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On the Chvatál-Complexity of Knapsack Problems

Gergely Kovács[a] Béla Vizvári[b]

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RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

[a]College for Modern Business Studies, Tatabánya, Stúdium tér 1, H-2800 Hungary

[b]RUTCOR, Rutgers University, bvizvari@rutcor.rutgers.edu, and Department of Industrial Engineering, Eastern Mediterranean University, Famagusta, Mersin 10, Turkey, bela.vizvari@emu.edu.tr, and Department of Operations Research, Eötvös Loránd University of Budapest, H-1088 Budapest, Pázmány Péter sétány 1/C, H-1117 Hungary, vizvari@math.elte.hu

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Abstract. There is a famous result of Chvatál giving a theoretical iterative procedure, which determines the integer hull of a polyhedral set. It starts from the polyhedral set itself and in each iteration new cuts are introduced. This result is considered the theory of the Gomory method of integer programming. The simplest integer programming problem is the knapsack problem. The number of iterations in Chvatáls procedure is investigated in the case of some special binary knapsack problems.

1 Introduction

The integer hull of a polyhedral set is the convex hull of the integer points of the set. If a polyhedral set is equal to its integer hull then any linear integer programming and linear programming problems having the same objective function have at least one common optimal solution. This is the underlying idea of the cutting plane methods.

[1] introduced a concept of generating cuts such that the cuts do not eliminate any integer solution. Assume that the polyhedral set is defined by the inequality system:

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}, \tag{1}$$

where $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{b}$ and $\mathbf{x}$ are vectors of m and n dimensions, respectively. Assume that $\underline{\lambda}$ is an m dimensional nonnegative vector such that the vector

$$\mathbf{A}^T \underline{\lambda}$$

is integer. Then all integer vectors $\mathbf{x}$ of the polyhedral set must satisfy the inequality

$$\underline{\lambda}^T \mathbf{A}\mathbf{x} \leq \left\lfloor \underline{\lambda}^T \mathbf{b} \right\rfloor. \tag{2}$$

In general (2) is a valid cut of the integer hull. Furthermore if $\underline{\lambda}^T \mathbf{b}$ is non-integer then it will cut off a part of the polyhedral set. It can be proven, see also [3], that:

1. there are only finite many significantly different Chvatál cuts of type (2) and

2. if the Chvatál cuts added to the set of inequalities (1) and in this way a new the polyhedral set is defined, and the whole procedure is repeated, then after finite many iterations the polyhedral set becomes equal to the integer hull.

In this paper the number of iterations is called Chvatál rank. From practical point of view it is important that how large can it be. To our best knowledge this problem has not been analyzed. The reason is that the integer programming problems are NP-complete, therefore the number of iterations should be great in general. On the other hand important subclasses can be those ones, where the number of iterations is small, e.g. it is only 1. Such a result can be hoped only in the case of problems, which have a simple structure. Thus one of the first candidates are special structured knapsack problems. In this paper binary knapsack problems are investigated.

The following notations and assumptions are used. Let n be the number of variables, i.e. the dimension of the problem. All variables are binary variables. The knapsack constraint is given in the form:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq b, \tag{3}$$

where $a_1, a_2, \ldots, a_n$ are positive integers. Furthermore

$$a_1 \leq a_2 \leq \cdots \leq a_n. \tag{4}$$

It follows from the binary property of the variables that the values of all components of an element of the integer hull are between 0 and 1. Therefore the initial polyhedral set is defined by the following system of inequalities, where the index of the inequality is used in the further analysis as it is indicated here:

index	**Right-Hand Side**		**Left-Hand Side**
0	$a_1x_1 + a_2x_2 + \cdots + a_nx_n$	$\leq$	b
1	x_1	$\leq$	1
2	x_2	$\leq$	1
	$\cdots$		
n	x_n	$\leq$	1
$n+1$	$-x_1$	$\leq$	0
$n+2$	$-x_2$	$\leq$	0
	$\cdots$		
$2n$	$-x_n$	$\leq$	0

Using the same index set the multipliers of the inequalities of this original constraint set are denoted by $\lambda_0, \ldots, \lambda_{2n}$.

In general the set of integer feasible solutions is empty if and only if $b < 0$. Then an inequality is required, which cannot be satisfied. It is obtained be the multipliers

$$\lambda_0 = 1,\ \lambda_1 = \lambda_2 = \cdots \lambda_n = 0,\ \lambda_{n+1} = a_1,\ \lambda_{n+2} = a_2, \cdots \lambda_{2n} = a_n$$

as then the resulted inequality has the classical form showing the unsatisfiability of the system:

$$0 \ = \ 0x_1 \ + \ 0x_2 \ + \ \cdots \ + \ 0x_n \ \leq \ b \ < \ 0.$$

Therefore it is assumed further on that $b \geq 0$.

In section 2 it is shown that up to 3 dimensions all binary knapsack problems have Chvatál rank at most 1. An example having a higher rank is provided in section 3 in dimension 4. This example is generalized and analyzed based on a result of [2] in section 4.

