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Method of Multivariate Lagrange Interpolation for Generating Bivariate Bonferroni-Type Inequalities

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Method of Multivariate Lagrange Interpolation for Generating Bivariate Bonferroni-Type Inequalities

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Abstract. Let $A_1, \ldots, A_N$ and $B_1, \ldots, B_M$ be two sequences of events and let $\nu_N(A)$ and $\nu_M(B)$ be the number of those A_i and B_j, respectively, that occur. We give a method, based on multivariate Lagrange interpolation, that yields linear bounds in terms of $S_{k,t}$, $k+t \leq m$ on the distribution of the vector $(\nu_N(A), \nu_M(B))$. The construction of the bounds can be carried out in a simple mechanical way. For the same value of m several inequalities can be generated, however, all of them are the best bounds for some values of $S_{k,t}$. Known bivariate Bonferroni-type inequalities are reconstructed and new inequalities are generated, too. The possible extensions are also discussed.

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1 Introduction

Let $A_1, \ldots, A_N$ and $B_1, \ldots, B_M$ be two sequences of events on the same probability space. Let $\nu_N(A)$ and $\nu_M(B)$ be the number of those A_i and B_j, respectively, that occur. Let us define the bivariate binomial moments: $S_{0,0} = 1$ and for other nonnegative integers k and l,

$$S_{k,t} = E\left[\binom{\nu_N(A)}{k}\binom{\nu_M(B)}{t}\right]. \tag{1.1}$$

It is easy to see (e.g., by the use of indicator variables) that

$$S_{k,t} = \sum_{\substack{1\le i_1<\cdots<i_k\le N\\ 1\le j_1<\cdots<j_t\le M}} P(A_{i_1}\cap\cdots\cap A_{i_k}\cap B_{j_1}\cap\cdots\cap B_{j_t}), \tag{1.2}$$

where in case k or t equals zero, the empty intersection that follows in (1.2) is replaced by Ω, the sample space.

Consider the following probabilities:

$$p(u, v; N, M) = P(\nu_N(A) = u, \nu_M(B) = v) \tag{1.3}$$

and

$$q(u, v; N, M) = P(\nu_N(A) \ge u, \nu_M(B) \ge v). \tag{1.4}$$

We are interested in finding bivariate Bonferroni-type inequalities. I.e., inequalities

$$\sum_{k=0}^{N}\sum_{t=0}^{M} c_{k,t}S_{k,t} \le r(u, v; N, M) \le \sum_{k=0}^{N}\sum_{t=0}^{M} d_{k,t}S_{k,t}, \tag{1.5}$$

where $r(u, v; N, M)$ can either be $p(u, v; N, M)$ or $q(u, v; N, M)$. The coefficients have to be independent of A_i and B_j, i.e., (1.5) holds for an arbitrary probability space and for the arbitrary choice of the A_i and B_j. The Bonferroni-type inequalities have several applications. They can be used, e.g., to estimate system reliability (see Habib and Szántai, 2000) or to estimate multivariate distributions (see Bukszár and Szántai, 2002). Further applications can be found in Galambos and Simonelli (1996).

Bivariate Bonferroni inequalities can be obtained in several ways. On one hand, they can be proved by the method of indicators, see e.g. Galambos and Xu (1993), Lee (1997). Another way is the method of polynomials, which is capable of not only proving but also generating new inequalities, see e.g. Galambos and Xu (1995). By the aid of the reduction formulas of Galambos and Xu (1995) and Simonelli (1999) further inequalities can be found.

In the present paper we introduce the method of multivariate Lagrange interpolation, which has turned out to be a very effective tool of generating and proving Bonferroni-type inequalities. On one hand the method yields a wide variety of Bonferroni-bounds, which are the best in a sense to be defined later. On the other hand the method is entirely mechanical. It can be implemented on computer, hence Bonferroni-bounds of several terms can also be constructed easily. In Section 3 inequalities of the literature are reconstructed and new, improved bounds are generated as well. The last section discusses the possible extensions of the method.

