} \setminus \mathcal{A} = \mathcal{R}(\mathcal{C} \setminus \mathcal{A})$;

(c) if $\mathcal{R}(\mathcal{A}) = \mathcal{A}$ and $\mathcal{B}$ is an exclusive subset of $\mathcal{C}$, then $\mathcal{B} \setminus \mathcal{A}$ is an essential subset of $\mathcal{C}$;

(d) if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, $\mathcal{A}$ is an essential subset of $\mathcal{B}$, and $\mathcal{B}$ is an exclusive subset of $\mathcal{C}$, then $\mathcal{A}$ is an essential subset of $\mathcal{C}$, as well;

(e) if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$, $\mathcal{A} \cap \mathcal{B} \neq \emptyset$, $\mathcal{A}$ is an essential subset of $\mathcal{C}$, and $\mathcal{B}$ is an exclusive subset of $\mathcal{C}$, then $\mathcal{A} \cap \mathcal{B}$ is also an essential subset of $\mathcal{C}$. ■

To see an interesting example of essential sets, let us consider again a Horn function h and return to the partition of the set $\mathcal{I}(h)$ into two subsets $\mathcal{I}(h) = \mathcal{H} \cup \mathcal{N}$ where $\mathcal{H}$ is the set of all pure Horn clauses in $\mathcal{I}(h)$ and $\mathcal{N}$ is the set of all negative clauses in $\mathcal{I}(h)$. Then, it is not hard to see that $\mathcal{N}$ is essential for h (since no two clauses in $\mathcal{N}$ are resolvable, the resolvent is in $\mathcal{N}$ only if *exactly* one of the parent clauses is in $\mathcal{N}$ and the other one is in $\mathcal{H}$).

A second important property of the partition $\mathcal{I}(h) = \mathcal{H} \cup \mathcal{N}$ states that if ϕ_1 and ϕ_2 are two distinct irredundant CNFs representing h, then ϕ_1 and ϕ_2 both contain the same number of negative clauses.

Proposition 6.3 ([7]) *Let ϕ_1 and ϕ_2 be two distinct irredundant CNFs of a Horn function h. Then $|\mathcal{C}(\phi_1) \cap \mathcal{N}| = |\mathcal{C}(\phi_2) \cap \mathcal{N}|$.*

The proof of this proposition is more or less based on the above mentioned fact that negative clauses form an essential set of h with no resolvable pair, and their complement (i.e. pure Horn clauses) forms an exclusive set of h (of course the original proof did not use this terminology). This observation leads to an obvious idea to generalize the statement of

Proposition 6.3. We will arrive at such a generalization in the end of this section. However, first we will prove a very useful min-max relation, characterizing CNF representations of a given function.

We shall start by proving a key theorem which shows that every essential set has one (or more) of its clauses present in every representation of f and moreover that this condition is not only necessary but also sufficient.

Theorem 6.4 *Let $\mathcal{C} \subseteq \mathcal{I}(f)$ be an arbitrary set of clauses. Then $\mathcal{C}$ represents f if and only if $\mathcal{C} \cap \mathcal{E} \neq \emptyset$ for every nonempty essential set of clauses $\mathcal{E} \subseteq \mathcal{I}(f)$.*

Proof. Let us assume that $\mathcal{C} \subseteq \mathcal{I}(f)$ represents f and $\mathcal{C} \cap \mathcal{E} = \emptyset$ for some nonempty essential set $\mathcal{E} \subseteq \mathcal{I}(f)$. That means that when we start making resolutions from the set $\mathcal{C}$ we can never get into $\mathcal{E}$, i.e. that $\mathcal{R}(\mathcal{C}) \cap \mathcal{E} = \emptyset$. However, $\mathcal{E} \subseteq \mathcal{I}(f) = \mathcal{R}(\mathcal{C})$ by Lemma 4.2, which is a contradiction. Therefore $\mathcal{C} \cap \mathcal{E} \neq \emptyset$ for every nonempty essential set of clauses $\mathcal{E} \subseteq \mathcal{I}(f)$.

To prove the opposite implication let us assume that $\mathcal{C} \subseteq \mathcal{I}(f)$ is a set of clauses which has a nonempty intersection with every nonempty essential set $\mathcal{E} \subseteq \mathcal{I}(f)$. First we shall show that this implies that $\mathcal{C}$ has nonempty intersections with special sets of clauses defined as follows. Let $t \in \{0,1\}^n$ (where n is the number of propositional variables in f) be an arbitrary vector. Then let us define

$$\mathcal{E}(t) = \{C \in \mathcal{I}(f) \mid C(t) = 0\}$$

where by $C(t) = 0$ we mean the following: if we substitute for the propositional variables of f the truth values according to the vector t then the clause C evaluates to zero (false). By definition, $\mathcal{E}(t) \subseteq \mathcal{I}(f)$ for every t. Let us now prove that $\mathcal{E}(t)$ is essential for every truth assignment t. For that let $C_1 = A \vee x$ and $C_2 = B \vee \overline{x} \vee y$ be arbitrary two resolvable clauses from $\mathcal{I}(f)$ such that $R(C_1, C_2) = A \vee B \vee y \in \mathcal{E}(t)$. This means that all literals in the set $A \cup B \cup \{y\}$ evaluate to zero under the truth assignment t, which in turn implies that exactly one of the clauses C_1 and C_2 evaluates to zero, depending on the value assigned to x. Therefore either $C_1 \in \mathcal{E}(t)$ or $C_2 \in \mathcal{E}(t)$ and thus $\mathcal{E}(t)$ is essential. Hence $\mathcal{C} \cap \mathcal{E}(t) \neq \emptyset$ holds for every truth assignment t such that $\mathcal{E}(t)$ is nonempty.

Let us now denote the CNF $\phi(\mathcal{C})$ by ϕ. We want to prove $\phi \equiv f$. The inequality $\phi \geq f$ trivially follows from the fact that $\mathcal{C} \subseteq \mathcal{I}(f)$. Thus it remains to be shown that also $\phi \leq f$ which is equivalent to proving that $f(t) = 0$ implies $\phi(t) = 0$ for every truth assignment t. So let t be an arbitrary assignment such that $f(t) = 0$. This means that there must exist a prime implicate C of f such that $C(t) = 0$. Therefore $\mathcal{E}(t)$ is nonempty and thus, as we have proved above, $\mathcal{E}(t)$ is essential and $\mathcal{C} \cap \mathcal{E}(t) \neq \emptyset$ holds, i.e. $\mathcal{C}$ contains a clause that evaluates to zero under the truth assignment t. However, that implies $\phi(t) = 0$ which completes the proof. ∎

Some of the sets $\mathcal{E}(t)$ used in the proof of Theorem 6.4 play quite an important role in the structure of $\mathcal{I}(f)$. Let us for a moment consider the lattice $\mathcal{L}$ of all subsets of $\mathcal{I}(f)$. Clearly, the property of being a representation of f is monotone in $\mathcal{L}$ (every superset of a representation is again a representation). The minimal sets in $\mathcal{L}$ which represent f are of course exactly all the irredundant representations of f. On the other hand, also the property of not being a representation of f is a monotone one in $\mathcal{L}$ (every subset of a non-representation is again a non-representation). Thus it is of some interest to understand what the maximal non-representations of f in $\mathcal{L}$ are. We summarize the properties of these sets in the following theorem.