2 The Case of Dimensions 1, 2, and 3

It follows from the assumptions that in the case of $n = 1$ there are only two possible sets of integer feasible solutions: $\{0, 1\}$, and $\{0\}$. For the first one the existing inequalities $0 \leq x \leq 1$ are determining the integer hull. In the second case the integer hull is the 0 point itself. It is obtained from $0 \leq x_1$, and its counter part generated by $\lambda_0 = 1/a_1$, $\lambda_1 = \lambda_2 = 0$. Thus the Chvatál rank is either 0 or 1.

Assume that $n = 2$. The it follows from (4) that the left-hand side of (3) is decreasing for the following sequence of binary vectors: $(1, 1)$, $(0, 1)$, $(1, 0)$, $(0, 0)$. Thus if a set of integer feasible solution consists of $k\,(1 \leq k \leq 4)$ vectors then these vectors are the last k vectors of the sequence. Hence it is not difficult to see that the Chvatál rank is again either 0 or 1.

If $n = 3$ then the maximal elements of the feasible solutions belong to one of the cases of the table below:

case	maximal vectors	inequalities of the feasible set
1	empty	$0 \leq -1$
2	$(0, 0, 0)$	$y_i \leq 0$
3	$(1, 0, 0)$	$y_2 \leq 0$, $y_3 \leq 0$
4	$(1, 0, 0)$, $(0, 1, 0)$	$y_1 + y_2 \leq 1$, $y_3 \leq 0$
5	$(1, 1, 0)$	$y_3 \leq 0$
6	$(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$	$y_1 + y_2 + y_3 \leq 1$
7	$(0, 0, 1)$, $(1, 1, 0)$	$y_1 + y_3 \leq 1$, $y_2 + y_3 \leq 1$
8	$(1, 1, 0)$, $(1, 0, 1)$	$y_2 + y_3 \leq 1$
9	$(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$	$y_1 + y_2 + y_3 \leq 2$
10	$(1, 1, 1)$	empty

Table 1. The non-trivial inequalities of the integer hall of the feasible set of knapsack problems in the 3-dimensional case.

The knapsack problem has a Chvatál rank at most 1 in all of the cases.

The statement can be shown in a trivial way for cases 1, 2, 3, 5, 10. E. g. in case 5 the multipliers are: $\lambda_0 = \frac{1}{a_3}, \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = \frac{a_1}{a_3}, \lambda_5 = \frac{a_2}{a_3}, \lambda_6 = 0$.

The other cases can be solved as follows.

Case 4: The inequality of $y_3 \leq 0$ as generated for case 5 and another cut with $\lambda_0 = \frac{1}{b}, \lambda_1 = 1 - \frac{a_1}{b}, \lambda_2 = 1 - \frac{a_2}{b}, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, \lambda_6 = \frac{a_3}{b}$.

Case 8: $\lambda_0 = \frac{1}{b}, \lambda_1 = 0, \lambda_2 = 1 - \frac{a_2}{b}, \lambda_3 = 1 - \frac{a_3}{b}, \lambda_4 = \frac{a_1}{b}, \lambda_5 = 0, \lambda_6 = 0$.

Case 9: $\lambda_0 = \frac{1}{b}, \lambda_1 = 1 - \frac{a_1}{b}, \lambda_2 = 1 - \frac{a_2}{b}, \lambda_3 = 1 - \frac{a_3}{b}, \lambda_4 = 0, \lambda_5 = 0, \lambda_6 = 0$.

Case 7: The cut of case 8 and another cut with $\lambda_0 = \frac{1}{b}, \lambda_1 = 1 - \frac{a_1}{b}, \lambda_2 = 0, \lambda_3 = 1 - \frac{a_3}{b}, \lambda_4 = 0, \lambda_5 = \frac{a_2}{b}, \lambda_6 = 0$.

Case 6: $\lambda_0 = \frac{1}{a_2}, \lambda_1 = 1 - \frac{a_1}{a_2}, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, \lambda_6 = \frac{a_3}{a_2} - 1$.

3 A Counterexample in Dimension 4

The 4-dimensional case is the first one, where there is a knapsack problem with an integer hull having a Chavtál rank higher than 1. Here is the example: $a_1 = 12$, $a_2 = 12$, $a_3 = 14$, $a_4 = 30$ and $b = 53$. It is easy to see that the maximal feasible solutions are the vectors $(1,1,1,0), (0,0,1,1), (0,1,0,1), (1,0,0,1)$. The hyperplane

$$y_1 + y_2 + y_3 + 2y_4 = 3$$

contains all of these maximal feasible points. Therefore

$$y_1 + y_2 + y_3 + 2y_4 \leq 3 \tag{5}$$

is a valid cut of the integer hull. If the cut could be generated then there were nonnegative λ's satisfying the following conditions:

$$\begin{array}{lcl} 12\lambda_0 + \lambda_1 - \lambda_5 & = & 1 \\ 12\lambda_0 + \lambda_2 - \lambda_6 & = & 1 \\ 14\lambda_0 + \lambda_3 - \lambda_7 & = & 1 \\ 30\lambda_0 + \lambda_4 - \lambda_8 & = & 2 \\ 53\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & < & 4 \end{array}$$