2 Method of Multivariate Lagrange Interpolation

In the following, we consider the special case of (1.5), where the binomial moments up to the total order m are given (assuming that $m \le min(N, M)$). I.e., we look for the coefficients $c_{k,t}$ and $d_{k,t}$ in the inequalities

$$\sum_{k=0}^{m}\sum_{t=0}^{m-k} c_{k,t}S_{k,t} \le r(u,v;N,M) \le \sum_{k=0}^{m}\sum_{t=0}^{m-k} d_{k,t}S_{k,t}, \tag{2.6}$$

where $r(u, v; N, M)$ can either be $p(u, v; N, M)$ or $q(u, v; N, M)$. This case is relevant in practice: usually the probabilities of the intersections of events are given up to a certain number of the events, hence the multivariate moments up to a certain total order can be calculated. Let us define the power moment

$$\mu_{k,t} = E[\nu_N(A)^k \nu_M(B)^t]. \tag{2.7}$$

Theorem 2.1

$$S_{k,t} = \sum_{i=0}^{k}\sum_{j=0}^{t} \frac{s(k,i)}{k!}\frac{s(t,j)}{t!}\mu_{i,j}, \tag{2.8}$$

$$\mu_{k,t} = \sum_{i=0}^{k}\sum_{j=0}^{t} S(k,i)i!S(t,j)j!S_{i,j}, \tag{2.9}$$

where $s(i,j)$ and $S(i,j)$ are the first and second kind Stirling numbers, respectively.

Proof.

$$S_{k,t} = E\left[\binom{\nu_N(A)}{k}\binom{\nu_M(B)}{t}\right] = E\left[\left(\sum_{i=0}^{k}\frac{s(k,i)}{k!}\nu_N(A)^i\right)\left(\sum_{j=0}^{t}\frac{s(t,j)}{t!}\nu_M(B)^j\right)\right]$$

$$= E\left[\sum_{i=0}^{k}\sum_{j=0}^{t}\frac{s(k,i)}{k!}\frac{s(t,j)}{t!}\nu_N(A)^i\nu_M(B)^j\right] = \sum_{i=0}^{k}\sum_{j=0}^{t}\frac{s(k,i)}{k!}\frac{s(t,j)}{t!}\mu_{i,j},$$

$$\mu_{k,t} = E\left[\nu_N(A)^k\nu_M(B)^t\right] = E\left[\left(\sum_{i=0}^{k}S(k,i)i!\binom{\nu_N(A)}{i}\right)\left(\sum_{j=0}^{t}S(t,j)j!\binom{\nu_M(B)}{j}\right)\right]$$

$$= E\left[\sum_{i=0}^{k}\sum_{j=0}^{t}S(k,i)i!S(t,j)j!\binom{\nu_N(A)}{i}\binom{\nu_M(B)}{j}\right] = \sum_{i=0}^{k}\sum_{j=0}^{t}S(k,i)i!S(t,j)j!S_{i,j}.$$

□

Corollary 2.2 *Inequalities (2.6) are equivalent to the following inequalities*

$$\sum_{k=0}^{m}\sum_{t=0}^{m-k} a_{k,t}\mu_{k,t} \le r(u,v;N,M) \le \sum_{k=0}^{m}\sum_{t=0}^{m-k} b_{k,t}\mu_{k,t}, \tag{2.10}$$

where

$$a_{k,t} = \sum_{i=k}^{m} \sum_{j=t}^{m-i} \frac{s(i,k)}{i!} \frac{s(j,t)}{j!} c_{i,j} \quad and \quad b_{k,t} = \sum_{i=k}^{m} \sum_{j=t}^{m-i} \frac{s(i,k)}{i!} \frac{s(j,t)}{j!} d_{i,j}.$$

If (2.10) is given then the coefficients of the corresponding inequalities (2.6) can be calculated by

$$c_{k,t} = \sum_{i=k}^{m} \sum_{j=t}^{m-i} S(i,k)k!S(j,t)t!a_{i,j} \quad and \quad d_{k,t} = \sum_{i=k}^{m} \sum_{j=t}^{m-i} S(i,k)k!S(j,t)t!b_{i,j}. \tag{2.11}$$

In the following we focus on the construction of inequalities (2.10). This is equivalent to finding inequalities of (2.6), because if inequalities in the form (2.10) are given then the coefficients of the corresponding inequalities in the form (2.6) can be calculated by (2.11).