Theorem 6.5 *Let $\mathcal{D} \subseteq \mathcal{I}(f)$ be a maximal (under inclusion) set of clauses not representing f. Then $\mathcal{D} = \mathcal{R}(\mathcal{D})$, the set $\mathcal{I}(f) \setminus \mathcal{D}$ is essential, and there exists a Boolean vector t such that $\mathcal{I}(f) \setminus \mathcal{D} = \mathcal{E}(t)$.*

Proof. Let us assume that there exist clauses $C_1, C_2 \in \mathcal{D}$ such that $C = R(C_1, C_2) \notin \mathcal{D}$. By the property of resolution (Lemma 2.2) we have $\phi(\mathcal{D}) = \phi(\mathcal{D} \cup \{C\})$, and hence $\mathcal{D} \cup \{C\}$ still does not represent f, which is a contradiction to the maximality of $\mathcal{D}$. Therefore $\mathcal{D} = \mathcal{R}(\mathcal{D})$ and by Lemma 6.2 part (c) (where we take $\mathcal{A} = \mathcal{D}$ and $\mathcal{B} = \mathcal{C} = \mathcal{I}(f)$) the set $\mathcal{I}(f) \setminus \mathcal{D}$ is essential.

To finish the proof let us consider the function $f' = \phi(\mathcal{D})$. Clearly $f \leq f'$ and since $f \neq f'$ there must exist a Boolean vector t such that $f(t) = 0$ and $f'(t) = 1$. Now consider the set $\mathcal{E}(t)$. Obviously $\mathcal{E}(t) \cap \mathcal{D} = \emptyset$ (all clauses in $\mathcal{D}$ must evaluate to 1 on t in order to make $f'(t) = 1$). So it remains to prove that $\mathcal{E}(t) \cup \mathcal{D} = \mathcal{I}(f)$, or in other words that every clause that evaluates to 1 on t is in $\mathcal{D}$. Assume by contradiction that there exists a clause $C \notin \mathcal{D}$ such that $C \in \mathcal{I}(f)$ and $C(t) = 1$. Clearly $\phi(\mathcal{D} \cup \{C\})(t) = 1$ while $f(t) = 0$ and so the set $\mathcal{D} \cup \{C\}$ still does not represent f, which is again a contradiction to the maximality of $\mathcal{D}$. ■

It follows from the proof of Theorem 6.5 that the maximal non-representations of f in $\mathcal{L}$ can be alternatively characterized as follows.

Corollary 6.6 *Set $\mathcal{D} \subseteq \mathcal{I}(f)$ of clauses is a maximal (under inclusion) set not representing f if and only if $\mathcal{D}$ is a maximal proper subset of $\mathcal{I}(f)$ closed under resolution.*

Let us now return to Theorem 6.4. It has an obvious corollary: if there exist nonempty essential sets $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k \subseteq \mathcal{I}(f)$ which are pairwise disjoint, then every representation of f must consist of at least k clauses. Hence, any collection of pairwise disjoint essential sets of clauses provides an easy lower bound on the size (i.e. on the number of clauses) of a minimal representation of f.

Definition 6.7 *Given a Boolean function f, let us denote by $\epsilon(f)$ the maximum number of pairwise disjoint nonempty essential subsets of $\mathcal{I}(f)$ and by $\sigma(f)$ the minimum number of clauses needed to represent f by a CNF.*

Using this notation, we can now formulate the above noted simple corollary of Theorem 6.4 very succintly as follows:

Corollary 6.8 *For every Boolean function f the following inequality holds*

$$\sigma(f) \geq \epsilon(f).$$

It remains to be seen whether the inequality in Corollary 6.8 is tight, or not. Since we have not found an example for which a strict inequality would hold we state the following conjecture.

Conjecture 6.9 *Let f be an arbitrary Boolean function. Then $\sigma(f) = \epsilon(f)$ holds for f.*

This conjecture may be perhaps too bold due to its generality, but there is some indication that it may be true at least for the class of Horn functions. In the next section we shall prove that Conjecture 6.9 is true for the two subclasses of Horn functions introduced in Section 3, namely for the class of acyclic Horn functions and the class of quasi-acyclic Horn functions.

Let us now come back to Theorem 6.4. The first part of the proof of Theorem 6.4, which shows that for an arbitrary $\mathcal{C}$ representing f and an arbitrary nonempty essential set $\mathcal{E}$, $\mathcal{C} \cap \mathcal{E} \neq \emptyset$ must hold, gives yet another corollary.

Corollary 6.10 *Let $\mathcal{E} \subseteq \mathcal{I}(f)$ be an arbitrary set of clauses. Then $\mathcal{E}$ is a nonempty essential set only if $\mathcal{E} \cap \mathcal{C} \neq \emptyset$ for every representation $\mathcal{C} \subseteq \mathcal{I}(f)$ of the function f.*

There is an obvious duality between Corollary 6.10 and Theorem 6.4 based on the reversal of roles between $\mathcal{E}$ and $\mathcal{C}$. However, unlike in Theorem 6.4 where the equivalence holds, only one implication (the "only if" part) is true in Corollary 6.10. The reason for this is the following. If $\mathcal{C}$ represents f and we add some clauses from $\mathcal{I}(f)$ to $\mathcal{C}$, the new set will still represent f. On the other hand, if $\mathcal{E}$ is an essential set and we add some clauses from $\mathcal{I}(f)$ to $\mathcal{E}$, the resulting set may not be essential. To avoid the presence of such "extra" clauses in $\mathcal{E}$ we shall add a minimality assumption. This minimality assumption simply means that we shall require not only that $\mathcal{E}$ intersect all representations of f (i.e. that $\mathcal{E}$ is a transversal of all representations) but also that $\mathcal{E}$ is a minimal set with this property (i.e. that $\mathcal{E}$ is a minimal transversal). With this additional assumption the reverse implication in Corollary 6.10 (the "if" part) becomes valid as well, making the duality with Theorem 6.4 work both ways (i.e., informally speaking, by the above corollary every essential set forms a transversal of all representations, and by the theorem below every minimal transversal of all representations forms an essential set).

