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On minimal and locally minimal true vectors and weakly monotone Boolean functions

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Abstract. A Boolean function is called weakly monotone if every its not minimal true vector has a true immediate predecessor, or in other words, if the sets of its minimal and locally minimal true vectors coincide. Each monotone function is weakly monotone but not vice versa.
Weakly monotone Boolean functions (and set-families corresponding to their true verctors) have applications in graph theory and game theory. For example, the induced subgraphs of a of a complementary connected or of a CIS *d*-graph form a weakly monotone family, while the subgraphs of a kernel-free directed graph do not; the subgames of a saddle point free matrix game form a weakly monotone family, while the subgames of a Nash equilibrium free bimatrix game do not. In this short note, we study complexity of verifying weak monotonicity. We show that the problem is polynomial for a disjunctive normal form (DFF) and co-NP-complete for a conjunctive normal form (CNF). We also observe that verifying satisfiability of a weakly monotone CNF or DNF remains hard.

key words minimal true vector, locally minimal true vector, predecessor, immediate predecessor, weakly monotone Boolean function, DNF, CNF, complexity, polynomial, co-NP-complete, satisfiability, tautology.

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1 Main concepts

Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$ of n variables, a vector $x = (x_1, \ldots, x_n) \in \{0,1\}^n$ is called a *true* (*false* vector of f if $f(x) = 1$ (respectively, $f(x) = 0$. Furthermore, x is a *minimal* true vector of f if all its predecessors are false; in other words, $f(x) = 1$, while $f(x') = 0$ whenever $x' \leq x$ and $x' \neq x$. Finally, x is a *locally minimal* true vector of f if all it immediate predecessors are false; in other words, if $f(x) = 1$, while $f(x - e_i) = 0$ for all $i \in supp(x) = \{i \in [n] \mid x_i \neq 0\}$, where $[n] = \{1, \ldots, n\}$ and e_i is a Boolean vector whose ith coordinate is 1, while all others are equal to 0.

Let $T = T(f),\ M = M(f),\ L = L(f) \subseteq \{0,1\}^n$ denote the sets of true, minimal true, and locally minimal true vectors of f, respectively.

Containments $M \subseteq L \subseteq T$ hold for every f, by the above definitions.

Remark 1 *It is also obvious from the definitions that the local minimality in f can be verified in polynomial time whenever f is given by a polynomial oracle. In contrast, even in this case verifying minimality is exponential. Indeed, the number of immediate predecessors of a vector $x \in \{0,1\}^n$ is n, while the number of all predecessors of x is 2^k, where $k = k(x) = \#\{i \in [n] \mid x_i = 1\}$.*

Boolean function f is *monotone* if $f(x') = 1$ whenever $x' \leq x$ and $f(x) = 1$.

Definition 1 *Let us call f* weakly monotone *if every its* not minimal *true vector has an immediate true predecessor, or in other words, if*

$x \in T \setminus M$ *implies that* $x - e_i \in T$ *for some* $i \in [n]$, *that is,* $x \notin L$.

The next two properties of weakly monotone functions immediately follow from the above definitions.

Claim 1 *A Boolean function f is weakly monotone if and only if the sets of its minimal and locally minimal vectors coincide: $M(f) = L(f)$.* □

Claim 2 *Every monotone Boolean function is weakly monotone.* □

Remark 2 *The concepts introduced above for the Boolean functions naturally extend the case of binary functions $f : P_1 \times \ldots P_n \to \{0,1\}$ defined on the product of arbitrary partially ordered sets. (posets); Claims 1 and 2 generalize the case of posets, too.*

Weakly monotone Boolean functions frequently appear in combinatorics; see [2, 3, 4] for their applications in graph and game theories. Let us note that verifying weak monotonicity of the Boolean functions corresponding to some interesting families of games or graphs might be non-trivial.