Inequalities (2.10) can be written as

$$E\left[\sum_{k=0}^{m} \sum_{t=0}^{m-k} a_{k,t} \nu_N(A)^k \nu_M(B)^t\right] \le E\left[I_{r(u,v;N,M)}\left(\nu_N(A), \nu_M(B)\right)\right] \le E\left[\sum_{k=0}^{m} \sum_{t=0}^{m-k} b_{k,t} \nu_N(A)^k \nu_M(B)^t\right], \tag{2.12}$$

where in case of $r(u,v;N,M) = p(u,v;N,M)$:

$$I_{p(u,v;N,M)}\left(z_1, z_2\right) = \begin{cases} 1 \text{ if } z_1 = u,\ z_2 = v, \\ 0 \text{ otherwise,} \end{cases}$$

and in case of $r(u,v;N,M) = q(u,v;N,M)$:

$$I_{q(u,v;N,M)}\left(z_1, z_2\right) = \begin{cases} 1 \text{ if } z_1 \ge u,\ z_2 \ge v, \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 2.3 *The left inequality of (2.12) is valid if and only if*

$$P_a(z_1, z_2) = \sum_{k=0}^{m} \sum_{t=0}^{m-k} a_{k,t} z_1^k z_2^t \le I_{r(u,v;N,M)}\left(z_1, z_2\right), \tag{2.13}$$

for all

$$z_1 \in Z_1 = \{0, 1, \dots, N\} \text{ and } z_2 \in Z_2 = \{0, 1, \dots, M\}. \tag{2.14}$$

Similarly, the right inequality of (2.12) is valid if and only if

$$P_b(z_1, z_2) = \sum_{k=0}^{m} \sum_{t=0}^{m-k} b_{k,t} z_1^k z_2^t \ge I_{r(u,v;N,M)}\left(z_1, z_2\right), \tag{2.15}$$

for all $(z_1, z_2) \in Z_1 \times Z_2$ *defined in (2.14).*

Proof. Trivial. □

Theorem 2.3 indicates that we should find polynomials which are smaller (greater) or equal to the value of the function $I_{r(u,v;N,M)}$ at the points of $Z_1 \times Z_2$ but *not necessarily at other (e.g., noninteger) values of* (z_1, z_2). In order to find those polynomials let us consider the following theorem on bivariate Lagrange interpolation. First we need the following

Definition 2.1 *Let*

$$Z_1 = \{z_{10}, z_{11}, \dots, z_{1N}\} \text{ and } Z_2 = \{z_{20}, z_{21}, \dots, z_{2M}\} \tag{2.16}$$

and $f(z_1, z_2),\ (z_1, z_2) \in Z = Z_1 \times Z_2$ *be a bivariate discrete function. Take the subset*

$$Z_{I_1 I_2} = \{z_{1i},\ i \in I_1\} \times \{z_{2i},\ i \in I_2\} = Z_{1I_1} \times Z_{2I_2}, \tag{2.17}$$

where $|I_j| = k_j + 1,\ j = 1, 2.$ *Then we can define the* (k_1, k_2)-order (bivariate) divided difference of f *on the set (2.17) in an iterative way. First we take the* k_1*th divided difference with respect to the first variable, then the* k_2*th divided difference with respect to the second variable. Let*

$$[z_{1i},\ i \in I_1; z_{2i},\ i \in I_2; f] \tag{2.18}$$

designate the (k_1, k_2)*-order divided difference. The sum* $k_1 + k_2$ *is called the total order of the divided difference.*

In order to make the definition easier to understand we present the following

Example 2.1

$$[z_{10}, z_{11}; z_{20}, z_{21}; f] = [z_{20}, z_{21}; \frac{f(z_{11}, z_2) - f(z_{10}, z_2)}{z_{11} - z_{10}}]$$

$$= \frac{\frac{f(z_{11}, z_{21}) - f(z_{10}, z_{21})}{z_{11} - z_{10}} - \frac{f(z_{11}, z_{20}) - f(z_{10}, z_{20})}{z_{11} - z_{10}}}{z_{21} - z_{20}}.$$

