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Abstract. A graph G is called CIS if each maximal clique intersects each maximal stable set in G, and is called almost CIS if it has a unique disjoint pair (C, S) consisting of a maximal clique C and a maximal stable set S. While it is still unknown if there exists a good structural characterization of all CIS graphs, in this note we prove the following Andrade-Boros-Gurvich conjecture: A graph is almost CIS if and only if it is a split graph with a unique split partition.

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Let $G = (V, E)$ be a graph. A *clique* of G is a set of pairwise adjacent vertices, and a *stable set* of G is a set of pairwise nonadjacent vertices. We call G a *CIS graph* if each maximal clique intersects each maximal stable set in G, where the adjective *maximal* is meant with respect to set-inclusion rather than size. The study of CIS graphs dates back to the 1960s when Grillet [8] proved that in every partially ordered set containing no quadruple (a, b, c, d) such that $a < b$, $c < d$, b covers c, and the remaining three pairs of elements are incomparable, each maximal chain meets each maximal antichain. With an attempt to generalize this theorem, Berge [2] made a conjecture and posed a research problem in terms of CIS graphs; see [9] for their solutions. Later, Chvátal [4, 9] proposed another conjecture concerning CIS graphs as a variation on Berge's problem, which was established independently by Andrade, Boros, and Gurvich [1] and Deng, Li, and Zang [5, 6]. We refer to [1] for an in-depth account of CIS graphs.

Despite considerable research effort, it is still unknown if there exists a good structural characterization of all CIS graphs. In this regard, Chvátal [4, 9] suggested the following problem.

Problem. *How difficult is it to recognize CIS graphs?*

As pointed out by Andrade, Boros, and Gurvich [1], CIS graphs somehow resemble perfect graphs in several ways; they also conjectured that this recognition problem, though very difficult, is polynomial-time solvable. On the other hand, given a graph G together with a specified maximal stable set S, it is co-NP-complete [9] to decide if S intersects every maximal clique of G.

By definition, if a graph G is not CIS, then it contains at least one disjoint pair (C, S) consisting of a maximal clique C and a maximal stable set S; such a (C, S) is called a *non-CIS pair* of G. Andrade, Boros, and Gurvich [1] proposed to call a graph *almost CIS* if it has a unique non-CIS pair, and discovered that almost CIS graphs are closely related to some well-known class of graphs. A graph is called *split* if its vertex set admits a partition (C, S), called a *split partition*, such that C is a clique and S is a stable set. As characterized by Foldes and Hammer [7], a graph is split if and only if it contains none of $2K_2$, C_4, and C_5 as an induced subgraph. Moreover, a split graph may have several split partitions; see, for instance, the graph obtained from a path $abcd$ by adding a fifth vertex e and making it adjacent to both b and c. It was shown in [3] that

Proposition 1. *A split graph has more than one split partition if and only if it is CIS.*

For completeness we include the proof of this statement from [3].

Proof. Let $G = (V, E)$ be a split graph with a split partition (A, B), where A is a clique and B is a stable set. We claim that (A, B) is the only possible non-CIS pair in G. To justify this, let C be a maximal clique different from A, and let S be a maximal stable set different from B. Clearly, C consists of a proper subset of A and a vertex u in B, and S consists of a proper subset of B and a vertex v in A. It is then a routine matter to check that $C \cap S = \{u\}$ if u and v are nonadjacent and $C \cap S = \{v\}$ otherwise. So the claim follows.

Suppose G admits a split partition (X, Y) other than (A, B). By the above claim, (X, Y) is also the only possible non-CIS pair in G. It follows that G contains no non-CIS pair. Hence G is a CIS graph. Conversely, suppose G is a CIS graph. Then either A is not a maximal clique or B is not a maximal stable set, say the former. Then there exists a vertex $u \in B$ such that $A \cup \{u\}$ is a clique. If $|B| \geq 2$, then $(A \cup \{u\}, B - \{u\})$ is a split partition of G other than (A, B). If $|B| = 1$, then G is a complete graph. So $(V - \{v\}, \{v\})$ is a split partition of G other than (A, B) for any vertex v in A. □

From Proposition 1 we see that every split graph with a unique split partition is almost CIS. Andrade, Boros, and Gurvich [1] conjectured that the converse of this statement also holds.

Conjecture. *Every almost CIS graph is a split graph with a unique split partition.*

Boros, Gurvich, and Zverovich [3] exhibited several nice properties enjoyed by almost CIS graphs and confirmed the conjecture for various graph classes. The purpose of this note is to present a proof of the whole conjecture.

Theorem. *A graph is almost CIS if and only if it is a split graph with a unique split partition.*

This characterization clearly yields a polynomial-time algorithm for recognizing almost CIS graphs. We remark that if CIS graphs could be recognized in polynomial time, then almost CIS graphs would surely play important roles in the structural characterization of these graphs.