The system has a feasible solution if and only if the optimal value of the linear programming program:

$$\begin{array}{rcl} \min 53\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & & \\ 12\lambda_0 + \lambda_1 - \lambda_5 & = & 1 \\ 12\lambda_0 + \lambda_2 - \lambda_6 & = & 1 \\ 14\lambda_0 + \lambda_3 - \lambda_7 & = & 1 \\ 30\lambda_0 + \lambda_4 - \lambda_8 & = & 2 \\ \lambda_0, \ldots \lambda_8 & \geq & 0 \end{array}$$

is less than 4. On the other hand the optimal solution is: $\lambda_0 = \frac{1}{15}$, $\lambda_1 = \lambda_2 = \frac{1}{5}$, $\lambda_3 = \frac{1}{15}$, $\lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 0$. The optimal objective function value is 4.

In general there are 28 different sets of maximal feasible solutions in dimension 4 if inequality (4) is satisfied. 10 out of 28 are equivalent to the 10 cases of the 3-dimension with the modification that the last component of all maximal feasible solutions is 0. Here are the further 18 cases:

11	(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)	$y_1 + y_2 + y_3 + y_4 \leq 1$
12	(0,0,1,0), (0,0,0,1), (1,1,0,0)	$y_1 + y_3 + y_4 \leq 1$, $y_2 + y_3 + y_4 \leq 1$
13	(0,0,0,1), (1,1,0,0), (1,0,1,0)	$y_1 + y_2 + y_3 + 2y_4 \leq 2$, $y_2 + y_3 + y_4 \leq 1$
14	(0,0,0,1), (1,1,0,0), (1,0,1,0), (0,1,1,0)	$y_1 + y_2 + y_3 + 2y_4 \leq 2$
15	(0,0,0,1), (1,1,1,0)	$y_1 + y_4 \leq 1$, $y_2 + y_4 \leq 1$, $y_3 + y_4 \leq 1$
16	(1,1,0,0), (1,0,1,0), (1,0,0,1)	$y_2 + y_3 + y_4 \leq 1$
17	(1,1,0,0), (1,0,1,0), (0,1,1,0), (1,0,0,1)	$y_1 + y_2 + y_3 + y_4 \leq 2$, $y_2 + y_4 \leq 1$, $y_3 + y_4 \leq 1$
18	(1,0,0,1), (1,1,1,0)	$y_2 + y_4 \leq 1$, $y_3 + y_4 \leq 1$
19	(1,1,0,0), (1,0,1,0), (0,1,1,0), (1,0,0,1), (0,1,0,1)	$y_1 + y_2 + y_3 + y_4 \leq 2$, $y_3 + y_4 \leq 1$
20	(1,0,0,1), (0,1,0,1), (1,1,1,0)	$y_1 + y_2 + y_4 \leq 2$, $y_3 + y_4 \leq 1$
21	(1,1,1,0), (1,1,0,1)	$y_3 + y_4 \leq 1$
22	(1,1,0,0), (1,0,1,0), (0,1,1,0) and (1,0,0,1), (0,1,0,1), (0,0,1,1)	$y_1 + y_2 + y_3 + y_4 \leq 2$
23	(1,0,0,1), (0,1,0,1), (0,0,1,1), (1,1,1,0)	$y_1 + y_2 + y_3 + 2y_4 \leq 3$
24	(0,0,1,1), (1,1,0,1)	$y_1 + y_3 + y_4 \leq 2$, $y_2 + y_3 + y_4 \leq 2$
25	(0,1,0,1), (1,1,1,0), (1,0,1,1)	$y_1 + y_2 + y_4 \leq 2$, $y_2 + y_3 + y_4 \leq 2$
26	(1,1,1,0), (1,1,0,1), (1,0,1,1)	$y_2 + y_3 + y_4 \leq 2$
27	(1,1,1,0), (1,1,0,1), (1,0,1,1), (0,1,1,1)	$y_1 + y_2 + y_3 + y_4 \leq 3$
28	(1,1,1,1)	$0 \leq y_i \leq 1 \;\; \imath = 1,2,3,4$

Table 2. The non-trivial inequalities of the integer hull of the feasible set of knapsack problems in the 4-dimensional case.

As it can be seen from Table 2 the above example belongs to case 23. All other cases have Chvatál rank 1. The proofs are similar to the 3-dimensional case, the details remain to the reader.

Even case 23 has Chvatál rank 1 for certain coefficients.