Theorem 2.4 *Let* Z_1 *and* Z_2 *be defined as in (2.16) and*

$$I = \{(i_1, i_2) | i_1 \geq 0,\ i_2 \geq 0 \text{ integer},\ i_1 + i_2 \leq m\}. \tag{2.19}$$

Then the Lagrange polynomial $L_I(z_1, z_2)$ *corresponding to the points* $Z_I = \{(z_{1i_1}, z_{2i_2}) | (i_1, i_2) \in I\}$ *is the following*

$$L_I(z_1, z_2) = \sum_{\substack{i_1 + i_2 \leq m \\ 0 \leq i_1, i_2}} [z_{10}, \dots z_{1i_1}; z_{20}, \dots z_{2i_2}; f] \prod_{j=1}^{2} \prod_{h=0}^{i_j - 1} (z_j - z_{jh}), \tag{2.20}$$

where $\prod_{h=0}^{i_j-1}(z_j - z_{jh}) = 1$ *for* $i_j = 0$*, by definition. The corresponding Lagrange residual can be written in the form*

$$\begin{aligned} R_I(z_1, z_1) &= f(z_1, z_2) - L_I(z_1, z_2) = \\ &= \sum_{\substack{i_1 + i_2 = m \\ 0 \leq i_1, i_2}} [z_{10}, \dots, z_{1i_1}, z_1; z_{20}, \dots, z_{2i_2}; f] \prod_{l=0}^{i_1} (z_1 - z_{1l}) \prod_{h=0}^{i_2 - 1} (z_2 - z_{2h}) \\ &\quad + [z_1; z_{20}, \dots, z_{2m}, z_2; f] \prod_{h=0}^{m} (z_2 - z_{2h}). \end{aligned} \tag{2.21}$$

Proof. The Lagrange polynomial of (2.20) is the classical result of the triangular case of the subscripts (i_1, i_2) of the interpolation points. The formula first was considered by Biermann (1903). However, the residual (2.21), in this form, was first given in the proof of Theorem 4.1 in Prékopa (1998). □

Despite the subscript set I is triangular, the geometric distribution of the interpolation points is not necessarily triangular, because in Theorem 2.4 there was no assumption on the orders of the elements $z_{10}, z_{11}, \ldots, z_{1N}$ and $z_{20}, z_{21}, \ldots, z_{2M}$.

Let us turn back to our case where

$$\begin{aligned} Z_1 &= \{0, 1, \ldots, N\} = \{z_{10}, z_{11}, \ldots, z_{1N}\}, \\ Z_2 &= \{0, 1, \ldots, M\} = \{z_{20}, z_{21}, \ldots, z_{2M}\}. \end{aligned} \tag{2.22}$$

The curly brackets define disordered sets, hence z_{10} does not have to equal 0, etc. In the following we define the orders of the elements in Z_1 and Z_2 in such a way that the Lagrange polynomial (2.20) of the bivariate function $I_{r(u,v;N,M)}(z_1, z_2)$ fulfills the equation (2.13) or (2.15).

We consider the following two cases:

$$r(u, v; N, M) = p(0, 0; N, M), \tag{2.23}$$

$$r(u, v; N, M) = q(1, 1; N, M). \tag{2.24}$$

We need the following

Theorem 2.5 *The divided differences of total order $m + 1$ of the function $I_{p(0,0;N,M)}(z_1, z_2)$ on $Z_1 \times Z_2$ are nonnegative (nonpositive) if m is odd (even), where Z_1, Z_2 are defined in (2.22).*

The same is true for the function $I_{q(1,1;N,M)}(z_1, z_2)$ on $Z_1 \times Z_2$, where Z_1, Z_2 are defined in (2.22).