The remainder of this note is devoted to a proof of this theorem. We shall repeatedly use the following trivial statement in the proof.

Proposition 2. *Let K and I be a maximal clique and a stable set in a graph G, respectively. If $K \cap I = \emptyset$ and each vertex in K has at least one neighbor in I, then $K \cap J = \emptyset$ for any stable set J containing I in G.*

Proof of the Theorem. Since the "if" part has already been established in [3], we only need to verify the "only if" part.

Throughout, let $G = (V, E)$ stand for a counterexample with the smallest number of vertices:

- G is an almost CIS graph;
- G is not a split graph with a unique split partition;
- every almost CIS graph having fewer vertices than G is a split graph with a unique split partition.

Let (C, S) be the unique non-CIS pair of G and let $\bar{G}$ denote the complement of G. It is easy to see that

(1) $\bar{G}$, an almost CIS graph with the unique non-CIS pair (S, C), is also a counterexample with the smallest number of vertices.

Set $A = V - (C \cup S)$. If $A = \emptyset$, then G would be a split graph with a split partition (C, S). Hence, by the assumption on G and Proposition 1, G would be a CIS graph, this contradiction implies $A \neq \emptyset$. We shall follow convention to let $N(v)$ denote the neighborhood of a vertex v in G and set $N_U(v) = N(v) \cap U$ for all subsets U of V.

(2) For any $a \in A$, there exists $s \in S$ such that $as \in E$ and $N_C(a) \subseteq N_C(s)$.

To justify this, let D be a maximal clique containing $N_C(a) \cup \{a\}$ in G. Then $D \cap S \neq \emptyset$ for (C, S) is the unique non-CIS pair of G. Clearly, the vertex s in $D \cap S$ has the property as described in (2).

It follows instantly from (2) that

(3) If $a \in A$ is adjacent to some $c \in C$, then a has at least one neighbor in $N_S(c)$.

Let us make some further observations.

(4) For any $s \in S$, one of the following holds:

(4.1) there exists $t \in S - \{s\}$ such that $N_C(s) \cup N_C(t) = C$;

(4.2) s is adjacent to all vertices in A.

To justify this, put $G_s = G - (N(s) \cup \{s\})$, $D = C - N_C(s)$, and $T = S - \{s\}$. Note that both D and T are contained in G_s. Let X be an arbitrary maximal stable set containing T in G_s. Then $X \cup \{s\}$ is a maximal stable set containing S in G. Since S itself is a maximal stable set in G, we must have $S = X \cup \{s\}$, which implies $X = T$ and hence

(4.3) T is a maximal stable set in G_s.

Now let us distinguish between two cases.

Case 1. G_s is a CIS graph.

From (4.3), we deduce that T intersects any maximal clique containing D in G_s. So some $t \in T$ is adjacent to all vertices in D, which implies $N_C(s) \cup N_C(t) = C$ and hence (4.1).

Case 2. G_s is not a CIS graph.

Let (K, U) be an arbitrary non-CIS pair of G_s and let L be an arbitrary maximal clique containing K in G. Observe that $L - K$ is fully contained in $N(s)$ and that $U \cup \{s\}$ is a maximal stable set in G, so $(L, U \cup \{s\})$ is a non-CIS pair of G. Since (C, S) is the unique such pair, we obtain $L = C$ and $U \cup \{s\} = S$. Hence $K = C - N(s) = C - N_C(s) = D$ and $U = S - \{s\} = T$; in other words, (D, T) is the unique non-CIS pair of G_s. Therefore G_s is an almost CIS graph. As $|V(G_s)| < |V(G)|$, the minimality of G implies that G_s is a split graph with the unique split partition (D, T).

Since $A = V(G) - (C \cup S) \subseteq V(G) - (D \cup T \cup \{s\}) = V(G) - V(G_s) - \{s\} = N(s)$, we conclude that s is adjacent to all vertices in A. This proves (4.2) and hence (4).

(5) There exist two vertices s and t in S such that $N_C(s) \cup N_C(t) = C$.

Otherwise, by (4) we have

(5.1) Each vertex in S is adjacent to each vertex in A.

Let a be a vertex in A. Then (2) guarantees the existence of some $x \in S$ such that $ax \in E$ and $N_C(a) \subseteq N_C(x)$. Take $y \in C - N_C(x)$ and $z \in N_S(y)$. From (5.1) we deduce that $az \in E$. Set $D = \{a, z\} \cup (N_C(a) \cap N_C(z))$ and $I = \{x, y\}$. Clearly, D is a clique and I is a stable set in G. Let K be an arbitrary maximal clique containing D in G. Then $K - D \subseteq A$. By (5.1), we have $K - D \subseteq N(x)$. So each vertex in K has at least one neighbor in I. From Proposition 2 it follows that K is disjoint from any maximal stable set containing I, contradicting the hypothesis that (C, S) is the unique non-CIS pair of G. Hence (5) holds.