Theorem 3.1 *The Chvatál rank of case 23 is higher than 1 if and only if*

$$a_1 + a_2 + a_3 \leq b \tag{6}$$
$$a_3 + a_4 \leq b \tag{7}$$
$$a_1 + a_2 + a_4 > b \tag{8}$$
$$a_3 < \frac{a_4}{2} \tag{9}$$
$$a_1 + a_2 + a_3 + \frac{a_4}{2} \leq b \tag{10}$$

Proof. We obtain the first set of conditions from the fact that the points $(1,1,1,0)$, $(0,0,1,1)$, $(0,1,0,1)$, $(1,0,0,1)$ are feasible. The first two points imply the inequalities (6) and (7). It follows from (4) that the similar inequalities of the third and fourth points are consequences of (7).

Any point, which is greater than at least one of these four points, must be infeasible. The minimal such points are $(1,0,1,1)$, $(0,1,1,1)$, $(1,1,0,1)$. It follows from (4) that the last one gives the minimal left-hand side. Thus it is enough to claim only inequality (8).

Case I. If $\frac{a_4}{2} \leq a_3$ then let $\lambda_0 = \frac{1}{a_3}$, $\lambda_1 = 1 - \frac{a_1}{a_3}$, $\lambda_2 = 1 - \frac{a_2}{a_3}$, $\lambda_3 = 0$, $\lambda_4 = 2 - \frac{a_4}{a_3}$, $\lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 0$. Then the coefficients of the left-hand side are $1, 1, 1, 2$. The value of the right-hand side is

$$\left(1 - \frac{a_1}{a_3}\right) + \left(1 - \frac{a_2}{a_3}\right) + \left(2 - \frac{a_4}{a_3}\right) + \frac{b}{a_3} = \frac{b - a_1 - a_2 - a_4}{a_3} + 4. \tag{11}$$

It follows from (8) that it is less than 4, i.e. it can be rounded down to 3. Thus (9) is a necessary condition of having a Chvatál rank greater than one.

Case II. If $a_3 < \frac{a_4}{2}$ then let $\lambda_0 = \frac{2}{a_4}$, $\lambda_1 = 1 - \frac{2a_1}{a_4}$, $\lambda_2 = 1 - \frac{2a_2}{a_4}$, $\lambda_3 = 1 - \frac{2a_3}{a_4}$, $\lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 0$.

Then the coefficients of the left-hand side are 1,1,1,2. The value of the right hand side is

$$\frac{2b}{a_4} + \left(1 - \frac{2a_1}{a_4}\right) + \left(1 - \frac{2a_2}{a_4}\right) + \left(1 - \frac{2a_3}{a_4}\right) = \frac{2b - 2a_1 - 2a_2 - 2a_3}{a_4} + 3. \tag{12}$$

This value is less than 4 if and only if

$$\frac{2b - 2a_1 - 2a_2 - 2a_3}{a_4} < 1, \tag{13}$$

what is equivalent to (10). Thus (6)-(10) are necessary conditions of having a Chvatál rank greater than 1.

It is easy to see that in any system of the λ multipliers giving a minimal right-hand side both λ_i and λ_{4+i} can not be positive ($i = 1, 2, 3, 4$). Thus the optimal choice of λ_i ($i = 1, 2, 3, 4$) if λ_0 is fixed,

$$\lambda_i = max\{1 - \lambda_0 a_i,\ 0\},$$

if $i = 1, 2, 3$ and

$$\lambda_4 = max\{2 - \lambda_0 a_4,\ 0\}.$$

Thus the right-hand side in an optimal solution is

$$\lambda_0 b + max\{1 - \lambda_0 a_1,\ 0\} + max\{1 - \lambda_0 a_2,\ 0\} +$$

$$+ max\{1 - \lambda_0 a_3,\ 0\} + max\{2 - \lambda_0 a_4,\ 0\}.$$

Its value is equal to

$$(\lambda_0 a_1 + max\{1 - \lambda_0 a_1,\ 0\}) + (\lambda_0 a_2 + max\{1 - \lambda_0 a_2,\ 0\}) +$$

$$+ (\lambda_0 a_3 + max\{1 - \lambda_0 a_3,\ 0\}) + \left(\lambda_0 \frac{a_4}{2} + 2max\{1 - \lambda_0 \frac{a_4}{2},\ 0\}\right) +$$

$$+ \lambda_0 (b - (a_1 + a_2 + a_3 + \frac{a_4}{2})).$$

Here the first three terms are at least 1 and the fourth term is at least $1 + max\{1 - \lambda_0 \frac{a_4}{2},\ 0\}$. Thus the sum of the first four terms is at least 4. If (10) holds then the total value of the right-hand side can not be less than 4, i.e. the cut (5) can not be generated in the first Chvatál iteration. Q.E.D.

4 A Special Class of Knapsack Problem

[2] has given a complete description of the integer hull of a special binary knapsack problem. The inequality of the problem is

$$x_1 + x_2 + \cdots + x_{m_1} + px_{m_1+1} + px_{m_1+2} + \cdots + px_{m_1+m_2} \le b, \tag{14}$$

where $m_1 \ge 2$, m_2, p, b are positive integers. The set of binary feasible solutions of (14) is denoted by F.