Proof. Let

$$f(z) = \begin{cases} 1, & \text{if } z = 0, \\ 0, & \text{if } z \geq 1. \end{cases} \tag{2.25}$$

Assume that $z_k > 0$, $k = 0, \ldots, i$. It is easy to see that

$$[0, z_1, \ldots, z_i; f] = \frac{(-1)^i}{\prod_1^i z_k}$$

and

$$[z_0, z_1, \ldots, z_i; f] = 0.$$

I.e., the even (odd) order divided differences of the function (2.25) are nonnegative (nonpositive). Considering that

$$I_{p(0,0;N,M)}(z_1, z_2) = f(z_1) f(z_2)$$

and

$$I_{q(1,1;N,M)}(z_1, z_2) = (1 - f(z_1))(1 - f(z_2))$$

the rest of the proof is straightforward. □

The following algorithm defines the values of $z_{10}, \dots, z_{1m}$ and $z_{20}, \dots, z_{2m}$.

Algorithm

Step 0. Initialize $t = 0$, $-1 \le q \le m$, $L = (0, 1, \dots, m-1-q)$ (In case of $q = m$ the set L is empty.), $U = (N, N-1, \dots, N-q)$. (In case of $q = -1$ the set U is empty.)

Step 1. Let $(z_{10}, \dots, z_{1m}) =$ (arbitrary merger of the sets L, U). Let $l^0 = 0$, $u^0 = M$.

Step 2. If $z_{1(m-t)} \in L$, then let $z_{2t} = l^t$, $l^{t+1} = l^t + 1$, $u^{t+1} = u^t$, and if $z_{1(m-t)} \in U$, then let $z_{2t} = u^t$, $u^{t+1} = u^t - 1$, $l^{t+1} = l^t$. If $t = m$ then stop. Else $t \leftarrow t+1$ and repeat Step 2.

Theorem 2.6 *Let the elements of the sequence $(z_{j0}, \dots, z_{jm})$, $j = 1, 2$ be defined by the Algorithm above. Furthermore let $(z_{1(m+1)}, \dots, z_{1N}) = (m-q, m-q+1, \dots, N-(q+1))$ and $(z_{2(m+1)}, \dots, z_{2M}) = (m-q, m-q+1, \dots, M-(q+1))$.*

Let $f(z_1, z_2)$ be a function on $Z_1 \times Z_2$, defined by (2.22). If the divided differences of total order $m+1$ of the function f are nonnegative and $q+1$ is even (odd) then $L_I(z_1, z_2)$, defined by (2.20), approximates the function below (above). I.e.,

$$f(z_1, z_2) \ge (\le) L_I(z_1, z_2), \ (z_1, z_2) \in Z_1 \times Z_2. \tag{2.26}$$

If the divided differences of total order $m+1$ of the function f are nonpositive then (2.26) holds with a reversed inequality sign.

Proof. It is enough to prove that $R_I(z_1, z_2)$ in (2.21) is nonnegative (nonpositive) for all $(z_1, z_2) \in Z_1 \times Z_2$. The divided differences in (2.21) are of total order $m+1$ hence they are nonnegative. Regarding the terms

$$\prod_{l=0}^{i_1} (z_1 - z_{1l}) \prod_{h=0}^{i_2-1} (z_2 - z_{2h}) \quad (i_1 + i_2 = m, \ i_1, i_2 \ge 0) \tag{2.27}$$

and

$$\prod_{h=0}^{m} (z_2 - z_{2h}), \tag{2.28}$$

they are zeros or they have exactly $q+1$ negative factors in case of $(z_1, z_2) \in Z_1 \times Z_2$. Hence (2.27) and (2.28) are nonnegative (nonpositive) if $q+1$ is even (odd). From this follows that the same is true for $R_I(z_1, z_2)$. If the divided differences of total order $m+1$ are nonpositive then the divided differences in (2.21) are nonpositive. The terms (2.27) and (2.28) remain nonnegative (nonpositive) hence $R_I(z_1, z_2)$ is nonpositive (nonnegative). □

The theorems above give us the following method of generating Bonferroni-bounds. Let $r(u, v; N, M) = p(0, 0; N, M)$ or $r(u, v; N, M) = q(1, 1; N, M)$. In both cases $I_{r(u,v;N,M)}(z_1, z_2)$ has nonnegative (nonpositive) divided differences of order $m+1$ if m is odd (even). Hence the algorithm of Theorem 2.6 gives Lagrange polynomials $R_I(z_1, z_2)$ approximating the function $I_{r(u,v;N,M)}(z_1, z_2)$ below and above. $R_I(z_1, z_2)$ fulfills the inequality (2.13) as the polynomial $P_a(z_1, z_2)$ or the inequality (2.15) as the polynomial $P_b(z_1, z_2)$, respectively. Taking the expected value of $P_a(z_1, z_2)$ or $P_b(z_1, z_2)$ and using the corresponding conversion formula of (2.11) the Bonferroni bound is given. The method is illustrated by a detailed example in the following section.