There are several examples from graph theory: Biconnected graphs, or more generally, complementary connected (CC) d-graphs, are weakly monotone. The last result was conjectured in [7] and proved in [2]. Moreover, the complementary family of the non-CC d-graphs is weakly monotone too [3]. It is also shown in [3] that family of the non-CIS d-graphs is weakly monotone, while the complementary family of the CIS d-graphs is not. Interestingly, the (locally) minimal CC and non-CC d-graphs coincide [3]. Weak monotonicity for the subgraphs of a kernel-free directed graph was conjectured in [6] and disproved in [1]; see also [2]. We refer the reader to [1, 2, 3] for the definitions and more details.

The next two examples are from game theory: weak monotonicity holds for the subgames of a saddle point free matrix game; in contrast, it fails for the subgames of a Nash equilibrium free bimatrix game; see [4] for more details.

2 Main results

In this short note, we consider complexity of verifying weak monotonicity of the explicitly given Boolean functions. We show that the problem is polynomial (respectively, co-NP-complete) when f is given by a disjunctive (respectively, conjunctive) normal form: DNF $D = \bigvee_{i=1}^{m} C_i$ (respectively, CNF $C = \bigwedge_{i=1}^{m} D_i$) of n Boolean variables $x_1, \ldots, x_n$, where D_i and C_i denote elementary disjunctions and conjunctions.

We also show that satisfiability (SAT) remains hard in both cases, i.e., it is NP-complete to recognize wether a weakly monotone DNF or CNF is satisfiable.

2.1 Verifying weak monotonicity of a DNF

Theorem 1 *Weak monotonicity of a DNF $D = \bigvee_{i=1}^{m} C_i$ can be verified in polynomial time.*

Proof. Let x^i be the minimum true vector of C_i for $i \in [m] = \{1, \ldots, m\}$, $X = \{x^i,\ i \in [m]\}$ denote the set of all these m vectors, and $M(X)$ be the set of all minimal vectors of X. Standardly, $M(D), L(D)$ and $T(D)$ denote the sets of minimal, locally minimal, and all true vectors of D. It is obvious that $M(X) = M(D)$. It is also clear that conditions of Definition **??** hold for every vector $x \in T(D) \setminus X$. Hence, it is sufficient to verify these conditions only for $X \setminus M(X)$. This can be executed in polynomial time, since each vector x has at most n immediate predecessors and verifying $x - e_i \in T(D)$ is trivial. □

2.2 Verifying weak monotonicity of a CNF

Theorem 2 *Verifying weak monotonicity of a CNF is co-NP-complete.*

Proof. We reduce the problem from satisfiability (SAT). Let $C = \bigwedge_{i=1}^{m} D_i$ be an arbitrary CNF of n variables $x_1, \ldots, x_n$. Let us apply a well-known trick (see, for example, [9]) and consider the following CNF C_{01} of m variables $y_1, \ldots, y_m$:

$C_{01} = (y_1 \vee \overline{y}_2)(y_2 \vee \overline{y}_3) \dots (y_{m-1} \vee \overline{y}_m)(y_m \vee \overline{y}_1)$.

It is easily seen that $C_{01}(y) = 1$ if and only if $y \equiv 0$ or $y \equiv 1$.

Furthermore, let us consider CNF $C' = C_{01} \bigwedge_{i=1}^{m} (\overline{y}_i \vee D_i)$ of $n + m$ variables (x, y). It is also not difficult to verify that $C'(x, y) = 1$ exactly in two cases: either if $y \equiv 0$ or $y \equiv 1$ and $C(x) = 1$. Thus, C' is weakly monotone if and only if C not is satisfiable. □

2.3 Satisfiability of a weakly monotone DNF or CNF

Let us say that a DNF D (respectively, CNF C) of n Boolean variables $x = (x_1, \dots, x_n)$ is *almost not satisfiable* if equation $D(x) = 0$ (respectively, $C(x) = 1$) holds for at most one $x \in \{0, 1\}^n$. (Sometimes (almost) not satisfiable DNFs are also called *(almost) tautological*.)

In [5] and [8], it was recently proven that SAT remains hard even for an almost satisfiable DNF or CNF. Furthermore, it is obvious that an almost satisfiable DNF or CNF is weakly monotone. Hence, SAT remains hard for a weakly monotone DNF or CNF.

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