Let $I \subseteq \{1, 2, \ldots, m_1 + m_2\}$ be an arbitrary index set. The sum of variables belonging to that set is $x(I)$. Let $T = \{m_1 + 1, \ldots, m_1 + m_2\}$ and $S \subseteq \{1, 2, \ldots, m_1\}$. Denote the number of element of S by s. Finally let q be a positive integer such that $1 \le q < p$.

$$h(s, q) = \max\{x(S) + qx(T) : x \in F\}.$$

The value of $h(s, q)$ is given by the formula

$$h(s,q) = \begin{cases} b & \text{if } s \ge b \\ \max\{s + q\lfloor \frac{b-s}{p} \rfloor, b - (p-q)\lceil \frac{b-s}{p} \rceil\} & \text{if } b > s \end{cases}$$

Among the last m_2 variables at most $l_{\max} = \min\left\{m_2, \left\lfloor \frac{b}{p} \right\rfloor\right\}$ can have value 1 in any feasible solution.

The main result of [2] is the following:

Theorem 4.1 *(A) The integer hull of the knapsack problem is described by the following system of inequalities:*

- *(14),*
- $x(T) \le l_{\max}$,
- $x(S) + qx(T) \le h(s, q)$, $\forall S : \emptyset \ne S \subseteq \{1, 2, \ldots, m_1\}$ *and* $\forall q : 1 \le q < p$,
- $0 \le x_i \le 1, i \in \{1, 2, \ldots, m_1 + m_2\}$.

(B) The inequality $x(S) + qx(T) \le h(s, q)$ defines a facet of the integer hull if and only if ($s > q$ or $s = q = 1$) and $s \in \{q + b - p, q + b - 2p, \ldots, q + b - pl_{\max}\}$.

4.1 The case $m_2 = 1$

First we shall discuss the case $m_2 = 1$ in a detailed way. We assume that $p \le b$ otherwise the problem is trivial. Then $l_{\max} = 1$. It follows from Theorem 4.1 that for each $q > 1$ the only value of s giving a facet of the integer hull is

$$s = q + b - p. \tag{15}$$

Hence it follows that $h(s, q) = s$.

The first question to be investigated is when has a facet with parameters satisfying (15) a Chvatál rank equal to 1. For the sake of simplicity we assume that $S = \{1, \ldots, s\}$. The best cut of this type what can be achieved in the first Chvatál iteration is described by the following linear program:

$$\begin{aligned}
\min\ b\lambda_0 + \lambda_1 + \lambda_2 + \ldots + \lambda_{m_1+1} & \\
\lambda_0 + \lambda_1 - \lambda_{m_1+2} &= 1 \\
\ldots & \\
\lambda_0 + \lambda_s - \lambda_{m_1+s+1} &= 1 \\
\lambda_0 + \lambda_{s+1} - \lambda_{m_1+s+2} &= 0 \\
\ldots & \\
\lambda_0 + \lambda_{m_1} - \lambda_{2m_1+1} &= 0 \\
p\lambda_0 + \lambda_{m_1+1} - \lambda_{2m_1+2} &= q \\
\lambda_0, \ldots \lambda_{2m_1+2} &\geq 0.
\end{aligned}$$

This linear programming problem will be analyzed by using simplex method. If the simplex method detects an optimal solution with an objective function value at least $s+1$ then the Chvatál rank is at least 2. These cases will be separated when we follow the path of optimization.

In what follows the negative of the objective function is maximized instead of the original objective function because of some technical reasons.

It is easy to see that the variables $\lambda_1, \ldots \lambda_{m_1+1}$ form a feasible basis. The appropriate simplex tableau is this, where all missing elements are 0:

	λ_0	λ_1		λ_s				λ_{m_1+1}	λ_{m_1+2}						λ_{2m_1+2}	RHS
λ_1	1	1							-1							1
$\vdots$			$\ddots$							$\ddots$						
λ_s	1			1							-1					1
λ_{s+1}	1				1							-1				0
$\vdots$						$\ddots$							$\ddots$			
λ_{m_1}	1						1							-1		0
λ_{m_1+1}	p							1							-1	q
OBF	$b-m_1-p$	0	$\cdots$	0	0	$\cdots$	0	0	1	$\cdots$	1	1	$\cdots$	1	1	$-q-s$

If $b \geq m_1+p$ then the integer hull is the complete unit cube and the problem is trivial. In that case the solution provided by the simplex tableau is optimal, i.e. the inequalities $x(S) + qx(T) \leq h(s,q)$ are not facet defining, but on the other hand the Chvatál rank is 0 as the polyhedral set is the unit cube.