About the quality of the bounds of the method we have the following

Theorem 2.7 *Assume that there exists a random vector with the support Z_I (where Z_I is defined by the Algorithm), which has the binomial moments with the given values of $S_{k,t}$, $k+t \le m$. Then the Bonferroni-bound yielded by the Lagrange polynomial $L_I(z_1, z_2)$ (i.e., by the method above) is the best linear bounding formula of the terms $S_{k,t}$, $k+t \le m$.*

Proof. Let us consider a lower bound. (The case of the upper bound can be proved in the same way.) Z_I is the set of the interpolation points, hence in (2.13)

$$P_a(z_1, z_2) = L_I(z_1, z_2) = \sum_{k=0}^{m} \sum_{t=0}^{m-k} a_{k,t} z_1^k z_2^t = I_{r(u,v;N,M)}\left(z_1, z_2\right) \text{ for } (z_1, z_2) \in Z_I.$$

If there exists a random vector $(\nu_N(A), \nu_M(B))$ that has values outside of Z_I with zero probability, then for this vector:

$$E\left[\sum_{k=0}^{m} \sum_{t=0}^{m-k} a_{k,t} \nu_N(A)^k \nu_M(B)^t\right] = E\left[I_{r(u,v;N,M)}\left(\nu_N(A), \nu_M(B)\right)\right].$$

From this:

$$\sum_{k=0}^{m} \sum_{t=0}^{m-k} a_{k,t} \mu_{k,t} = r(u, v; N, M) \Longrightarrow \sum_{k=0}^{m} \sum_{t=0}^{m-k} c_{k,t} S_{k,t} = r(u, v; N, M).$$

I.e., there exists a distribution with the given binomial moments where the Bonferroni-inequality holds with equality, hence it cannot be improved. □

3 Bonferroni-type inequalities

Example 3.1 *Let us construct an upper bound for $p(0, 0; N, M)$ where the maximum of the total order of the bivariate moments is $m = 2$. The $m+1$st divided differences of the function $f(z_1, z_2) = I_{p(0,0;N,M)}(z_1, z_2)$ are nonpositive, hence in the Algorithm $q + 1$ has to be even. One possible choice of Step 1 can be:*

$$(z_{10}, z_{11}, z_{12}) = (N, 0, N - 1).$$

Step 2 gives the following values for the second components:

$$(z_{20}, z_{21}, z_{22}) = (M, 0, M - 1).$$

Considering the coefficients of $L_I(z_1, z_2)$ in (2.20):

$$[z_{10}; z_{20}; f] = [N; M; I_{p(0,0;N,M)}] = I_{p(0,0;N,M)}(N, M) = 0$$

$$[z_{10}; z_{20}, z_{21}; f] = [N; M, 0; I_{p(0,0;N,M)}] = 0 \times \frac{(-1)}{M} = 0$$

$$[z_{10}; z_{20}, z_{21}, z_{22}; f] = [N; M, 0, M-1; I_{p(0,0;N,M)}] = 0 \times \frac{(-1)^2}{M(M-1)} = 0$$

$$[z_{10}, z_{11}; z_{20}; f] = [N, 0; M; I_{p(0,0;N,M)}] = \frac{(-1)}{N} \times 0 = 0$$

$$[z_{10}, z_{11}; z_{20}, z_{21}; f] = [N, 0; M, 0; I_{p(0,0;N,M)}] = \frac{(-1)}{N} \times \frac{(-1)}{M} = \frac{1}{NM}$$

$$[z_{10}, z_{11}, z_{12}; z_{20}; f] = [N, 0, N-1; M; I_{p(0,0;N,M)}] = \frac{(-1)^2}{N(N_1)} \times 0 = 0.$$

