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On Acyclicity of Games with Cycles

Daniel Andersson[a] Vladimir Gurvich[b]
Thomas Dueholm Hansen[c]

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RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

[a] Dept. of Computer Science, Aarhus University, koda@cs.au.dk
[b] RUTCOR, Rutgers University, gurvich@rutcor.rutgers.edu
[c] Dept. of Computer Science, Aarhus University, tdh@cs.au.dk

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Abstract. We study restricted improvement cycles (ri-cycles) in finite positional n-person games with perfect information modeled by directed graphs (digraphs) that may contain directed cycles (di-cycles). We obtain criteria of restricted improvement acyclicity (ri-acyclicity) in two cases: for $n = 2$ and for acyclic digraphs. We also provide several examples that outline the limits of these criteria and show that, essentially, there are no other ri-acyclic cases. We also discuss connections between ri-acyclicity and some open problems related to Nash-solvability.

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1 Main Concepts and Results

1.1 Games in Normal Form

Game Forms; Strategies and Preference Profiles. Given a set of players $I = \{1, \dots, n\}$, a set of strategies X_i for each $i \in I$, and a set of outcomes A, a mapping $g : X \to A$, where $X = \prod_{i \in I} X_i$, is called a *game form.* In this paper, we restrict ourselves to *finite* game forms, i.e., we assume I, A and X to be finite. A vector $x = (x_i, i \in I) \in \prod_{i \in I} X_i = X$ is called a *strategy profile.*

Furthermore, let $u : I \times A \to \mathbb{R}$ be a utility function. Standardly, the value $u(i, a)$ (or $u_i(a)$) is interpreted as the payoff to player $i \in I$ in case of the outcome $a \in A$. In figures, the notation $a <_i b$ means $u_i(a) < u_i(b)$.

Sometimes, it is convenient to exclude ties. Accordingly, u is called a *preference profile* if the mapping u_i is injective for each $i \in I$. In this case, u_i defines a complete order over A. This order describes the preferences of player $i \in I$.

A pair (g, u) is called a *game in normal form.*

Improvement Cycles and Acyclicity. In a game (g, u), an *improvement cycle (im-cycle)* is defined as a sequence of k strategy profiles $\{x^1, \dots, x^k\} \subseteq X$ such that x^j and x^{j+1} coincide in all coordinates but one $i = i(j)$ and, moreover, $u_i(x^{j+1}) > u_i(x^j)$, that is, the corresponding player $i = i(j) \in I$ profits by substituting strategy x_i^{j+1} for x_i^j.

We assume that sums are taken modulo k, that is, $k+1 = 1$; in other words, the sequence of the obtained profiles forms a cycle $x^1, \dots, x^k, x^1$.

A game (g, u) is called *im-acyclic* if it has no im-cycles. A game form g is called *im-acyclic* if for each u the corresponding game (g, u) is im-acyclic.

We call x^{j+1} an improvement with respect to x^j for player $i = i(j)$. We call it a *best reply* (BR) improvement if player i cannot get a strictly better result provided all other players keep their strategies. Correspondingly, we introduce the concepts of a BR im-cycle and BR im-acyclicity. Obviously, im-acyclicity implies BR im-acyclicity but not vice versa.

Nash Equilibria and Acyclicity. Given a game (g, u), a strategy profile $x \in X$ is called a *Nash equilibrium* (NE) if $u_i(x) \geq u_i(x')$ for each $i \in I$, whenever $x'_j = x_j$ for all $j \in I \setminus \{i\}$. In other words, x is a NE if no player can get a strictly better result by substituting a new strategy (x'_i for x_i) when all other players keep their old strategies.

Conversely, if x is not a NE then some player i can make such a change. In particular, i can choose a best reply. Hence, a NE-free game (g, u) has a BR im-cycle.

Remark 1. Let us mention that the above implication holds only for finite games and the inverse one does not hold at all.

A game (g, u) is called *Nash-solvable* if it has a NE. A game form g is called *Nash-solvable* if for each u the corresponding game (g, u) has a NE.

1.2 Positional Games with Perfect Information

Games in Positional Form. Let $G = (V, E)$ be a finite directed graph (digraph) whose vertices $v \in V$ and directed edges $e \in E$ are called *positions* and *moves*, respectively. The edge $e = (v', v'')$ is a move from position v' to v''. Let $out(v)$ and $in(v)$ denote the sets of moves from and to v, respectively.

Position $v \in V$ is called *terminal* if $out(v) = \emptyset$, that is, there are no moves from v. Let V_T denote the set of all terminal positions.

Let us also fix an initial position $v_0 \in V \setminus V_T$. A directed path from v_0 to a position $v \in V_T$ (respectively, to $v \in V \setminus V_T$) is called a finite *play* (respectively, *a debut*).

Furthermore, let $I = \{1, \ldots, n\}$ be the set of players and $D : V \setminus V_T \to I$ be a decision mapping. We will say that the player $i = D(v) \in I$ makes a decision (move) in a position $v = D^{-1}(i) \in V_i$. Equivalently, D is defined by a partition of positions $D : V = V_1 \cup \ldots \cup V_n \cup V_T$. In this paper we do not consider random moves.

The triplet $\mathcal{G} = (G, D, v_0)$ is called a *positional game form.*

Cycles, Outcomes, and Utility Functions. Let C denote the set of simple (that is, without self-intersections) directed cycles (di-cycles) in G. The set of outcomes A can be defined in two ways:

(i) $A = V_T \cup C$, that is, each terminal and each di-cycle is a separate outcome.
(ii) $A = V_T \cup \{C\}$, that is, each terminal is an outcome and all di-cycles define one special outcome. We will denote it by $c = \{C\}$.

Case (i) was considered in [2] for two-person games ($n = 2$); see Section 1.3 for more details. In this paper, we analyze case (ii) for n-person games.

Remark 2. Let us mention that as early as in 1912, Zermelo already considered case (ii) for the zero-sum two-person games in his pioneering work [11], where the game of Chess was chosen as a basic example. Obviously, the corresponding graph contains di-cycles: One appears whenever a position is repeated in a play. By definition, all cycles are treated as one outcome — a draw. More precisely, Chess results in a draw whenever the same position appears three times in a play. Yet, this difference does not matter, since we are going to restrict ourselves to positional (stationary) strategies; see Remark 3.

Standardly, a mapping $u : I \times A \to \mathbb{R}$ defines a utility function. Let us remark that players can rank outcome c arbitrarily in their preferences. In contrast, in [1] it was assumed that cycle $c \in A$ is the worst outcome for all players $i \in I$.

Positional Games in Normal Form. The triplet $\mathcal{G} = (G, D, v_0)$ and the quadruple $(G, D, v_0, u) = (\mathcal{G}, u)$ are called the *positional form* and the *positional game*, respectively. Every positional game can also be represented in *normal form*, as described below.

A mapping $x : V \setminus V_T \to E$ that assigns to every non-terminal position v a move $e \in out\ (v)$ from this position is called a *situation or strategy profile.* A *strategy* of player

$i \in I$ is the restriction $x_i : V_i \to E$ of x to $V_i = D^{-1}(i)$. In other words, the set of strategy profiles $X = \prod_{i \in I} X_i$ is the direct product of sets of strategies of all players.

Remark 3. A strategy x_i of a player $i \in I$ is interpreted as a decision plan for every position $v \in V_i$. Let us remark that, by definition, the decision in v can depend only on v itself but not on the preceding positions and moves, that is, not on the debut. In other words, we restrict the players to their positional strategies.

Each strategy profile $x \in X$ uniquely defines a play $p(x)$ that starts in v_0 and then follows the moves prescribed by x. The play either ends in a terminal of V_T or results in a cycle, $a(x) = c$. Thus, we obtain a game form $g_{\mathcal{G}} \to A$, which is called the *normal form* of $\mathcal{G}$.

This game form is standardly represented by an n-dimensional table whose entries are outcomes from $A = V_T \cup \{c\}$; see examples in Figures 1, 6 and 8. The pair $(g_{\mathcal{G}}, u)$ is called the *normal form* of the positional game $(\mathcal{G}, u)$.