If $m_1 + p > b$ then the only candidate to enter the basis is variable λ_0. If $m_1 > s$ then any of the variables $\lambda_{s+1}, \ldots, \lambda_{m_1}$ may leave the basis. Assume that λ_{s+1} leaves the basis. Then the new simplex tableau is this:

	λ_0	λ_1		λ_s	λ_{s+1}				λ_{m_1+1}	λ_{m_1+2}							λ_{2m_1+2}	RHS
λ_1	0	1			-1					-1			1					1
$\vdots$			$\ddots$								$\ddots$							
λ_s	0			1	-1							-1	1					1
λ_0	1				1								-1					0
λ_{s+2}	0				-1	1							1	-1				0
$\vdots$							$\ddots$								$\ddots$			
λ_{m_1}	0				-1			1					1			-1		0
λ_{m_1+1}	0				$-p$				1				p				-1	q
OBF	0	0	$\cdots$	0	$-b+m_1+p$	0	$\cdots$	0	0	1	$\cdots$	1	$b+1-m_1-p$	1	$\cdots$	1	1	$-q-s$

There is only one potential candidate to enter the basis as $b+1-m_1-p = s+p-q+1-m_1-p = s-m_1+1-q \le 0$ as $m_1 \ge s$ and $q \ge 1$. The only case when the current basis is optimal, is $m_1 = 1$, $q = 1$. It implies that $b+1 = m_1+p$. Then the maximal feasible solutions are the following vectors: (a) the first m_1 components are 1, and the last component is zero, (b) the last component is 1 and $b-p = m_1-1$ components among the first m_1 are 1, i.e. the maximal feasible binary solutions are such that they have only zero component, their all other components are 1. Thus this case is a generalization of case 27 of Table 2. Hence the only inequality what must be generated to obtain the integer hull is

$$\sum_{j=1}^{m_1+1} x_j \le m_1.$$

It can be generated by the following weights:

$$\lambda_0 = \frac{1}{p},\ \lambda_1 = \cdots = \lambda_{m_1} = \frac{p-1}{p},\ \lambda_{m_1+1} = \cdots = \lambda_{2m_1+2} = 0.$$

Then all coefficients of the left-hand side are 1 in the generated inequality and the right-hand side is

$$\frac{b}{p} + m_1\frac{p-1}{p} \ = \ m_1 + \frac{p-1}{p},$$

which can be truncated to m_1.

Now assume that $m_1 + p > b$. In general the inequality $b+u-m_1-p < 0$ is a consequence of (15) if and only if $u < m_1 - s + q$. After the entering of variable λ_{m_1+s+2} into the basis the next candidate will be variable λ_{m_1+s+3} with reduced cost $b+2-m_1-p$. The simplex method is iterating in this way until all of the variables from λ_{m_1+s+2} to λ_{2m_1} enter the basis. At this moment the simplex tableau is as follows:

	λ_0	λ_1 λ_s		λ_{m_1}	λ_{m_1+1}			λ_{2m_1+1}	λ_{2m_1+2}	RHS
λ_1	0	1		-1		-1		1		1
$\vdots$		$\ddots$				$\ddots$				
λ_s	0	1		-1		-1		1		1
λ_0	1			1				-1		0
λ_{m_1+s+2}	0		-1	1			1	-1		0
$\vdots$			$\ddots$				$\ddots$			
λ_{2m_1}	0		-1	1			1	-1		0
λ_{m_1+1}	0			$-p$	1			p	-1	q
OBF	0	$0 \ \cdots \ 0$	$1 \ \cdots \ 1$	$q+1$	0	$1 \ \cdots \ 1$	$0 \ \cdots \ 0$	$-q$	1	$-q-s$

The next variable to enter the basis is λ_{2m_1+1}. As $q < p$ variable λ_{m_1+1} is leaving the basis. The new simplex tableau is this:

	λ_0	λ_1 λ_s		λ_{m_1+1}			λ_{2m_1+2}	RHS
λ_1	0	1		$-\frac{1}{p}$	-1		$\frac{1}{p}$	$1-\frac{q}{p}$
$\vdots$		$\ddots$			$\ddots$			
λ_s	0	1		$-\frac{1}{p}$	-1		$\frac{1}{p}$	$1-\frac{q}{p}$
λ_0	1			$\frac{1}{p}$			$-\frac{1}{p}$	$\frac{q}{p}$
λ_{m_1+s+2}	0		-1	$\frac{1}{p}$		1	$-\frac{1}{p}$	$\frac{q}{p}$
$\vdots$			$\ddots$			$\ddots$		
λ_{2m_1}	0		-1	$\frac{1}{p}$		1	$-\frac{1}{p}$	$\frac{q}{p}$
OBF	0	$0 \ \cdots \ 0$	$1 \ \cdots \ 1$	$\frac{q}{p}$	$1 \ \cdots \ 1$	$0 \ \cdots \ 0$	$1-\frac{q}{p}$	$\frac{q^2}{p}-q-s$

This is the optimal simplex tableau and the optimal objective function value is

$$-\frac{q^2}{p} + q + s.$$

Thus the Chvatál rank of the facet defining cut is 1 if and only if

$$-\frac{q^2}{p} + q + s < h(s, q) + 1 = s + 1.$$

This is equivalent to the inequality

$$q^2 - pq + p > 0. \tag{16}$$

If $m_1 + p > b$ and $s = m_1$ then the only candidate to enter the basis is again variable λ_0. Then according to the ratio test only λ_{m_1+1} may leave the basis. After pivoting the new simplex tableau is just the same as the optimal tableau of the previous case:

	λ_0	λ_1		λ_{m_1}	λ_{m_1+1}				λ_{2m_1+2}	RHS
λ_1	0	1			$-\frac{1}{p}$	-1			$\frac{1}{p}$	$1-\frac{q}{p}$
$\vdots$			$\ddots$				$\ddots$			
λ_{m_1}	0			1	$-\frac{1}{p}$			-1	$\frac{1}{p}$	$1-\frac{q}{p}$
λ_0	1				$\frac{1}{p}$				$-\frac{1}{p}$	$\frac{q}{p}$
OBF	0	0	$\cdots$	0	$\frac{q}{p}$	1	$\cdots$	1	$1-\frac{q}{p}$	$\frac{q^2}{p}-q-s$

Hence the Chvatál rank is 1 if and only if (16) holds. Thus the following lemma has been proven:

Lemma 4.1 *Let m_1, p, and b be positive integers such that $m_1 + p > b + 1$. Then the Chvatál rank of the integer hull of the set*

$$\left\{\mathbf{x} \in \mathbb{R}^{m_1+1} \mid x_1 + \cdots + x_{m_1} + p x_{m_1+1} \;\le\; b;\;\; 0 \le x_i \le 1,\; i = 1, \ldots, m_1\right\} \tag{17}$$

is 1 if and only if a positive integer q with $q < p$ exists such that (16) holds.

If p is fixed then the left-hand side of (16) can be considered as a single variable quadratic function of q, say $f(q)$. The roots of $f(q)$ are:

$$q_1 = \frac{p + \sqrt{p^2 - 4p}}{2}, \qquad q_2 = \frac{p - \sqrt{p^2 - 4p}}{2}.$$

Hence if $p = 3$ then the equation $f(q) = 0$ has only complex roots, i.e. $f(q)$ lies completely above the horizontal axis. Thus for $p = 3$ the Chvatál rank of the integer hull is 1 independently from the value of m_1. It is easy to see that if $p \ge 4$ then the roots have the following properties:

$$p - 1 \ge q_1, \qquad 2 \ge q_2.$$

Inequality (16) is satisfied if $q = 1$ or $q = p - 1$ and it is not satisfied for any integer between 1 and $p - 1$. For example if $p \ge 4$ then the cuts of type

$$x_{i_1} + \cdots + x_{i_s} + 2x_{m_1+1} \;\le\; s$$

with $s = b - p + 2$, have a Chvatál rank at least 2. This fact leads to the following theorem.

Theorem 4.2 *Let m_1, p, and b be positive integers such that $m_1 + p > b + 1$ and $p \ge 4$. Then the Chvatál rank of the integer hull of the set (17) is at least 2.*

The main content of the theorem is that although the set defined in (17) has one of the simplest definitions among the sets of binary vectors, its Chvatál rank is still large.

4.1.1 An upper bound of the Chvatál rank

In this subsection an inductive procedure is given to generate all of the facet defining cuts. To any induction two things have to be shown: the iterative step and the fact that the conditions of the start of the iterative procedure, are satisfied. We shall discuss the inductive algorithm in this order.

First assume that the facet defining cuts for $q = i$ and $q = p - i$, where $i < \frac{p}{2}$, are exiting and have been already generated. We shall show that in this position the facet defining inequalities for $q = i + 1$ and $q = p - i - 1$ can be generated in one Chvatál iteration.

Case $q = i + 1$**.** We got a facet defining inequality if $s = b - p + i + 1$. For the sake of simplicity assume that $S = \{1, \ldots, s\}$. According to the assumption we have all inequalities with $s_0 = s - 1$ and $q_0 = q - 1$. The set S has s subsets with s_0 number of elements. We shall consider the appropriate s inequalities with parameters s_0 and q_0. All of these inequalities will have the same weight, say u, in the generation of the required inequality. Only one inequality will have a weight different from 0 among the inequalities belonging to the parameters $s_1 = b - i$, $q_1 = p - i$. Assume that the appropriate set is $S_1 = \{1, \ldots, b + i\}$. The weight of this inequality is denoted by v. Finally the weights of the inequalities $-x_j \le 0$ ($j = s + 1, \ldots, b - i$) are again v.

The value of u and v will be chosen such that the coefficients of the left-hand side of required inequality are obtained. The coefficient 1 is obtained from the 1's of $s - 1$ inequalities all having the weight u and from the 1 of the last inequality with weight v. Thus the equation

$$(s - 1)u + v = 1 \tag{18}$$

must hold. The coefficient of x_{m_1+1} is $i + 1$ in the required inequality. In the same position the coefficients of the inequalities used in the generation are i and $p - i$, respectively. Hence we obtain the equation

$$siu + (p - i)v = i + 1. \tag{19}$$

The solution of the system (18)-(19):

$$u = \frac{p - 2i - 1}{(s - 1)(p - 2i) - i}, \qquad v = 1 - (s - 1)u = \frac{s - i - 1}{(s - 1)(p - 2i) - i}. \tag{20}$$