Hence

$$I_{p(0,0;N,M)}(z_1, z_2) \leq P_b(z_1, z_2) = L_I(z_1, z_2) = \frac{1}{NM}(z_1-N)(z_2-M) = 1-\frac{1}{N}z_1-\frac{1}{M}z_2+\frac{1}{NM}z_1z_2. \tag{3.1}$$

Taking the expected values of $I_{p(0,0;N,M)}(\nu_N(A), \nu_M(B))$ and $P_b(\nu_N(A), \nu_M(B))$, we have:

$$p(0, 0; N, M) \leq 1 - \frac{1}{N}\mu_{1,0} - \frac{1}{M}\mu_{0,1} + \frac{1}{NM}\mu_{1,1}. \tag{3.2}$$

Since in this case $\mu_{1,0} = S_{1,0}$, $\mu_{0,1} = S_{0,1}$ and $\mu_{1,1} = S_{1,1}$, i.e., $d_{k,t} = b_{k,t}$ in (2.11), we get:

$$p(0, 0; N, M) \leq 1 - \frac{1}{N}S_{1,0} - \frac{1}{M}S_{0,1} + \frac{1}{NM}S_{1,1}. \tag{3.3}$$

Inequality (3.3) is the same as Inequality (I) in Galambos and Xu (1995).

If we would like to get a lower bound we can choose, e.g., the case

$$(z_{10}, z_{11}, z_{12}) = (0, N, 1)$$

then Step 2 gives

$$(z_{20}, z_{21}, z_{22}) = (0, M, 1).$$

Following the same way as before we have the inequality

$$p(0, 0; N, M) \geq 1 - \frac{N+1}{N}\mu_{1,0} - \frac{M+1}{M}\mu_{0,1} + \frac{1}{N}\mu_{2,0} + \frac{1}{M}\mu_{0,2} + \frac{1}{NM}\mu_{1,1}. \tag{3.4}$$

In order to get the bound in the terms of binomial moments we have to apply (2.11).

$$\begin{aligned}
c_{0,0} &= 1 \cdot 1 \cdot a_{0,0} + 0 = 1 \cdot 1 \cdot 1 = 1 \\
c_{0,1} &= 1 \cdot 1 \cdot a_{0,1} + 1 \cdot 1 \cdot a_{0,2} = \left(-\frac{M+1}{M}\right) + \left(\frac{1}{M}\right) = -1 \\
c_{0,2} &= 1 \cdot 2 \cdot a_{0,2} = \frac{2}{N} \\
c_{1,0} &= 1 \cdot 1 \cdot a_{1,0} + 1 \cdot 1 \cdot a_{2,0} = \left(-\frac{N+1}{N}\right) + \left(\frac{1}{N}\right) = -1 \\
c_{1,1} &= 1 \cdot 1 \cdot a_{1,1} = \frac{1}{NM} \\
c_{2,0} &= 2 \cdot 1 \cdot a_{2,0} = \frac{2}{M}.
\end{aligned} \tag{3.5}$$

Hence we have:

$$p(0,0;N,M) \geq 1 - S_{1,0} - S_{0,1} + \frac{2}{N} S_{2,0} + \frac{2}{M} S_{0,2} + \frac{1}{NM} S_{1,1}. \tag{3.6}$$

Example 3.2 *Consider bounds for $q(1,1;N,M)$ in case of $m = 3$. The $m+1$st divided differences of $I_q(1,1;N,M)(z_1,z_2)$ are nonnegative, hence if $q+1$ is odd (even) then the Algorithm gives an upper (lower) bound. Regarding the upper bounds we consider the following two cases:*

(a) $(z_{10}, z_{11}, z_{12}, z_{13}) = (0, N, 1, 2) \Longrightarrow (z_{20}, z_{21}, z_{22}, z_{23}) = (0, 1, M, 2)$