1.3 On Nash-Solvability of Positional Game Forms

In [2], Nash-solvability of positional game forms was considered for case (i); each di-cycle is a separate outcome. An explicit characterization of Nash-solvability was obtained for the two-person ($n = 2$) game forms whose digraphs are *bidirected*: $(v', v'') \in E$ if and only if $(v'', v') \in E$.

In [1], case (ii); all dicycles form one outcome c, was considered with an additional assumption:

(ii') c is ranked worst by all players.

Under this additional assumption Nash-solvability was proven in the following three cases:

(a) Two-person games ($n = |I| = 2$).
(b) Games with at most three outcomes ($p = |A| \leq 3$).
(c) Play-once games, in which each player controls only one position ($|V_i| = 1$ for every $i \in I$).

Also, the following conjecture was raised:

Conjecture 1. ([1]) In case (ii') Nash-solvability always holds.

This Conjecture would be implied by the following statement:

Every im-cycle contains a di-cycle, or more precisely:

Every im-cycle $\mathcal{X} = \{x^1, \ldots, x^k\} \subseteq X$ contains a strategy profile x^j such that the corresponding play $p(x^j)$ results in a di-cycle.

Indeed, Conjecture 1 would follow, since the outcome $c \in A$ being the worst one for all players, belongs to no im-cycle. However, the example of Section 2.3 will show that such an approach fails.

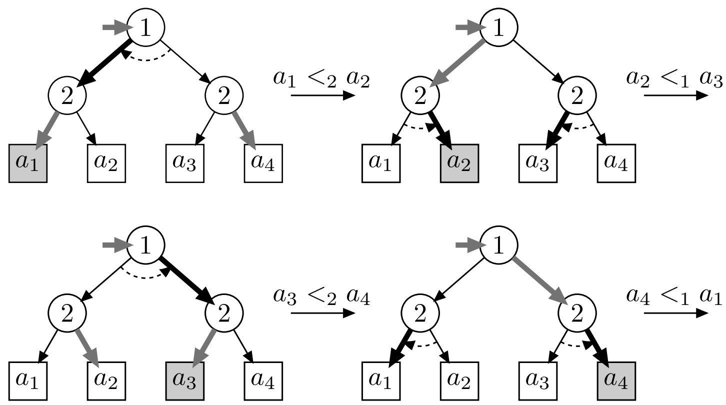
Nevertheless, Conjecture 1 is not disproved. Moreover, a stronger conjecture was recently suggested by Gimbert and Sørensen (private communications). They assumed that condition (ii') is not needed.

Conjecture 2. Every positional game is Nash-solvable, in case (ii).

They gave a simple and elegant proof for the two-person case. With their permission, we reproduce it in Section 5.

1.4 Restricted Improvement Cycles and Acyclicity

Improvement Cycles in Trees. Kukushkin [8, 9] was the first to consider im-cycles in positional games. He restricted himself to trees and observed that even in this case im-cycles can exist. Let us recall his introductory example from [8]. The example can be found in Figure 1. The preference constraint required to change from one strategy profile to the next is shown above each transition-arrow, and the players' preferences are displayed at the bottom left corner. At the bottom right corner the normal form representation of the im-cycle is displayed.



$1: \ a_2 < a_3, \ a_4 < a_1$

$2: \ a_1 < a_2, \ a_3 < a_4$

$$\begin{array}{c} \to 2 \\ \downarrow \\ 1 \end{array} \begin{bmatrix} a_1 & a_1 & \rightarrow & a_2 & a_2 \\ & \uparrow & & \downarrow & \\ a_3 & a_4 & \leftarrow & a_3 & a_4 \end{bmatrix}$$

Fig. 1. Im-cycle in a tree.

Indeed, it is easy to verify that the following four strategy profiles

$$x^1 = (x_1^1, x_2^2), \ x^2 = (x_1^1, x_2^3), \ x^3 = (x_1^2, x_2^3), \ x^4 = (x_1^2, x_2^2) \in X$$

form an im-cycle whenever

$$u_1(a_2) < u_1(a_3), u_1(a_4) < u_1(a_1) \quad \text{and} \quad u_2(a_1) < u_2(a_2), u_2(a_3) < u_2(a_4),$$

where $g(x^j) = a_j$ for $j = 1, 2, 3, 4$.

Yet, it is also easy to see that some unnecessary changes of strategies take place in this im-cycle. For example, let us consider transition from $x^1 = (x_1^1, x_2^2)$ to $x^2 = (x_1^1, x_2^3)$. Player

1 keeps her strategy x_1^1, while 2 substitutes x_2^3 for x_2^2 and gets a profit, since $g(x_1^1, x_2^2) = a_1$, $g(x_1^1, x_2^3) = a_2$, and $u_2(a_1) < u_2(a_2)$. However, x_2^2 chooses a_1 and a_4, while x_2^3 chooses a_2 and a_3. Switching from a_1 to a_2 is a reasonable action for player 2, since $u_2(a_1) < u_2(a_2)$. In contrast, simultaneously switching from a_4 to a_3 cannot serve any practical purpose, since the decision is changed outside the actual play ($p(x^1)$ that led to a_1). It is clear that such changes make no sense, yet, they can prepare im-cycles.

In [8], Kukushkin introduced the concept of *restricted improvements* (ri). In particular, he proved that positional games on trees become ri-acyclic if players are not allowed to change their decisions outside the actual play. For completeness, we will sketch his simple and elegant proof in Section 3.1, where we also mention some related results and problems.

Since we consider arbitrary finite digraphs (not only trees), let us define accurately several types of restrictions for this more general case.

Inside Play Restriction. Given a positional game form $\mathcal{G} = (G, D, v_0)$ and strategy profile $x^0 = (x_i^0, i \in I) \in X$, let us consider the corresponding play $p_0 = p(x^0)$ and outcome $a_0 = a(x^0) \in A$. This outcome is either a terminal, $a_0 \in V_T$, or a cycle, $a_0 = c$.

Let us consider the strategy x_i^0 of a player $i \in I$. He is allowed to change his decision in any position v_1 from p_0. This change will result in a new strategy profile x^1, play $p_1 = p(x^1)$, and outcome $a_1 = a(x^1) \in A$.

Then, player i may proceed, changing his strategy further. Now, he is only allowed to change the decision in any position v_2 that is located *after* v_1 in p_1, etc., until a position v_k, strategy profile x^k, play $p_k = p(x^k)$, and outcome $a_k = a(x^k) \in A$ appears; see Figure 2, where $k = 3$.

Equivalently, we can say that all positions $v_1, \ldots, v_k$ belong to one play.

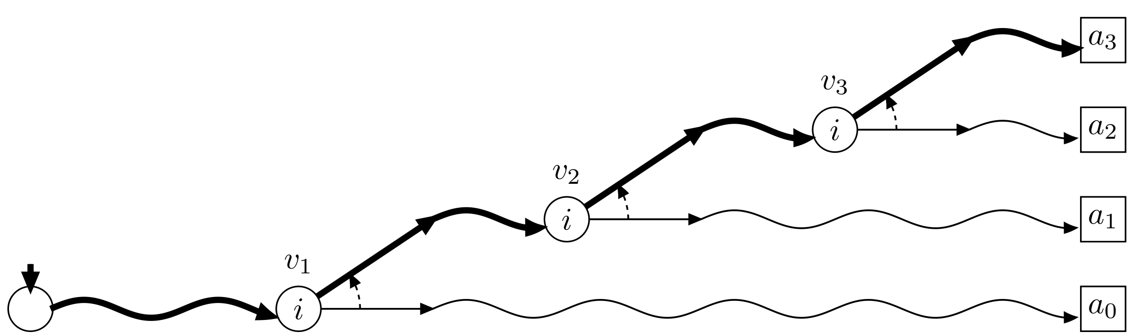


Fig. 2. Inside play restriction.