Theorem 6.11 *Let $\mathcal{E}$ be an arbitrary minimal (under inclusion) subset of $\mathcal{I}(f)$ such that $\mathcal{E} \cap \mathcal{C} \neq \emptyset$ for every $\mathcal{C} \subseteq \mathcal{I}(f)$ which represents f. Then $\mathcal{E}$ is an essential set of clauses.*

Proof. Let us assume by contradiction that $\mathcal{E}$ is not essential, i.e. that there exist two resolvable clauses $C_1, C_2 \in \mathcal{I}(f)$ such that $C_1, C_2 \notin \mathcal{E}$ but $C = R(C_1, C_2) \in \mathcal{E}$. Let us consider the set Σ of all representations of f which contain the clause C, i.e. let

$$\Sigma = \{\mathcal{C} \mid \mathcal{R}(\mathcal{C}) = \mathcal{I}(f) \text{ and } C \in \mathcal{C}\}.$$

If every $\mathcal{C} \in \Sigma$ would intersect $\mathcal{E}$ in two or more clauses then we could remove C from $\mathcal{E}$ and still maintain the property that $\mathcal{E}$ intersects all representations of f. However, this would be a contradiction to the minimality of $\mathcal{E}$. Therefore there must exist a representation $\mathcal{C}'$ of

f in the set Σ which intersects $\mathcal{E}$ in exactly one clause, i.e. such that $\mathcal{C}' \cap \mathcal{E} = \{C\}$. Let us now construct a set of clauses

$$\mathcal{C}'' = \mathcal{C}' \setminus \{C\} \cup \{C_1, C_2\}.$$

Clearly $\mathcal{C}'' \subseteq \mathcal{I}(f)$. Moreover, since $C = R(C_1, C_2)$ it is obvious that $\mathcal{R}(\mathcal{C}'') = \mathcal{R}(\mathcal{C}) = \mathcal{I}(f)$, i.e., $\mathcal{C}''$ represents f. However, $\mathcal{C}'' \cap \mathcal{E} = \emptyset$ which is a contradiction to the choice of $\mathcal{E}$. ∎

Let us finish this section by proving a generalization of Proposition 6.3, which also provides a partial evidence for the validity of Conjecture 6.9.

Theorem 6.12 *Given a Boolean function f, let $\mathcal{X} \subseteq \mathcal{I}(f)$ be an exclusive subset of f such that no two clauses from $\mathcal{E} = \mathcal{I}(f) \setminus \mathcal{R}(\mathcal{X})$ are resolvable. Then, there exists an integer $k = k(\mathcal{E}) > 0$, and pairwise disjoint essential subsets $\mathcal{Q}_j \subseteq \mathcal{E}$, $j = 1, .., k$ such that $|\mathcal{Q}_j \cap \mathcal{C}| = 1$ for $j = 1, ..., k$ and $|(\mathcal{E} \setminus \bigcup_{j=1}^{k} \mathcal{Q}_j) \cap \mathcal{C}| = 0$ for any irredundant set $\mathcal{C} \subseteq \mathcal{I}(f)$ of clauses representing f.*

Proof. Let us observe that $\mathcal{E}$ is an essential set by (c) of Lemma 6.2. Furthermore, the property that no two clauses of the essential family $\mathcal{E}$ are resolvable implies that if $R(A, B) \in \mathcal{E}$ for some resolvable clauses $A, B \in \mathcal{I}(f)$, then exactly one of these clauses belongs to $\mathcal{E}$.

Let us then first define a directed graph $\mathbf{H}$, the vertices of which are the clauses in $\mathcal{E}$, and where (A, B) is a directed arc for $A, B \in \mathcal{E}$ if and only if $B \in \mathcal{R}(\mathcal{X} \cup \{A\})$. Let us next consider a strong component $\mathcal{Q} \subseteq \mathcal{E}$ of $\mathbf{H}$, which is an initial component, i.e., for which there exists no arc (A, B) of $\mathbf{H}$ such that $A \in \mathcal{E} \setminus \mathcal{Q}$ and $B \in \mathcal{Q}$. We claim that $\mathcal{Q}$ is an essential set of f. To see this, let us consider a pair of resolvable clauses $A, B \in \mathcal{I}(f)$ for which $C = R(A, B) \in \mathcal{Q}$. Since $C \in \mathcal{Q} \subseteq \mathcal{E}$ and since $\mathcal{E}$ is essential with no two of its clauses resolvable, we must have exactly one of A and B belong to $\mathcal{E}$, as we observed above. Say, we have $A \in \mathcal{E}$ and $B \in \mathcal{R}(\mathcal{X})$. Then, we have $C \in \mathcal{R}(\mathcal{X} \cup \{A\})$, and thus by the definition of $\mathbf{H}$ we must have (A, C) as an arc of $\mathbf{H}$. Since we assumed that $\mathcal{Q}$ is an initial set, with no arcs entering it, we must have $A \in \mathcal{Q}$, showing that $\mathcal{Q}$ is indeed essential.

Let us next consider all the initial strong components $\mathcal{Q}_1$, ..., $\mathcal{Q}_k$ of $\mathbf{H}$. We claim that for any irredundant representation $\mathcal{C}$ of f we must have $|\mathcal{C} \cap \mathcal{Q}_j| = 1$ for $j = 1, ..., k$ and $|\mathcal{C} \cap \mathcal{E}| = k$, from which the statement readily follows.

To see this, let us observe first that for all subsets $\mathcal{C} \subseteq \mathcal{I}(f)$ representing f we must have $\mathcal{C} \cap \mathcal{Q}_i \neq \emptyset$ for $i = 1, ..., k$ by Theorem 6.4, since all the sets $\mathcal{Q}_i$, $i = 1, ..., k$ are essential, as we observed above. Let us fix now an irredundant representation $\mathcal{C}$ of f, and let us choose clauses $C_j \in \mathcal{Q}_j \cap \mathcal{C}$ for $j = 1, ..., k$, arbitrarily. Since $\mathcal{X}$ is exclusive we have $\mathcal{R}(\mathcal{X}) = \mathcal{R}(\mathcal{X} \cap \mathcal{C})$ by Lemma 5.4. Furthermore, since $\mathbf{H}$ is transitively closed, every clause $C \in \mathcal{E}$ is reachable by an arc from the set $\{C_1, ..., C_k\}$, implying $\mathcal{E} \subseteq \mathcal{R}(\mathcal{X} \cup \{C_1, ..., C_k\})$. Thus, by applying Lemma 4.4 we get $\mathcal{I}(f) = \mathcal{E} \cup \mathcal{R}(\mathcal{X}) \subseteq \mathcal{R}(\mathcal{X} \cup \{C_1, ..., C_k\}) = \mathcal{R}((\mathcal{X} \cap \mathcal{C}) \cup \{C_1, ..., C_k\}) \subseteq \mathcal{R}(\mathcal{C}) = \mathcal{I}(f)$, implying that $(\mathcal{X} \cap \mathcal{C}) \cup \{C_1, ..., C_k\} \subseteq \mathcal{C}$ is a representation of f. Since $\mathcal{C}$ is assumed to be irredundant, equality follows, implying $\mathcal{C} \setminus \mathcal{X} = \mathcal{C} \cap \mathcal{E} = \{C_1, ..., C_k\}$, from which the claim follows. ∎

Of course, it is clear that Theorem 6.12 implies the following corollary which more closely resembles the statement of Proposition 6.3.

Corollary 6.13 *Let f, $\mathcal{X}$, and $\mathcal{E}$ be as in the statement of Theorem 6.12, and let ϕ_1 and ϕ_2 be two distinct irredundant CNFs of f. Then $|\mathcal{C}(\phi_1) \cap \mathcal{E}| = |\mathcal{C}(\phi_2) \cap \mathcal{E}|$.*

Clearly, Proposition 6.3 is just a special case of Corollary 6.13 if we set $\mathcal{X}$ to be the set of all pure Horn clauses in $\mathcal{I}(f)$ (in this case $\mathcal{X} = \mathcal{R}(\mathcal{X})$) and $\mathcal{E}$ to be the set of all negative clauses in $\mathcal{I}(f)$.