It is easy to see that both u and v are nonnegative. These multipliers give a Chvatál rank 1 if and only if

$$us(s - 1) + v(b - i) < s + 1. \tag{21}$$

By substituting (18) the left-hand side of (21) can be reformulated as follows:

$$us(s - 1) + v(b - i) \;=\; us(s - 1) + (1 - (s - 1)u)(b - i) \;=\; u(s - 1)(s - b + i) + b - i.$$

Hence (21) is equivalent to

$$u(s - 1)(s - b + i) + b - i - s = (u(s - 1) - 1)(s - b + i) = v(b - s - i) < 1$$

As $s = b - p + i + 1$ the equation $b - s - i = p - 2i - 1$ holds. Thus

$$v(b - s - i) = v(p - 2i - 1) = \frac{s - i - 1}{(s - 1)(p - 2i) - i}(p - 2i - 1) < 1 \tag{22}$$

The value of s is at least 2 as $b > p$ and $i \ge 1$. From the assumption that $i < \frac{p}{2}$ it follows that denominator is positive. Then (22) holds if and only if

$$sp - 2si - s - ip + 2i^2 + i - p + 2i + 1 < sp - 2si - p + 2i - i \tag{23}$$

It can be reformulated as

$$0 < i(p - 2i - 1) + s - i - 1, \tag{24}$$

which is true as $s - i - 1 = b - p$.

Case $q = p - i - 1$**.** The inequality to be generated has the parameters $s = b - i - 1$ and $q = p - i - 1$. For the sake of simplicity it is assumed that $S = \{1, \ldots, b - i - 1\}$. The pairs of parameters used in the previous case, i.e. $s_0 = b - p + i$, $q_0 = i$ and $s_1 = b - i$, $q_1 = p - i$, are used again. The generation is made by the following construction. The facet defining inequalities with s_0 and q_0 such that the index set of their 1's is a subset of S, are used with the same weight, say u. The facet defining inequality with parameters s_1, q_1 and $S_1 = \{1, \ldots, b - i\}$ and the inequality $-x_{b-i} \leq 0$ have the weight v.

Each 1 of the required inequality is in

$$\binom{b - i - 2}{b - p + i - 1}$$

inequalities of the used ones of type s_0, q_0. Hence the equation

$$\binom{b - i - 2}{b - p + i - 1} u + v = 1 \tag{25}$$

is obtained. The number of inequalities of the first type is

$$\binom{b - i - 1}{b - p + i}$$

in the construction. Thus to get the coefficient of x_{m_1+1} the equation

$$\binom{b - i - 1}{b - p + i} u + (p - i)v = p - i - 1 \tag{26}$$

must be satisfied. The solution of the (25)-(26) system is

$$u = \frac{1}{\binom{b-i-2}{b-p+i-1}\left(p - i - \frac{b-i-1}{b-p+i} i\right)}, \qquad v = 1 - \frac{1}{p - i - \frac{b-i-1}{b-p+i} i}. \tag{27}$$

It is easy to see that

$$\frac{1}{p - i - \frac{b-i-1}{b-p+i} i} > 0.$$

Hence it can be proven that both u and v are nonnegative. The right-hand side of the generated inequality is

$$RHS = \binom{b - i - 1}{b - p + i}(b - p + i)u + (b - i)v.$$

The construction gives the required Chvatál cut if and only if $RHS < b - i$. Let

$$A = \binom{b - i - 2}{b - p + i - 1} \quad \text{and} \quad B = \frac{1}{p - 2i - \frac{b-i-1}{b-p+i} i}.$$

Then

$$RHS = \frac{B}{A} A \frac{b - i - 1}{b - p + i}(b - p + i) + (b - i) - B(b - i) = b - i - B < b - i.$$

What is remained to show is that the algorithm can be started. If $q = 1$ then $s = b - p + 1$ and from the assumption $m_1 + p > b$ it follows that $m_1 \geq s \geq 2$. If $q = p - 1$ then $s = b - 1$, which can be greater than m_1. In that case the inequality

$$x_1 + \cdots + x_{m_1} + (p-1)x_{m_1+1} \leq b - 1 \tag{28}$$

is equivalent to (17). (28) can be generated by the following multipliers:

$$\lambda_0 = \frac{p-1}{p},\ \lambda_1 = \lambda_2 = \cdots = \lambda_{m_1} = \frac{1}{p},\ \lambda_{m_1+1} = \cdots = \lambda_{2m_1+2} = 0.$$

The same procedure can be repeated until the appropriate s will not exceed m_1.

These results can be summarize in the following statement.

Theorem 4.3 *Let m_1, p, and b be positive integers such that $m_1 + p > b + 1$ and $p \geq 4$. Then the Chvatál rank of the integer hull of the set (17) is at most*

$$\left\lfloor \frac{p}{2} \right\rfloor.$$

4.2 The case $m_2 > 1$

It can be shown that the facet defining inequality has Chvatál rank 1 if and only if (16) holds, in a similar way by analyzing the simplex method. The details remain to the reader.

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