Let us remark that, by construction, obtained plays $\{p_0, p_1, \ldots, p_k\}$ are pairwise distinct. In contrast, the corresponding outcomes $\{a_0, a_1, \ldots, a_k\}$ can coincide and some of them might be di-cycles, that is, equal to $c \in A$.

Whenever the acting player i substitutes the strategy x_i^k, defined above, for the original strategy x_i^0, we say that this is an *inside play deviation*, or in other words, that this change of decision in x satisfies the *inside play restriction*.

It is easy, but important, to notice that this restriction, in fact, does not limit the power of a player. More precisely, if a player i can reach an outcome a_k from x by a deviation then i can also reach a_k by an inside play deviation.

From now on, we will consider only such *inside play restricted* (or just *restricted*, in short) deviations and, in particular, only restricted improvements (ri). We will talk about ri-cycles and ri-acyclicity rather than im-cycles and im-acyclicity, respectively.

Types of Improvements. We define the following four types of improvements:

Standard improvement (or just improvement): $u_i(a_k) > u_i(a_0)$;
Strong improvement: $u_i(a_k) > u_i(a_j)$ for $j = 0, 1, \dots, k-1$;
Last step improvement: $u_i(a_k) > u_i(a_{k-1})$;
Best reply (BR) improvement: a_k is the best outcome that player i can reach from x (as noted above, the inside play restriction does not restrict the set of reachable outcomes).

Obviously, each best reply or strong improvement is a standard improvement. Furthermore, it is easy to verify that no other containments hold between the above four classes.

For example, a last step improvement might not be an improvement and vice versa. Similarly, a BR-improvement might not be strong, since equalities $u_i(a_k) = u_i(a_j)$ can hold for some $j = 0, 1, \dots, k-1$. Conversely, a strong improvement might not be a BR.

We will consider ri-cycles and ri-acyclicity specifying in each case a type of improvement from the above list.

Let us remark that any type of ri-acyclicity still implies Nash-solvability. Indeed, if a positional game has no NE then for every strategy profile $x \in X$ there is a player $i \in I$ who can improve x by x'. In particular, i can always choose a strong BR restricted improvement. Then, x' is not a NE, either; etc. Since we consider only finite games, such an iterative procedure will result in a strong BR ri-cycle. Equivalently, if we assume that there is no such cycle then the considered game is Nash-solvable; in other words, already BR strong ri-acyclicity implies Nash-solvability.

1.5 Sufficient Conditions for Ri-acyclicity

We start with Kukushkin's result for trees.

Theorem 1. *([8]). Positional games on trees have no restricted standard improvement cycles.*

After trees, it is natural to consider acyclic digraphs. The next criterion is also suggested by Kukushkin (private communications).

Theorem 2. *Positional games on acyclic digraphs have no restricted last step improvement cycles.*

Let us remark that Theorem 1 does not result immediately from Theorem 2, since standard improvements might not be last step improvements.

Finally, for two-person positional games (that can have di-cycles) the following statement holds.

Theorem 3. *Two-person positional games have no restricted strong improvement cycles.*

Let us remark that this statement implies Nash-solvability of two-person positional games; see Section 5 for an independent proof.

We prove Theorems 1, 2, 3, in Sections 3.1, 3.2, 3.3, respectively.

2 Examples of Ri-cycles

In this paper, we emphasize negative results showing that it is unlikely to strengthen one of the above theorems or obtain other criteria of ri-acyclicity.

2.1 Examples Limiting Theorems 2 and 3

The example in Figure 3 shows that for both Theorems, 2 and 3, the specified type of improvement is essential. Indeed, this example shows that a two-person game on an acyclic digraph can have a ri-cycle. However, it is not difficult to see that in this ri-cycle, not all improvements are strong. Moreover, some of them are not even the last step improvements.

Thus, all conditions of Theorems 2 and 3 are essential.

Furthermore, if in Theorem 3 we substitute BR improvement for strong improvement, the modified statement will not hold. Indeed, the example in Figure 4 shows a two-person positional game with a BR ri-cycle in which not all improvements are strong.

2.2 Preference Acyclicity

By definition, every change of strategy must result in an improvement for the corresponding player. Hence, each im-cycle implies a set of preferences for each player. Obviously, these sets of preferences must be acyclic. Thus, we obtain one more type of acyclicity. Let us call it *preference acyclicity* (pr-acyclicity). For example, the ri-cycle in Figures 3 implies

$$u_1(a_1) < u_1(a_2) < u_1(a_3) < u_1(a_4),$$

$$u_2(a_4) < u_2(a_2) < u_2(a_3) < u_2(a_1),$$

while the one in Figure 4 implies: $u_1(c) < u_1(a_1),\ u_2(a_1) < u_2(c)$.

2.3 On c-free Ri-cycles

In Section 1.3, we demonstrated that Conjecture 1 on Nash-solvability would result from the following statement

(i) **There are no c-free im-cycles**.

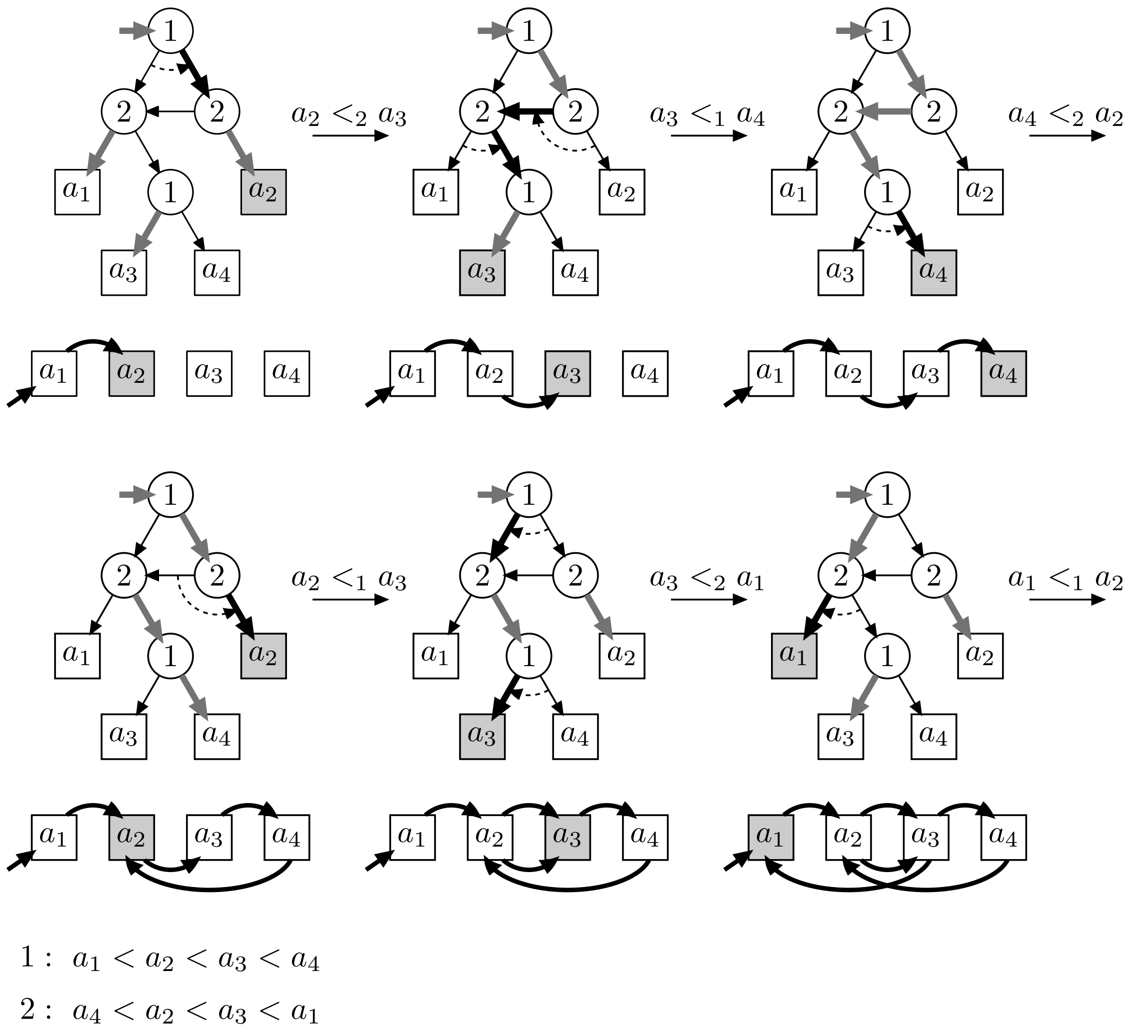


Fig. 3. 2-person ri-cycle in acyclic digraph.