Note that not every essential set $\mathcal{E}$ of implicates with no resolution inside has the properties claimed in Theorem 6.12. In other words, the condition that the complement of $\mathcal{E}$ is a resolution closure of an exclusive set cannot be neglected. A good example for this observation is the following: given any function f, its negative implicates in $\mathcal{I}(f)$ obviously form an essential set with no resolution inside. However, if f is not Horn, it may happen that the non-negative implicates in $\mathcal{I}(f)$ do not form a resolution closure of an exclusive set, and the properties claimed in Theorem 6.12 fail to hold. To see this, consider the following two CNFs:

$$\mathcal{C}_1 = (x \vee z)(\overline{x} \vee \overline{z})(x \vee \overline{y}),$$

$$\mathcal{C}_2 = (x \vee z)(\overline{x} \vee \overline{z})(\overline{y} \vee \overline{z}).$$

It is not hard to verify that they represent the same function and both are irredundant, but while $\mathcal{C}_1$ contains one negative clause, $\mathcal{C}_2$ contains two negative clauses.

7 Disjoint essential sets for Horn functions

In this section we shall restrict our attention to Horn functions only, in particular to the subclasses of quadratic Horn, acyclic Horn, and quasi-acyclic Horn functions. We shall show that Conjecture 6.9 is true for all of the mentioned subclasses of Horn functions. We shall proceed as follows: after some simple preprocessing (getting rid of unit implicates) we shall use Theorem 6.12 to show that we can in fact concentrate only on pure Horn functions. Then we shall prove Conjecture 6.9 for quadratic pure Horn and acyclic pure Horn functions. Finally, we will use a combination of the last two proofs to verify Conjecture 6.9 for quasi-acyclic pure Horn functions.

By standard Boolean terminology, a *unit* clause is a clause consisting of exactly one literal. If x or $\overline{x}$ is a unit prime implicate of a Boolean function f, then clearly no other prime implicates of f may contain the variable x (negated or not). This implies that also in $\mathcal{I}(f)$ the variable x appears only in the unit clause and nowhere else, which in turn means that this clause constitutes a trivial exclusive (and hence also essential) set (it cannot be derived by resolution from any other clauses in $\mathcal{I}(f)$.

It follows that any Horn function f can be decomposed into a conjunction of unit clauses f_1 and a Horn function f_2 which has no unit prime implicates, in such a way that f_1 and f_2

are defined on disjoint sets of variables, and $f = f_1 \wedge f_2$. Of course, Conjecture 6.9 can be verified independently for f_1 and f_2 (due to the disjointness of their sets of variables) and it trivially holds for f_1 by the above considerations. Therefore we can from now on restrict our attention (without loss of generality) solely to Horn functions with no unit prime implicates.

Let h be a Horn function and $\mathcal{C} \subseteq \mathcal{I}(h)$ a minimum (and therefore irredundant) set of clauses representing h such that $C_1, C_2, \dots, C_k$ are all the negative clauses in $\mathcal{C}$. Then Theorem 6.12 guarantees the existence of pairwise disjoint essential sets of negative implicates $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_k$ such that $C_j \in \mathcal{Q}_j$ for $j = 1, ..., k$. Furthermore, if p is the pure Horn component of h (which is represented by the pure Horn clauses in C by Definition 5.6) then $\mathcal{I}(p)$ consists only of pure Horn clauses (since all prime implicates of a pure Horn function are pure Horn and a resolution of two pure Horn clauses is also pure Horn as recalled in Section 3), and thus $\mathcal{Q}_j \cap \mathcal{E} = \emptyset$ for every $j = 1, ..., k$ and every subset $\mathcal{E}$ of $\mathcal{I}(p)$. This implies that if we manage to prove Conjecture 6.9 for p by exhibiting pairwise disjoint essential subsets $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_\ell$ of $\mathcal{I}(p)$, where ℓ is the number of pure Horn clauses in $\mathcal{C}$, then the sets $\mathcal{E}_i$, $i = 1, ..., \ell$, together with the sets $\mathcal{Q}_j$, $j = 1, ..., k$, prove Conjecture 6.9 also for h.

Therefore we shall from now on restrict our attention to pure Horn functions (with no unit prime implicates) only. Let us start by introducing a very useful technique for verifying that a given clause is an implicate of a given pure Horn function.

Let η be a pure Horn CNF of a pure Horn function h. We shall define a *forward chaining* procedure which associates to any subset Q of the propositional variables of h a set M in the following way. The procedure takes as input the subset Q of propositional variables, initializes the set $M = Q$, and at each step it looks for a pure Horn clause $S \vee y$ in η such that $S \subseteq M$, and $y \notin M$. If such a clause is found, the propositional variable y is included into M, and the search is repeated as many times as possible.

In the relational database terminology the propositional variables in M are said to be "chained" to the subset Q (see e.g. [5]). In the expert systems terminology the usage of the clause $S \vee y$ is called "firing the rule" $\bigwedge_{x \in S} x \to y$ (see e.g. [8]).

FORWARD CHAINING PROCEDURE($\mathcal{C}, Q$)

Input:	A set $\mathcal{C}$ of pure Horn clauses, and a subset Q of propositional variables.
Initialization:	**Set** $M = Q$.
Main Step:	**While** $\exists\, C \in \mathcal{C} : \mathit{Subg}(C) \subseteq M$ and $\mathit{Head}(C) \notin M$ **do** $M = M \cup \{\mathit{Head}(C)\}$.
Stop:	**Output** $FC_{\mathcal{C}}(Q) = M$.

If the forward chaining procedure subsequently uses clauses $C_1, \dots, C_k$ (in this order) to enlarge the set M (starting with set Q), we say that the sequence of clauses $C_1, \dots, C_k$ forms a *forward chaining derivation* of $Head(C_k)$ from Q. The sequence is called an *irredundant* forward chaining derivation of $Head(C_k)$ from Q, if no proper subsequence of $C_1, \ldots, C_k$ forms a forward chaining derivation of $Head(C_k)$ from Q. The following lemma, proved in [9], shows how the above procedure can help in determining whether a given clause is an implicate of a given CNF, or not.

Lemma 7.1 *Given a set $\mathcal{C}$ of pure Horn clauses, a subset Q of its propositional variables, and another its variable y, we have $y \in FC_{\mathcal{C}}(Q)$ if and only if $Q \vee y \in \mathcal{R}(\mathcal{C})$.*

In what follows we will frequently not distinguish between CNFs and their sets of clauses, and thus for $\mathcal{C} = \mathcal{C}(\eta)$ we shall write both $FC_\eta(Q) = FC_{\mathcal{C}}(Q)$. If η' and η'' are two distinct CNF representations of a given pure Horn function h and if Q is an arbitrary subset of the propositional variables, then by Lemma 7.1 $FC_{\eta'}(Q) = FC_{\eta''}(Q)$ because η' and η'' have the same set of implicates. Therefore, the set of propositional variables reachable from Q by forward chaining depends only on the underlying function rather than on a particular CNF representation. For this reason, we shall also use the expression $FC_h(Q)$ instead of $FC_\eta(Q)$ whenever we do not want to refer to a specific CNF.