(6) There exist two vertices c and d in C such that $N_S(c) \cap N_S(d) = \emptyset$.

To justify this, we turn to considering $\bar{G}$. In view of (1) and (5) (with respect to $\bar{G}$ now), there exist two vertices c and d in C such that $(S - N_S(c)) \cup (S - N_S(d)) = S$, implying $N_S(c) \cap N_S(d) = \emptyset$. So (6) is established.

(7) There exist $a \in A$ and $b \in S$ such that $ab \in E$ and $N_C(a) \cup N_C(b) = C$.

To see this, let s and t be two vertices in S such that $N_C(s) \cup N_C(t) = C$ (recall (5)) and let u be a vertex in A. If $I = \{s, t, u\}$ is a stable set in G then, by Proposition 2, any maximal stable set containing I in G would be disjoint from C, contradicting the hypothesis that (C, S) is the unique non-CIS pair of G. So u is adjacent to s or t, say the former. Clearly, we may assume that $N_C(u) \cup N_C(s) \neq C$, for otherwise, setting $\{a, b\} = \{u, s\}$, we are done. Next, observe that the clique $N_C(s) \cup \{s\}$ is not maximal in G, for otherwise, take $v \in C - (N_C(u) \cup N_C(s))$. Then each vertex in $N_C(s) \cup \{s\}$ is adjacent to u or v. Thus, by Proposition 2, the maximal clique $N_C(s) \cup \{s\}$ is disjoint from any maximal stable set containing $\{u, v\}$ in G, a contradiction. Therefore, there exists $w \in A$ which is adjacent to all vertices in $N_C(s)\cup\{s\}$. It follows that $N_C(s) \subseteq N_C(w)$ and hence we have $N_C(w)\cup N_C(t) = C$ as well. Now, from Proposition 2, we deduce that $wt \in E$. Setting $\{a, b\} = \{w, t\}$, we are done. Thus (7) follows.

Let a, b, c, d be the four vertices as exhibited in (7) and (6) and let s be the vertex in S as specified in (2). Since $C = N_C(a) \cup N_C(b) \subseteq N_C(s) \cup N_C(b)$, we have $s \neq b$. As $N_S(c) \cap N_S(d) = \emptyset$, renaming vertices if necessary, we may assume that $c \in N_C(s) - N_C(b)$ and $d \in N_C(b) - N_C(s)$. Consequently, $c \in N_C(a) - N_C(b)$ and $d \in N_C(b) - N_C(a)$. Let K be a maximal clique containing $N_C(b) \cup \{b\}$ in G and set $B = K - (N_C(b) \cup \{b\})$. Clearly, $B \subseteq A$. Since $d \in K$ and $ad \notin E$, we have $a \notin K$ and hence $a \notin B$. We claim that

(8) a is adjacent to all vertices in B.

Assume the contrary: $ax \notin E$ for some $x \in B$. In view of K, we obtain $N_C(b) \subseteq N_C(x)$. So $C = N_C(a) \cup N_C(b) \subseteq N_C(a) \cup N_C(x)$. By Proposition 2, C is disjoint from any maximal stable set containing $\{a, x\}$ in G, this contradiction justifies claim (8).

Set $D = \{a, b\} \cup B \cup (N_C(a) \cap N_C(b))$ and $T = N_S(c) \cup \{d\}$. Clearly, D and T are disjoint. By (7) and (8), D is a clique and, by (6), T is a stable set in G.

(9) Each vertex in D has at least one neighbor in T.

Since $ac \in E$, from (3) we see that a has at least one neighbor in $N_S(c)$. Note that $D - \{a\}$ is contained in $K - \{d\}$, so all vertices in $D - \{a\}$ are adjacent to d. Thus (9) holds.

Since (C, S) is the unique non-CIS pair of G, by (9) there exists a vertex $e \notin D$ such that $D \cup \{e\}$ is a clique and $T \cup \{e\}$ is a stable set in G. In view of the edge eb, we have $e \notin S$. Since $ea \in E$ and $da \notin E$, we get $e \neq d$. So $ed \notin E$ as $d \in T$, which implies $e \notin C$. Therefore $e \in A - (B \cup \{a\})$. Since e is adjacent to no vertex in $N_S(c)$, by (3) we obtain $ec \notin E$. Finally, observe that each vertex in K is adjacent to either e or c, so by Proposition 2 the maximal clique K is disjoint from any maximal stable set containing $\{e, c\}$, contradicting the hypothesis that (C, S) is the unique non-CIS pair of G.

This completes the proof of our theorem. □

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