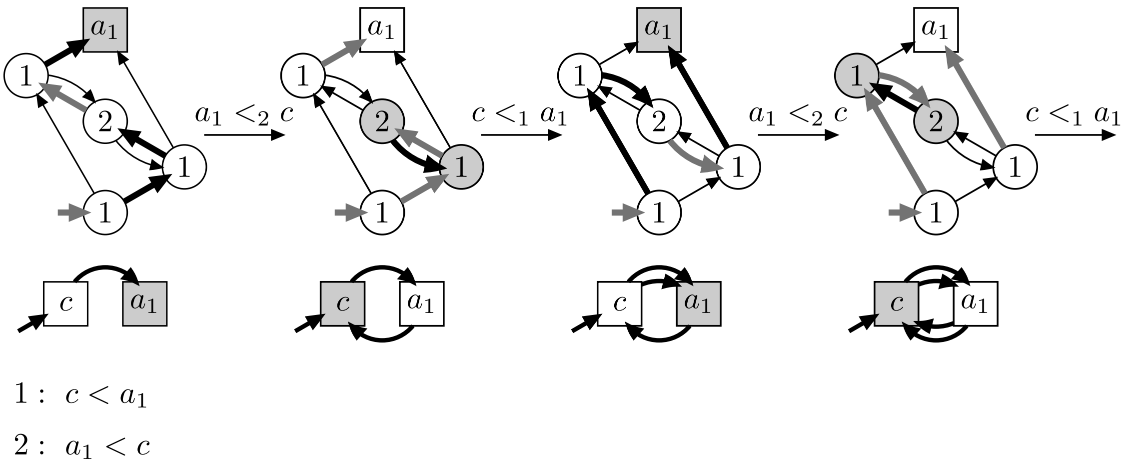


Fig. 4. 2-person BR ri-cycle in graph with cycles.

Of course, (i) fails. As we know now, im-cycles exist already in trees (see Figure 1), which do not have di-cycles. However, let us substitute (i) by the similar but much weaker statement

(ii) **Every restricted BR strong im-cycle contains a di-cycle**.

One can derive Conjecture 1 from (ii), as easily as from (i).

Unfortunately, (ii) also fails. Indeed, let us consider the ri-cycle in Figure 5. This game is play-once; each player controls only one position. Moreover, there are only two possible moves in each position. For this reason, every ri-cycle in this game is BR and strong.

There are seven players ($n = 7$) in this example, yet, by teaming up players in coalitions we can reduce the number of players to four while the improvements remain BR and strong. Indeed, this can be done by forming three coalitions $\{1, 7\}, \{3, 5\}$, $\{4, 6\}$ and merging the preferences of the coalitionists. The required extra constraints on the preferences of the coalitions. are also shown in Figure 5.

It is easy to see that a pr-cycle appears whenever any three players form a coalition. Hence, the number of coalitions cannot be reduced below 4, and it is, in fact, not possible to form 4 coalitions in any other way while keeping improvements BR and strong.

Obviously, for the two-person case, Theorem 3 implies (ii). Yet, for $n = 2$ Conjectures 1 and 2 are known to be true; see Section 3.

Remark 4. We should confess that our original motivation fails. It is hardly possible to derive new results on Nash-solvability from ri-acyclicity. Although, ri-acyclicity is much weaker than im-acyclicity, it is still too much stronger than Nash-solvability. In general, by Theorems 2 and 3, ri-acyclicity holds for $n = 2$ and for acyclic digraphs. Yet, for these two cases Nash-solvability is known. It is still possible that (ii) (and, hence, Conjecture 1) holds for $n = 3$, too.

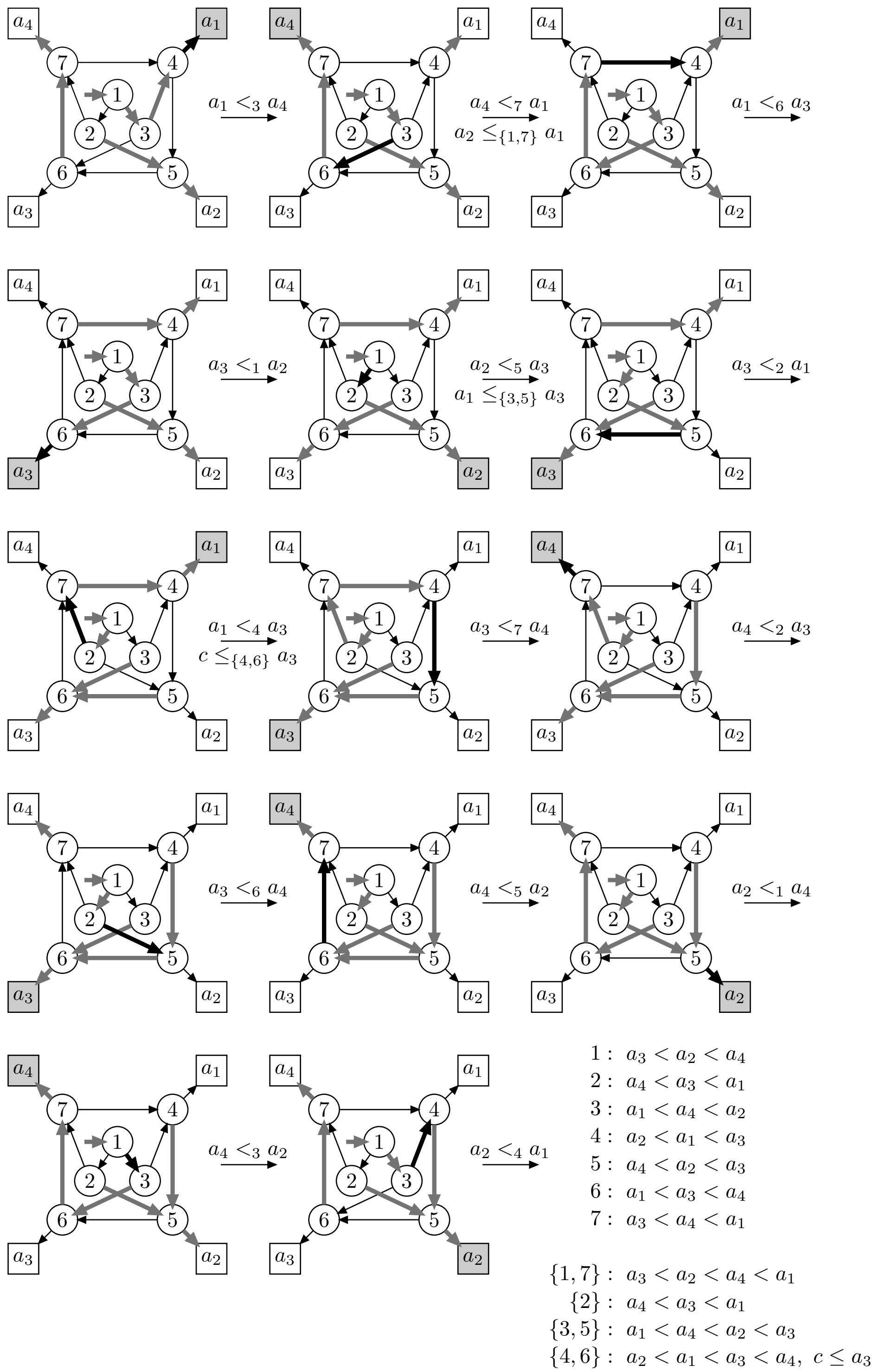


Fig. 5. c-free strong BR ri-cycle.