Now let us return to essential sets of implicates. A key role in the upcoming proofs will be played by essential sets which were used in the proof of Theorem 6.4. Given a Boolean function f on n variables and a Boolean vector $t \in \{0,1\}^n$, we shall denote by $\mathcal{E}(t)$ the following set:

$$\mathcal{E}(t) = \{C \in \mathcal{I}(f) \,|\, C(t) = 0\}.$$

In the proof of Theorem 6.4 we have already seen that each such set $\mathcal{E}(t)$ is essential (it may be empty). Using these sets we can show the following easy observation.

Lemma 7.2 *Let f be a Boolean function and let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\} \subseteq \mathcal{I}(f)$ be an irredundant set of clauses representing f. Then for each $i = 1, \ldots, m$ there exists an essential set $\mathcal{E}_i$, for which $\mathcal{E}_i \cap \mathcal{C} = \{C_i\}$.*

Proof. Take $i \in \{1, \ldots, m\}$ arbitrarily. Since $\mathcal{C}$ is irredundant, there exists at least one Boolean vector t such that $C_i(t) = 0$ and $C_j(t) = 1$ for all $j \neq i$. Thus $\mathcal{E}_i$ can be set to $\mathcal{E}(t)$. ■

It is clear that the sets $\mathcal{E}_i$, $i = 1, \ldots, m$, in the above proof need not be disjoint. However, we shall show that if f is a quadratic pure Horn, an acyclic pure Horn, or a quasi-acyclic pure Horn function and $\mathcal{C}$ is its minimum representation, then we can find Boolean vectors $t_1, \ldots, t_m$ such that the sets $\mathcal{E}_i = \mathcal{E}(t_i)$, $i = 1, \ldots, m$ in the above proof are pairwise disjoint, proving Conjecture 6.9 for f.

In the remainder of this section we shall frequently use the implication graph $\mathbf{G}_f = (\mathbf{N}, \mathbf{A})$ of f defined in Section 3 (see Definition 3.3). Let us introduce some further notation. By K_x we shall denote that strong component of $\mathbf{G}_f$ which contains variable x. Given a topological order τ of strong components of $\mathbf{G}_f$, we shall denote by $K_x <_\tau K_y$ ($K_x \leq_\tau K_y$ resp.) that K_x is strictly less (less or equal resp.) than K_y in the order τ. Note that by the definition of topological order, the existence of an arc in $\mathbf{G}_f$ from some variable in K_x to some variable in K_y implies $K_x \leq_\tau K_y$. Finally, for a Boolean vector t (a truth assignment to the variables of f) and a variable x, $t[x]$ will denote the element of t which corresponds to x. Now we are ready to prove some useful properties connecting forward chaining and implication graphs.

Lemma 7.3 *Let $v \in FC_f(S)$ and let clauses $C_1, \ldots, C_k \subseteq \mathcal{I}(f)$ form an irredundant forward chaining derivation of v from S. Let x be an arbitrary variable used in clause C_i for some $1 \leq i \leq k$. Then either $x = v$ or (x, v) is an arc in $\mathbf{G}_f$.*

Proof. By the definition of an irredundant forward chaining derivation it follows that $\forall 1 \leq i \leq k-1 \,:\, Head(C_i) \in \bigcup_{j=i+1}^{k} Subg(C_j)$ and that $Head(C_k) = v$. By Lemma 3.4 we know that for every C_i each of its subgoals is connected by an arc in $\mathbf{G}_f$ to its head. A simple inductive argument going backward from C_k to C_i then shows that x is connected to v by a directed path (possibly of length zero if $x = Head(C_k) = v$) in $\mathbf{G}_f$, which together with the fact that $\mathbf{G}_f$ is a transitively closed graph finishes the proof. ■

Lemma 7.4 *Let $\mathcal{C}$ be an irredundant and prime representation of a pure Horn function f. Let $C = A \vee x \in \mathcal{C}$ and $D = B \vee y \in \mathcal{I}(f)$. If $y \notin FC_{\mathcal{C}\setminus\{C\}}(B)$, then $A \subseteq FC_{\mathcal{C}\setminus\{C\}}(B)$ and either $x = y$ or (x, y) is an arc in $\mathbf{G}_f$.*

Proof. Since $D \in \mathcal{I}(f)$, Lemma 7.1 guarantees that $y \in FC_{\mathcal{C}}(B)$. Hence there exists an irredundant forward chaining derivation $C_1, \ldots, C_k$ of y from B using clauses from $\mathcal{C} \subseteq \mathcal{I}(f)$. Moreover, since $y \notin FC_{\mathcal{C}\setminus\{C\}}(B)$, each derivation of y from B must use C and thus $C_i = C$ for some $1 \leq i \leq k$ which directly implies $A \subseteq FC_{\mathcal{C}\setminus\{C\}}(B)$, and using Lemma 7.3 it also implies that either $x = y$ or (x, y) is an arc in $\mathbf{G}_f$. ■

Lemma 7.5 *Let $\mathcal{C}$ be an irredundant and prime representation of a pure Horn function f, and let A,B be two sets of variables of f such that $A \subseteq FC_{\mathcal{C}}(B)$. Furthermore, let $A' \subseteq A$. Then $A' \subseteq FC_{\mathcal{C}}(B')$, where $B' = B \cap (A' \cup \{x \mid \exists a \in A' : (x, a)$ is an arc in $\mathbf{G}_f\})$.*

Proof. Let $a \in A'$ be arbitrary. Since $A \subseteq FC_{\mathcal{C}}(B)$ there exists an irredundant forward chaining derivation $C_1, \ldots, C_k$ of a from B using clauses from $\mathcal{C} \subseteq \mathcal{I}(f)$. By Lemma 7.3, every variable x used in $C_1, \ldots, C_k$ fulfils either $x = a$ or (x, a) is an arc in $\mathbf{G}_f$. Thus $a \in FC_{\mathcal{C}}(B \cap (\{a\} \cup \{x \mid (x, a)$ is an arc in $\mathbf{G}_f\}))$, from which the claim follows. ■

Now we have all the necessary tools to prove Conjecture 6.9 for the class of quadratic pure Horn functions.