However, ri-acyclicity is of independent (of Nash-solvability) interest. In this paper, we study ri-acyclicity for the case when each terminal is a separate outcome, while all di-cycles form one special outcome. For the alternative case, when each terminal and each di-cycle is a separate outcome, Nash-solvability was considered in [2], while ri-acyclicity was never studied.

2.4 Flower Games: Ri-cycles and Nash-Solvability

Flower Positional Game Forms. A positional game form $\mathcal{G} = (G, D, v_0)$ will be called a *flower* if there is a (chordless) di-cycle C in G that contains all positions, except the initial one, v_0, and the terminals, V_T; furthermore, we assume that there are only moves from v_0 to C and from C to V_T; see examples in Figures 6, 8 and 9.

By definition, C is a unique di-cycle in G. Nevertheless, it is enough to make flower games very different from acyclic games; see [1] (where flower games are referred to as St. George games). Here we consider several examples of ri-cycles in flower game forms of 3 and 4 players; see Figures 6, 8 and 9. Let us note that the game forms of Figures 6 and 8 are play-once: each player is in control of one position, that is, $n = |I| = |V \setminus V_T| = 3$ or 4, respectively. In fact, Figure 9 can also be turned into a six-person play-once flower game.

Flower Three-Person Game Form. Positional and normal forms of a three-person flower game are given in Figure 6. This game form is ri-acyclic. Indeed, it is not difficult to verify that an im-cycle in it would result in a pr-cycle for one of the players. Yet, there is a ri-path of length 7 (that is, a Hamiltonian im-path).

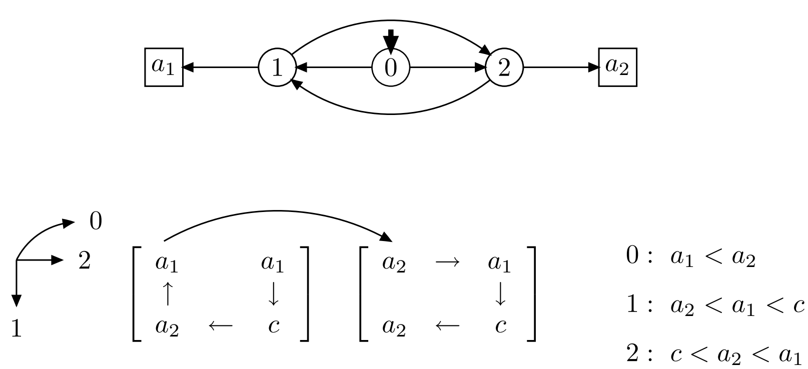
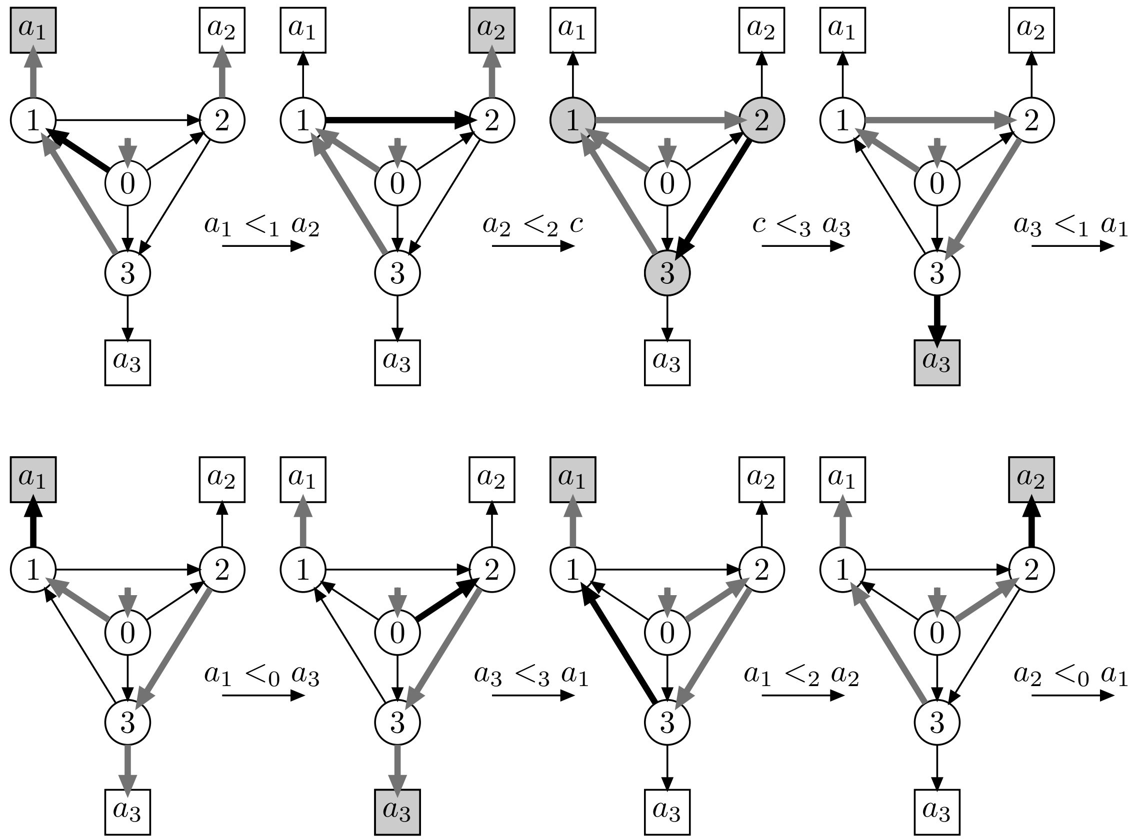


Fig. 6. Hamiltonian im-path.

Flower Four-Person Game Form. Positional and normal forms of a four-person flower game are given in Figures 7 and 8, respectively, where a ri-cycle is shown. Obviously, it is a strong and BR ri-cycle, since there are only two possible moves in every position. However, it contains c.

The number of players can be reduced by forming the coalition $\{1,2\}$ or $\{1,3\}$. However, in the first case the obtained ri-cycle is not BR, though it is strong, whereas a non-restricted improvement appears in the second case.



$0: \ a_2 < a_1 < a_3$
$1: \ a_3 < a_1 < a_2$
$2: \ a_1 < a_2 < c$
$3: \ c < a_3 < a_1$

Fig. 7. Positional form of a ri-cycle in the flower game form with 4 players.

Moreover, no c-free ri-cycle can exist in this four-person flower game form. To see this, let us consider the graph of its normal form shown in Figure 8. It is not difficult to verify that, up to isomorphism, there is only one ri-cycle, shown above. All other "ri-cycles" are fake, since they imply pr-cycles; see the second graph in Figure 8.

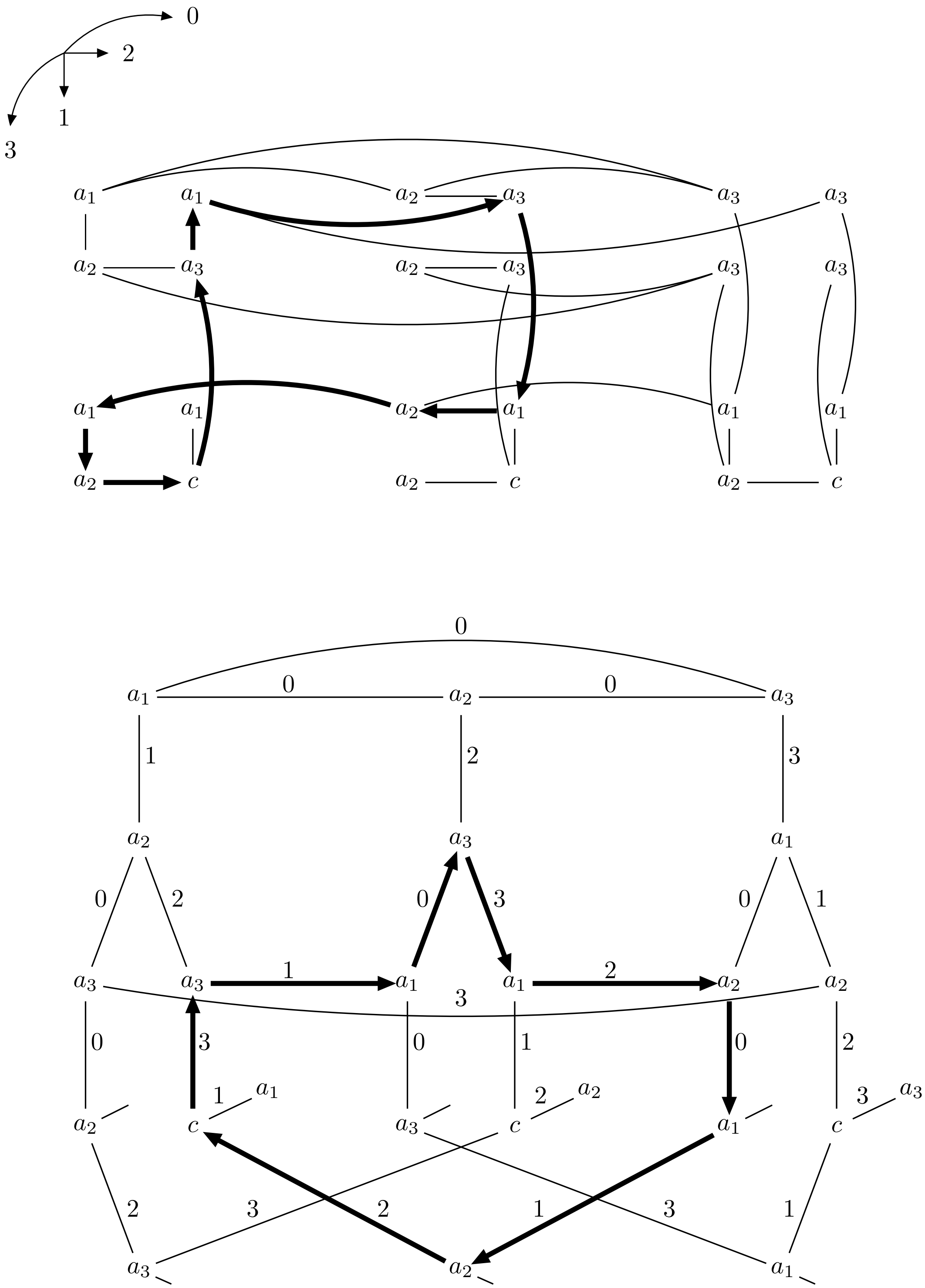


Fig. 8. Normal form and unfolded normal form of a ri-cycle in the flower game form with 4 players.

On BR Ri-cycles in Three-Person Flower Games. In Section 2.3 we gave an example of a c-free strong BR ri-cycle in a four-person game. Yet, the existence of such ri-cycle in a three-person game remains open. However, a strong BR ri-cycle that contains c can exist already in a three-person flower game; see Figure 9.

Nash-Solvability of Flower Games. In this section we assume without loss of generality that v_0 is controlled by player 1 and that every position v in C has exactly two moves: one along C and the other to a terminal $a = a_v \in V_T$. Indeed, if a player has several terminal moves from one position then, obviously, all but one, which leads to the best terminal, can be eliminated.

We will call positions in C gates and, given a strategy profile x, we call gate $v \in C$ open (closed) if move (v, a_v) is chosen (not chosen) by x.

First, let us consider the simple case when player 1 controls only v_0 and later we will reduce Nash-solvability of flower games in general to this case.

Lemma 1. *Flower games in which player* 1 *controls only* v_0 *are Nash-solvable.*

Proof. Let us assume that there is a move from v_0 to each position of C. In general, the proof will remain almost the same, except for a few more cases; see below.

The following alternative holds: (i) either for each position $v \in C$ the corresponding player $i = D(v)$ prefers c to $a = a_v$, or (ii) there is a $v' \in C$ such that $i' = D(v')$ prefers $a' = a_{v'}$ to c. If a player controls several such positions then let a' be his best outcome.

In case (i), each strategy profile such that all gates are closed is a NE. In case (ii), the following strategy profile x is a NE: Player 1 moves from v_0 to v', the gate v' is open, and all other gates are closed. □

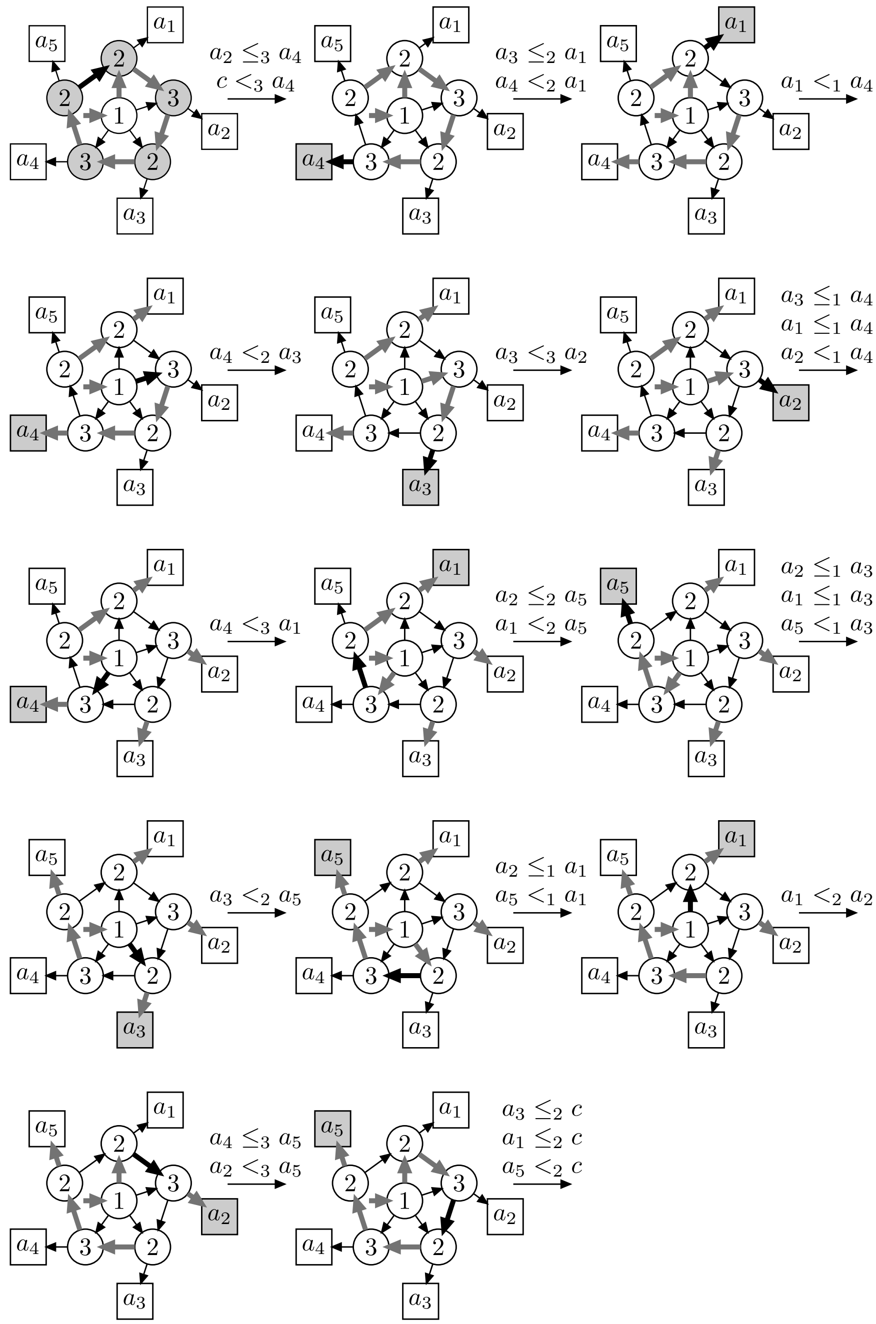
Theorem 4. *Flower games are Nash-solvable.*

Proof. By Lemma 1, it is sufficient to show that flower games are Nash-solvable if and only if flower games in which player 1 controls only v_0 are Nash-solvable.