Theorem 7.6 *Let f be a quadratic pure Horn function on n variables. Let m be the number of clauses in a minimum quadratic pure Horn CNF representing function f. Then there exist m pairwise disjoint essential sets of implicates of f.*

Proof. Consider a minimum set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\} \subseteq \mathcal{I}(f)$ representing f (such a set can be constructed in polynomial time either by the algorithm for a transitive reduction of a directed graph [15] or by the minimization algorithm for quasi-acyclic functions [11]). Recall that every prime implicate of a quadratic pure Horn function is a quadratic pure Horn clause, and a resolvent of two quadratic pure Horn clauses is again a quadratic pure Horn clause. It follows that not only $\mathcal{C}$ but also $\mathcal{I}(f)$ consists only of quadratic pure Horn clauses. Let us consider the implication graph $\mathbf{G}_f$ and let us fix a topological order τ of strong components of $\mathbf{G}_f$. There are two types of clauses in $\mathcal{C}$: for a clause $C_i = \overline{x} \vee y$ either $K_x <_\tau K_y$ (clause of type (A)) or $K_x = K_y$ (clause of type (B)). Given a clause $C_i = \overline{x} \vee y$ we define a set $\mathcal{E}_i$ in the following way:

$$\mathcal{E}_i = \begin{cases} \{(\overline{u} \vee v) \in I(f) \mid u \in K_x \wedge v \in K_y\} & \text{if } K_x <_\tau K_y \text{ (set of type (A))} \\ \{(\overline{z} \vee y) \in I(f) \mid z \in K_x = K_y \wedge z \neq y\} & \text{if } K_x = K_y \text{ (set of type (B))} \end{cases}$$

If we think of quadratic pure Horn clauses as arcs in $\mathbf{G}_f$ then $\mathcal{E}_i$ of type (A) is a complete bipartite subgraph of all arcs going from K_x to K_y and $\mathcal{E}_i$ of type (B) is a star subgraph of all arcs in $K_x = K_y$ entering y.

Let us first show that the sets $\mathcal{E}_i$, $i = 1, \ldots, m$ are pairwise disjoint sets of implicates of f. Obviously, a set of type (A) can never intersect a set of type (B). Two sets of type (A) intersect only if $\mathcal{C}$ contains two clauses $C_i = \overline{x} \vee y, C_j = \overline{u} \vee v$ of type (A) such that $K_x = K_u$ and $K_y = K_v$ (in fact, in such a case the sets $\mathcal{E}_i$ and $\mathcal{E}_j$ not only intersect but are equal). However, this is a contradiction to the irredundancy of $\mathcal{C}$, since C_i together with clauses of type (B) in $\mathcal{C}$ spanning K_x and K_y imply C_j. To see that no two sets of type (B) intersect we have to use the minimality of $\mathcal{C}$. Indeed, in a minimum representation each strong component of $\mathbf{G}_f$ must be spanned by a simple cycle of clauses (arcs) from $\mathcal{C}$ (see [15] or [11] for a proof of this simple fact). This means that for every y in a strong component of $\mathbf{G}_f$ of size larger than one, $\mathcal{C}$ contains exactly one clause (arc) of type B entering y. This in turn implies that all sets $\mathcal{E}_i$ of type B are pairwise disjoint.

It remains to show that each $\mathcal{E}_i$, $i = 1, \ldots, m$, forms an essential set of clauses. For each C_i we shall define a Boolean vector $t_i \in \{0, 1\}^n$ and show that $\mathcal{E}_i = \mathcal{E}(t_i)$ for $i = 1, \ldots, m$, which will finish the proof since each $\mathcal{E}(t_i)$ is essential by the proof of Theorem 6.4.

Let us start with the simpler case when $C_i = \overline{x} \vee y \in \mathcal{C}$ is of type (B). We define t_i in the following way:

$$t_i[z] = \begin{cases} 1 & \text{if } z \neq y \text{ and } (K_y =) K_x \leq_\tau K_z \\ 0 & \text{if } z = y \text{ or } K_z <_\tau K_x (= K_y) \end{cases}$$

Informally speaking, all variables in strong components "before" $K_x = K_y$ in order τ are assigned value 0, all variables in strong components "after" $K_x = K_y$ in order τ are assigned value 1, and within $K_x = K_y$ only y is assigned value 0 while all other variables are assigned value 1. It is clear from the definition of t_i that $\mathcal{E}_i \subseteq \mathcal{E}(t_i)$, so we only have to show the opposite inclusion. Consider a clause $C \in \mathcal{E}(t_i)$. Since by definition $\mathcal{E}(t_i) \subseteq \mathcal{I}(f)$, C is quadratic pure Horn and we may write $C = \overline{u} \vee v$. Moreover, by Lemma 3.4 (u, v) is an arc in $\mathbf{G}_f$ and hence clearly $K_u \leq_\tau K_v$. The fact that $C(t_i) = 0$ implies $t_i[u] = 1$ and $t_i[v] = 0$. Hence $u \neq y$ and $K_x \leq_\tau K_u$ (all "ones" are in or "after" K_x), similarly $v = y$ or $K_v <_\tau K_y = K_x$ (all "zeros" except of y are "before" K_x). If $v \neq y$, then $K_v <_\tau K_x \leq_\tau K_u$ which is a contradiction to $K_u \leq_\tau K_v$, and thus $v = y$. Putting all this together we get $K_u \leq_\tau K_v$, $K_x \leq_\tau K_u$, and $K_v = K_y = K_x$, which implies $K_u = K_v = K_y = K_x$ and hence $C \in \mathcal{E}_i$.

If $C_i = \overline{x} \vee y \in \mathcal{C}$ is of type (A) let us moreover assume that the topological order τ of strong components of $\mathbf{G}_f$ minimizes the number of strong components which are before K_y (among all topological orders). Note that in this case for any variable u for which $K_u \leq_\tau K_y$ the arc $(u, y)$is in $\mathbf{G}_f$. This time we define t_i in the following way:

$$t_i[z] = \begin{cases} 1 & \text{if } z \in K_x \text{ or } K_y <_\tau K_z \\ 0 & \text{if } K_z \neq K_x \text{ and } K_z \leq_\tau K_y \end{cases}$$

Note that (x, y) is an arc in $\mathbf{G}_f$, which implies $K_x <_\tau K_y$. Informally speaking, all variables in K_y and in strong components "before" K_y in order τ except of K_x are assigned value 0, while all variables in K_x and in strong components "after" K_y in order τ are assigned value 1. Again, it is clear from the definition of t_i that $\mathcal{E}_i \subseteq \mathcal{E}(t_i)$, so we only have to show the opposite inclusion. Let us take a clause $C \in \mathcal{E}(t_i)$. As in the previous case, C can be written as $C = \overline{u} \vee v$ where (u, v) is an arc in $\mathbf{G}_f$, $t_i[u] = 1$ and $t_i[v] = 0$. This assignment implies that $u \in K_x$ or $K_y <_\tau K_u$, similarly $K_v \neq K_x$ and $K_v \leq_\tau K_y$. Since (u, v) is an arc in $\mathbf{G}_f$, $K_u \leq_\tau K_v \leq_\tau K_y$ and thus $u \in K_x$ (since all "ones" not in K_x are "after" K_y). Therefore $K_x = K_u <_\tau K_v$. Proving that $K_v = K_y$ (and hence $C \in \mathcal{E}_i$) takes a bit more work. We have to show that no strong component which was assigned the value 0 (and thus could contain v) except of K_y is reachable by an arc form K_x. For this we shall use the special choice of τ and the irredundancy of $\mathcal{C}$. Let us by contradiction assume that $K_x = K_u <_\tau K_v <_\tau K_y$. By the special choice of τ we get that (v, y) is an arc in $\mathbf{G}_f$, and hence $\overline{v} \vee y$ is a prime implicate of f. Since $u \in K_x$, $\overline{x} \vee v$ is also a prime implicate of f. The following three observations now finish the proof:

- The fact $\overline{x} \vee v \in \mathcal{I}(f)$ implies $v \in FC_{\mathcal{C}}(\{x\})$ and in fact also $v \in FC_{\mathcal{C}\setminus\{C_i\}}(\{x\})$ since otherwise Lemma 7.4 implies that either $y = v$ or (y, v) is an arc in $\mathbf{G}_f$, which is not possible since $K_v <_\tau K_y$.