We will give an indirect proof. Let $(\mathcal{G}, u)$ be a NE-free flower game. Moreover, let us assume that it is *minimal*, that is, a NE appears whenever we delete any move from $\mathcal{G}$. This assumption implies that for each gate e, there is a strategy profile x^1 such that this gate is closed but it is opened by a BR restricted improvement x^2. Since the game is NE-free, there is an infinite sequence $\mathcal{X} = \{x^1, x^2, \ldots\}$ of such BR restricted improvements. Then, it follows from Theorem 2 that gate $e = (v, a_v)$ will be closed again by a profile $x^k \in \mathcal{X}$. Indeed, if we delete e, the reduced graph is acyclic.

Now, let us assume that gate $e = (v, a)$ is controlled by player 1. Let $e' = (v', a')$ be the closest predecessor of v in C such that there is a move from v_0 to v'. Opening v, player 1 can at the same time choose the move (v_0, v').

Clearly, until v will be closed again no gate between v' and v in C, including v' itself, will be opened. Indeed, otherwise the corresponding gate could be closed again by no sequence $\mathcal{X}$ of restricted best replies. Since player 1 already performed a BR, the next one must be



Feasible total order:

1 : $a_5 < a_2 < a_1 < a_3 < a_4 < c$

2 : $a_4 < a_3 < a_1 < a_2 < a_5 < c$

3 : $c < a_3 < a_2 < a_4 < a_1 < a_5$

Fig. 9. Strong BR ri-cycle in 3-person flower game.

performed by another player. However, these players control only the gates between v' and v in C. Hence, one of them will be opened.

Thus, we obtain a contradiction. Indeed, if a NE-free flower game has a gate of player 1, it will never be required to open. By deleting such gates repeatedly one gets a NE-free flower game that has no gates of player 1. □

3 Proofs of Theorems 1, 2, and 3

3.1 Ri-acyclicity for Trees. Proof of Theorem 1

As we know, im-cycles can exist even for trees (see Section 1.4) but ri-cycles cannot. Here we sketch the proof from [8].

Given a (directed) tree $G = (V, E)$ and an n-person positional game $(\mathcal{G}, u) = (G, D, v_0, u)$, let $p_i = \sum_{v \in V_i}(|out(v)| - 1)$ for every player $i \in I = \{1, \ldots, n\}$. It is not difficult to verify that $1 + \sum_{i=1}^{n} p_i = p = |V_T|$.

Let us fix a strategy profile $x = (x_1, \ldots, x_n)$. To every move $e = (v, v')$ which is *not* chosen by x let us assign the outcome $a = a(e, x)$ which x would result in starting from v'. it is easy to see that these outcomes together with $a(x)$ form a partition of V_T.

Given a player $i \in I$, let us assign p_i numbers $u_i(a(e, x))$ for all $e = (v, v')$ not chosen by x_i, where $v \in V_i$. Let us order these numbers in monotone non-increasing order and denote the obtained p_i-dimensional vector $y_i(x)$.

Let player $i \in I$ substitute a restricted improvement x'_i for x_i; see Section 1.4. The new strategy profile x' results in an outcome $a_k \in A = V_T$ which is strictly better for i than the former outcome $a_0 = a(x)$. Let us consider vectors $y_j(x)$ and $y_j(x')$ for all $j \in I$. It is not difficult to verify that these two vectors are equal for each $j \in I$, except $j = i$, while $y_i(x)$ and $y_i(x')$, for the acting player i, differ by only one number: $u_i(a_k)$ in $y_i(x')$ substitutes for $u_i(a_0)$ in $y_i(x)$. The new number is strictly larger than the old one, because, by assumption of Theorem 1, x'_i is an improvement with respect to x_i for player i. Thus, vectors y_j for all $j \neq i$ remain unchanged, while y_i becomes strictly larger. Hence, no ri-cycle can appear. □

Yet, there are ri-paths. An interesting question: what is the length of the longest ri-path? Given $n = |I|$, $p = |A|$, and p_i such that $\sum_{i=1}^{n} p_i = p - 1 \geq n \geq 1$, the above proof of Theorem 1 implies the following upper bound: $\sum_{i=1}^{n} p_i(p - p_i)$.

It would also be interesting to get an example with a high lower bound.

3.2 Last Step Ri-acyclicity for Acyclic Digraphs. Proof of Theorem 2

Given positional game $(\mathcal{G}, u) = (G, D, v_0, u)$ whose digraph $G = (V, E)$ is acyclic, let us order positions of V so that $v < v'$ whenever there is a directed path from v to v'. To do so, let us assign to each position $v \in V$ the length of a longest di-path from v_0 to v and then order arbitrarily positions with equal numbers.

Given a strategy profile x, let us, for every $i \in I$, assign to each position $v \in V_i$ the outcome $a(v, x)$ which x would result in starting from v and the number $u_i(a(v, x))$. These numbers form a $|V \setminus V_T|$-dimensional vector $y(x)$ whose coordinates are assigned to positions $v \in V \setminus V_T$. Since these positions are ordered, we can introduce the inverse lexicographic order over such vectors y.

Let a player $i \in I$ choose a last step ri-deviation x'_i from x_i. Then, $y(x') > y(x)$, since the last changed coordinate increased: $u_i(a_k) > u_i(a_{k-1})$. Hence, no last step ri-cycle can exist. □

3.3 Strong Ri-acyclicity of Two-Person Games. Proof of Theorem 3

Given a two-person positional game (G, D, v_0, u) and a strategy profile x such that in the resulting play $p = p(x)$ the terminal move (v, a) belongs to a player $i \in I$, a strong improvement x'_i results in a terminal $a' = p(x')$ such that $u_i(a') > u_i(a)$. (This holds for n-person games, as well.)

Given a strong ri-cycle $\mathcal{X} = \{x^1, \ldots, x^k\} \in X$, let us assume, without any loss of generality, that game (G, D, v_0, u) is minimal with respect to $\mathcal{X}$, that is, no move can be deleted from $\mathcal{G}$ without breaking the cycle $\mathcal{X}$.

Let us consider the multi-digraph $\mathcal{E}$ whose vertices are outcomes $a \in A$ and directed edges are pairs $(a_j, a_{j+1}) \in A \times A$, where $a_j = a(x^j)$ and $j = 1, \ldots, k$. It is easy to see that $\mathcal{E}$ is a *Eulerian* multi-digraph, that is, strongly connected and for each vertex its in-degree in and out-degree are equal.

Remark 5. If we restrict ourselves to BR ri-cycles then $\mathcal{E}$ will be two-colored, that is, all its edges are naturally partitioned in two classes E_1 and E_2 corresponding to the deviations of players 1 and 2; see, for example, Figures 3 and 4, where edges of these two classes are shown above and below, respectively. Obviously, each BR ri-cycle $\mathcal{X}$ is associated with an Eulerian circuit in which these two classes alternate. Then obviously, for each vertex v, the following two equalities hold:

$$|in(v) \cap E_1| = |out(v) \cap E_2| \quad \text{and} \quad |in(v) \cap E_2| = |out(v) \cap E_1|$$

However, Theorem 3 claims strong, but not necessarily BR, acyclicity.

Furthermore, both subgraphs $G(E_1)$ and $G(E_2)$ (induced by E_1 and E_2, respectively) are acyclic, since otherwise a pr-cycle would appear in $\mathcal{X}$.

Hence, there is a vertex a^1 whose out-degree in $G(E_1)$ and in-degree in $G(E_2)$ both equal 0. In fact, an outcome a^1 most preferred by player 1 over all a_j, $j = 1, \ldots, k$, must have this property. (Let us remark that we do not exclude ties in preferences. If there are several best outcomes of player 1 then a^1 can be any of them.)

Similarly, we define a vertex a^2 whose in-degree in $G(E_1)$ and out-degree in $G(E_2)$ both equal 0.

Let us remark that either a^1 or a^2, but not both, might be equal to c. Thus, without loss of generality, let us assume that a^1 is a terminal outcome.

Either player 1 or 2 must have a move leading directly to a^1. In the first case, such a move cannot be changed by $\mathcal{X}$, since $u_1(a_j) \leq u_1(a^1)$ for all $j = 1, \ldots, k$.

Let us also recall that a^1 has no incoming edges of E_2. Hence, in $\mathcal{X}$, player 2 never makes an improvement that results in a^1.

It follows that whoever has a move leading directly to a^1 will either make it always or never. In both cases we obtain a contradiction with the minimality of $\mathcal{G}$. □

4 Laziness Restriction

In addition to the inside play restriction, let us consider the following closely related but stronger restriction.

Let player i substitute strategy x'_i for x_i to get a new outcome $a' = a(x')$ instead of $a = a(x)$. We call such a deviation *lazy*, or say that it satisfies the *laziness restriction*, if it minimizes the number of positions in which player i changes the decision to reach a'.

Let us note that the corresponding strategy x'_i might not be unique.

Obviously, each lazy deviation satisfies the inside play restriction.

Furthermore, if a lazy deviation is an improvement, $u_i(a) < u_i(a')$, then this improvement is strong.

Proposition 1. *Given a strategy profile x, a target outcome $a' \in A$, and a player $i \in I$, the problem of finding a lazy deviation from x_i to x'_i such that $a(x') = a'$ (and x' is obtained from x by substituting x'_i for x_i) reduces to the shortest directed path problem.*

Proof. Let us assign a length $d(e)$ to each directed edge $e \in E$ as follows: $d(e) = 0$ if move e is prescribed by x, $d(e) = 1$ for every other possible move of the acting player i, and $d(e) = \infty$ for all other edges. Then let us consider two cases: (i) $a' \in V_T$ is a terminal and (ii) $a' = c$.

In case (i), a shortest di-path from v_0 to a' defines a desired x'_i, and vice versa. Case (ii), $a' = c$, is a little more complicated.

First, for every directed edge $e = (v, v') \in E$, let us find a shortest di-cycle C_e that contains e and its length d_e. This problem is easily reducible to the shortest di-path problem, too. The following reduction works for an arbitrary weighted digraph $G = (V, E)$. Given a directed edge $e = (v, v') \in E$, let us find a shortest di-path from v' to v. In case of non-negative weights, this can be done by Dijkstra's algorithm.

Then, it is also easy to find a shortest di-cycle C_v through a given vertex $v \in V$ and its length d_v; obviously, $d_v = \min_{v' \in V}(d_e \mid e = (v, v'))$.

Then, let us apply Dijkstra's algorithm again to find a shortest path p_v from v_0 to every vertex $v \in V$ and its length d^0_v.

Finally, let us find a vertex v^* in which $\min_{v \in V}(d^0_v + d_v)$ is reached. It is clear that the corresponding shortest di-path p_{v^*} and di-cycle C_{v^*} define the desired new strategy x'_i. □

5 Nash-Solvability of Two-Person Positional Game Forms

If $n = 2$ and $c \in A$ is the worst outcome for both players, Nash-solvability was proven in [1]. In fact, the last assumption is not necessary: even if outcome c is ranked by two players arbitrarily, Nash-solvability still holds. This observation was recently made by Gimbert and Sørensen.

A two-person game form g is called:

Nash-solvable if for every utility function $u : \{1,2\} \times A \to \mathbb{R}$ the obtained game (g,u) has a Nash equilibrium.

zero-sum-solvable if for each zero-sum utility function ($u_1(a) + u_2(a) = 0$ for all $a \in A$) the obtained zero-sum game (g,u) has a Nash equilibrium, which is called a saddle point for zero-sum games.

$\pm$-*solvable* if zero-sum solvability holds for each u that takes only values: $+1$ and -1.

Necessary and sufficient conditions for zero-sum solvability were obtained by Edmonds and Fulkerson [3] in 1970; see also [5]. Somewhat surprisingly, these conditions remain necessary and sufficient for $\pm$-solvability and for Nash-solvability, as well; in other words, all three above types of solvability are equivalent, in case of two-person game forms [6]; see also [7] and Appendix 1 of [2].

Proposition 2. *([4]). Each two-person positional game form in which all di-cycles form one outcome is Nash-solvable.*

Proof. Let $\mathcal{G} = (G, D, v_0, u)$ be a two-person zero-sum positional game, where $u : I \times A \to \{-1, +1\}$ is a zero-sum ± 1 utility function. Let $A_i \subseteq A$ denote the outcomes winning for player $i \in I = \{1,2\}$. Without any loss of generality we can assume that $c \in A_1$, that is, $u_1(c) = 1$, while $u_2(c) = -1$. Let $V^2 \subseteq V$ denote the set of positions in which player 2 can enforce a terminal from A_2. Then, obviously, player 2 wins whenever $v_0 \in V^2$. Let us prove that player 1 wins otherwise, when $v_0 \in V^1 = V \setminus V^2$.

Indeed, if $v \in V^1 \cap V_2$ then $v' \in V^1$ for every move (v, v') of player 2; if $v \in V^1 \cap V_1$ then player 1 has a move (v, v') such that $v' \in V_1$. Let player 1 choose such a move for every position $v \in V^1 \cap V_1$ and an arbitrary move in each remaining position $v \in V^2 \cap V_1$. This rule defines a strategy x_1. Let us fix an arbitrary strategy x_2 of player 2 and consider the profile $x = (x_1, x_2)$. Obviously, play $p(x)$ cannot come to V_2 if $v_0 \in V_1$. Hence, for the outcome $a = a(x)$ we have: either $a \in V^1$ or $a = c$. In both cases player 1 wins. Thus, the game is Nash-solvable. $\square$

Let us recall that this result also follows immediately from Theorem 3.

Finally, let us remark that, already for two-person games, Nash equilibria can be unique but not subgame perfect. This can be seen in Figure 10.

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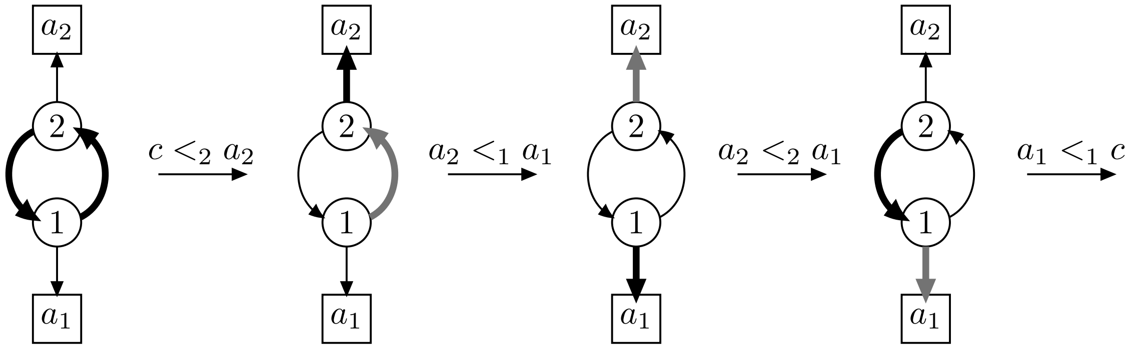

$1: \; a_2 < a_1 < c$

$2: \; c < a_2 < a_1$

Fig. 10. Two-person game with no subgame perfect positional strategies. The improvements do not obey any inside play restriction, since there is no fixed starting position.

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