- The fact $\overline{v} \vee y \in \mathcal{I}(f)$ implies $y \in FC_{\mathcal{C}}(\{v\})$ and in fact also $y \in FC_{\mathcal{C}\setminus\{C_i\}}(\{v\})$ since otherwise Lemma 7.4 implies that $\{x\} \subseteq FC_{\mathcal{C}\setminus\{C_i\}}(\{v\})$, which in turn implies that there is a directed path from v to x in $\mathbf{G}_f$. Again, this is not possible since $K_x <_\tau K_v$.

- Putting the above two facts together gives $y \in FC_{\mathcal{C}\setminus\{C_i\}}(x)$ which is a contradiction to the irredundancy of $\mathcal{C}$. ■

A similar result proving Conjecture 6.9 can be derived for the class of acyclic pure Horn functions.

Theorem 7.7 *Let f be an acyclic pure Horn function on n variables. Let m be the number of clauses in the minimum acyclic pure Horn CNF representing function f. Then there exist m pairwise disjoint essential sets of implicates of f.*

Proof. The proof is in many ways similar to the proof of Theorem 7.6. This time, since all strong components of $\mathbf{G}_f$ are singletons, the clauses of type (B) do not exists, on the other hand clauses of type (A) are no longer restricted to quadratic ones.

Again consider a minimum prime set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\} \subseteq \mathcal{I}(f)$ representing f, which is (as proved in [9]) in fact the unique irredundant and prime CNF representing f. As in the previous proof we shall define for each $i = 1, \ldots, m$ a Boolean vector $t_i \in \{0,1\}^n$ and show that $C_i \in \mathcal{E}(t_i)$ and that the sets $\mathcal{E}(t_i)$, $i = 1, \ldots, m$ are pairwise disjoint.

Since $\mathbf{G}_f$ is acyclic, we can topologically order its vertices, i.e. variables of f. So let τ be a topological order of variables, which will be fixed for the rest of proof. Let us consider $C_i = X \vee y$ where $X = \{x_1, \ldots, x_k\}$ and let t_i be defined as follows:

$$t_i[z] = \begin{cases} 1 & \text{if } z \in FC_{\mathcal{C}\setminus\{C_i\}}(X) \text{ or } y <_\tau z \\ 0 & \text{otherwise} \end{cases}$$

Note that this definition is a generalization of the corresponding definition (for a clause of type (A)) in the proof of Theorem 7.6. In the rest of this proof we shall proceed as follows. First we shall observe that $C_i \in \mathcal{E}(t_i)$. Then we shall show that $\mathcal{E}(t_i) \subseteq \{A \vee y \mid X \subseteq A\}$, which will make it easy to prove the disjointness of the sets $\mathcal{E}(t_i)$, $i = 1, \ldots, m$.

Let us start by observing that $C_i(t_i) = 0$. Clearly $x_j \in FC_{\mathcal{C}\setminus\{C_i\}}(X)$ and hence $t_i[x_j] = 1$ for $j = 1, \ldots, k_i$. Since $\mathcal{C}$ is irredundant, $y \notin FC_{\mathcal{C}\setminus\{C_i\}}(X)$ and hence $t_i[y] = 0$. By combining these two observations we get $C_i(t_i) = 0$ and therefore $C_i \in \mathcal{E}(t_i)$.

Now let us consider an arbitrary $C = A \vee b \in \mathcal{E}(t_i)$ where $A = \{a_1, \ldots, a_\ell\}$ and let us assume without loss of generality that $a_1 <_\tau a_2 <_\tau \cdots <_\tau a_\ell$ and $x_1 <_\tau x_2 <_\tau \cdots <_\tau x_k$. Let us show that $b = y$ and $X \subseteq A$. The fact that $C(t_i) = 0$ means that $t_i[a_j] = 1$ for $j = 1, \ldots, \ell$ and $t_i[b] = 0$. By the definition of t_i this implies for every $j = 1, \ldots, \ell$

$$a_j \in FC_{\mathcal{C}\setminus\{C_i\}}(X) \text{ or } y <_\tau a_j \qquad (i)$$

while

$$b \notin FC_{\mathcal{C}\setminus\{C_i\}}(X) \text{ and } b \leq_\tau y. \qquad (ii)$$

Since $C \in \mathcal{I}(f)$, Lemma 3.4 implies that (a_j, b) is an arc in $\mathbf{G}_f$ for each $j = 1, \ldots, \ell$ and so $a_j <_\tau b \leq_\tau y$. This makes the option $y <_\tau a_j$ in (i) impossible and therefore

$$A \subseteq FC_{\mathcal{C}\setminus\{C_i\}}(X). \qquad (iii)$$

Now (ii) and (iii) together clearly imply that

$$b \notin FC_{\mathcal{C}\setminus\{C_i\}}(A), \qquad (iv)$$

since otherwise $b \in FC_{\mathcal{C}\setminus\{C_i\}}(X)$, which is not the case. From (*iv*) we get using Lemma 7.4 that

$$X \subseteq FC_{\mathcal{C}\setminus\{C_i\}}(A), \qquad (v)$$

and also that either $y = b$ or (y, b) is an arc in $\mathbf{G}_f$ and hence $y \leq_\tau b$. However, in (*ii*) we have shown $b \leq_\tau y$ which together imply $y = b$.

Now we shall show that $X \subseteq A$. Let us assume by contradiction that there exists a variable $x_j \notin A$ and let j be minimum with this property. Let $A' \subseteq A$ be the set of all variables less than x_j with respect to τ and let $X' = \{x_1, \ldots, x_{j-1}\}$. Obviously, (*iii*) and Lemma 7.5 imply $A' \subseteq FC_{\mathcal{C}\setminus\{C_i\}}(X')$ which in turn implies $FC_{\mathcal{C}\setminus\{C_i\}}(A') \subseteq FC_{\mathcal{C}\setminus\{C_i\}}(X')$. Similarly, (*v*) and Lemma 7.5 imply $x_j \in FC_{\mathcal{C}\setminus\{C_i\}}(A')$. However, that together implies $x_j \in FC_{\mathcal{C}\setminus\{C_i\}}(X')$ which is a contradiction to the assumed primality of C_i. Hence $X \subseteq A$.

Now we know that when a clause C belongs to $\mathcal{E}(t_i)$, then it is of the form $A \vee y$ where $X \subseteq A$. Let us assume by contradiction that $C \in \mathcal{E}(t_i) \cap \mathcal{E}(t_j)$ for some $i \neq j$. The only possibility is that $C_i = X_i \vee y$, $C_j = X_j \vee y$, and $X_i, X_j \subseteq A$. Moreover, we know from (*iii*) that $A \subseteq FC_{\mathcal{C}\setminus\{C_i\}}(X_i)$. Since $C_j \in \mathcal{C}$ and $C_j \neq C_i$ we have $y \in FC_{\mathcal{C}\setminus\{C_i\}}(X_j)$. But since $X_j \subseteq A$, we also have $y \in FC_{\mathcal{C}\setminus\{C_i\}}(A) \subseteq FC_{\mathcal{C}\setminus\{C_i\}}(X_i)$ which is a contradiction to irredundancy of $\mathcal{C}$, since C_i could be dropped without changing the function. ∎

It should be noted that if $\mathcal{C}$ is a quadratic pure Horn acyclic CNF representing a quadratic pure Horn acyclic function f, then it is not hard to observe that the proof of Theorem 7.7 shows that $\mathcal{E}_i = \mathcal{E}(t_i) = \{C_i\}$, since $\mathcal{I}(f)$ consists of only quadratic pure Horn clauses.

By combining the proofs of Theorem 7.6 and Theorem 7.7 we can now prove the same result for the class of quasi-acyclic pure Horn functions.

Corollary 7.8 *Let f be a quasi-acyclic pure Horn function. Let m be number of clauses in the minimum quasi-acyclic pure Horn CNF representing function f. Then there exist m pairwise disjoint essential sets of implicates of f.*

Proof. Let us consider a minimum prime set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\} \subseteq \mathcal{I}(f)$ representing f with the following properties:

- In each strong component Q of $\mathbf{G}_f$ one of its variables (denoted x_Q) is chosen as its "representative", and all clauses which contain variables from several different strong components (clauses of type A using the terminology of Theorem 7.6) contain no variables from Q except for x_Q.

- Each strong component Q of $\mathbf{G}_f$ of size k is spanned by k quadratic clauses from $\mathcal{C}$ which form a cycle (clauses of type B).

It was shown in [11] that there always exists a minimal CNF with the above properties. As in the previous proof we shall define for each $i = 1, \ldots, m$ a Boolean vector $t_i \in \{0, 1\}^n$ and show that $C_i \in \mathcal{E}(t_i)$ and that the sets $\mathcal{E}(t_i)$, $i = 1, \ldots, m$ are pairwise disjoint. We shall proceed as follows:

- If C_i is of type A we define t_i as in the proof of Theorem 7.7. To prove that $C_i \in \mathcal{E}(t_i)$ and the sets $\mathcal{E}(t_i)$ for all clauses of type A are disjoint, it suffices to follow line by line the proof of Theorem 7.7. The topological order used this time is a topological order of the strong components of $\mathbf{G}_f$ (or equivalently of the representative variables). The only difference is that the sets obtained by forward chaining include with every representative variable also all other variables in the given strong component (all its logically equivalent "copies"), but the proof remains valid.

- If C_i is of type B we define t_i as in the proof of Theorem 7.6 (for clauses of type B). Again, the fact that $C_i \in \mathcal{E}(t_i)$ and the sets $\mathcal{E}(t_i)$ for all clauses of type A are disjoint follows directly from the proof of Theorem 7.6. The proof uses the fact that $\mathcal{I}(f)$ contains only quadratic clauses. That is no longer true in the quasi-acyclic case, however what is true (and is sufficient for the validity of the proof) is that every clause in $\mathcal{I}(f)$ that contains a head and a subgoal from the same strong component of $\mathbf{G}_f$ is a quadratic clause (cannot contain any additional literals). Thus $\mathcal{E}(t_i)$ for a clause of type B consists (as before) only of quadratic clauses which represent arcs inside the strong component entering the head of C_i. This last observation also proves the "mixed" disjointness of sets $\mathcal{E}(t_i)$ for every pair of clauses of type A and B.

■

8 Conclusions

The main results of this paper are presented in Sections 5, 6, and 7. Section 5 introduces the notion of an exclusive set of implicates of a Boolean function and derives many properties that these sets possess. The most important property is proved in Theorem 5.5. Loosely speaking, given two different CNF representations $\mathcal{C}_1$ and $\mathcal{C}_2$ of a Boolean function f and an exclusive set $\mathcal{X}$ of implicates of f, the set of implicates in $\mathcal{C}_1$ that belong to $\mathcal{X}$ and the set of implicates in $\mathcal{C}_2$ that belong to $\mathcal{X}$ represent the same subfunction of f. Given $\mathcal{X}$, this subfunction is uniquely defined, and it is called the $\mathcal{X}$-component of f (or an exclusive component of f if $\mathcal{X}$ is clear from the context). The properties of exclusive components are summarized in Corollaries 5.7 and 5.8. The above results have a nice application in Boolean minimization. Indeed, if $\mathcal{X}$ is an exclusive set and $\mathcal{C}$ is the input CNF for the minimization problem, one can extract the sub-CNF which represents the $\mathcal{X}$-component of f, find its shortest CNF representation, and then insert this new sub-CNF back into the input CNF. That suggests a decomposition approach for minimization algorithms. Whenever one can find an exclusive subset of clauses of a given function or several pairwise disjoint exclusive subsets of clauses of a given function, it is possible to decompose the minimization problem, solve the subproblems separately, and then compose the obtained solutions back together. This approach is used by the authors of this paper in an upcoming manuscript for a polynomial time minimization of a subclass of Horn functions which properly includes the classes of quadratic, acyclic, and quasi-acyclic Horn functions.

Section 6 then introduces the notion of an essential set of implicates of a Boolean function. The main results presented in Theorems 6.4 and 6.11 state a nice duality (or orthogonality) between representations of a function f and essential sets of implicates of f. A set of clauses represents f if and only if it intersects every nonempty essential set of f. On the other hand, a set of clauses is essential only if it intersects all representations. Moreover, if a set of clauses intersects all representations and is minimal with this property, then it is essential. A simple corollary of these results provides the following lower bound on the length of CNF representations: given k pairwise disjoint nonempty essential sets of implicates of f, it is clear that every CNF representation of f contains at least k clauses. We conjecture (in Conjecture 6.9) that this lower bound is in fact tight, namely that the number of clauses in a shortest representation of f equals the maximum number of pairwise disjoint nonempty essential subsets of implicates of f.

Finally, in Section 7 we prove the validity of Conjecture 6.9 for the classes of quadratic, acyclic, and quasi-acyclic Horn functions. It should be noted that these results can be easily extended to the corresponding subclasses of renameable Horn functions. Given a Horn CNF $\mathcal{C}$, one can in linear time decide whether $\mathcal{C}$ is renameable Horn, and in the affirmative case output a set S of variables, such that switching (complementing) the variables in S produces a Horn CNF. In case this CNF falls into one of the above mentioned subclasses of Horn functions, one can find the appropriate Boolean vectors defining disjoint essential families as described in Section 7. After complementing the components of these vectors that correspond to the set S, one obtains disjoint essential sets of implicates of the original function. This observation also implies that Conjecture 6.9 holds for the entire class of quadratic functions as it is well known that a quadratic CNF is either renameable Horn or identically zero, the latter being a trivial case in which there are no prime implicates and hence also no essential sets.

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