(b) $(z_{10}, z_{11}, z_{12}, z_{13}) = (0, 1, N, 2) \Longrightarrow (z_{20}, z_{21}, z_{22}, z_{23}) = (0, M, 1, 2)$

The corresponding upper bounds are:

(a)

$$q(1,1;N,M) \leq S_{11} - \frac{2}{N} S_{21} - \frac{2}{NM} S_{12},$$

(b)

$$q(1,1;N,M) \leq S_{11} - \frac{2}{M} S_{12} - \frac{2}{NM} S_{21}.$$

This gives the result of Theorem 2 of Lee (1997):

$$q(1,1;N,M) \leq min\left(S_{11} - \frac{2}{N} S_{21} - \frac{2}{NM} S_{12}, S_{11} - \frac{2}{M} S_{12} - \frac{2}{NM} S_{21}\right) \tag{3.7}$$

Lower bounds can be given by the application of the Algorithm for the following cases:

(a) $(z_{10}, z_{11}, z_{12}, z_{13}) = (0, 1, 2, 3) \Longrightarrow (z_{20}, z_{21}, z_{22}, z_{23}) = (0, 1, 2, 3)$

(b) $(z_{10}, z_{11}, z_{12}, z_{13}) = (0, N, N-1, 1) \Longrightarrow (z_{20}, z_{21}, z_{22}, z_{23}) = (0, M, M-1, 1)$

The corresponding bounds are:

(a)

$$q(1,1;N,M) \geq S_{11} - S_{12} - S_{21},$$

(b)

$$q(1,1;N,M) \geq \frac{3}{NM} S_{11} - \frac{2}{NM(M-1)} S_{12} - \frac{2}{NM(N-1)} S_{21}.$$

This gives the result:

$$q(1,1;N,M) \geq max \left(S_{11} - S_{12} - S_{21}, \frac{3}{NM} S_{11} - \frac{2}{NM(M-1)} S_{12} - \frac{2}{NM(N-1)} S_{21} \right) \tag{3.8}$$

Example 3.3 *If $m = 4$ then all $m+1$st divided differences of $I_q(1,1;N,M)(z_1, z_2)$ are nonpositive.*

(a) Applying the Algorithm for the case:
$(z_{10}, z_{11}, z_{12}, z_{13}, z_{14}) = (0, 1, N, 2, 3) \Longrightarrow (z_{20}, z_{21}, z_{22}, z_{23}, z_{24}) = (0, 1, M, 2, 3)$,

(b) and applying the Algorithm for the case:
$(z_{10}, z_{11}, z_{12}, z_{13}, , z_{14}) = (0, N, 1, N-1, 2) \Longrightarrow (z_{20}, z_{21}, z_{22}, z_{23}, , z_{24}) = (0, M, 1, M-1, 2)$,

we get the following lower and upper bounds, respectively:

(a)

$$q(1,1;N,M) \geq S_{11} - S_{12} - S_{21} + \frac{3}{M} S_{13} + \frac{3}{N} S_{31} + \frac{4}{NM} S_{22}, \tag{3.9}$$

(b)

$$\begin{aligned} q(1,1;N,M) \quad \leq \quad & S_{11} - \frac{2(NM-N+M-2)}{NM(M-1)} S_{12} - \frac{2(NM-M+N-2)}{NM(N-1)} S_{21} \\ & + \frac{6}{NM(M-1)} S_{13} + \frac{6}{NM(N-1)} S_{31} + \frac{4}{NM} S_{22}. \end{aligned} \tag{3.10}$$

Considering (3.9) it improves the classical lower bound $S_{11} - S_{12} - S_{21}$, i.e.

$$q(1,1;N,M) \geq S_{11} - S_{12} - S_{21} + \frac{3}{M} S_{13} + \frac{3}{N} S_{31} + \frac{4}{NM} S_{22} \geq S_{11} - S_{12} - S_{21}.$$

It is also easy to see that (3.10) improves the upper bound of Galambos and Xu (1993):

$$\begin{aligned} S_{11} - \frac{2(NM-N+M-2)}{NM(M-1)}S_{12} - \frac{2(NM-M+N-2)}{NM(N-1)}S_{21} \\ + \frac{6}{NM(M-1)}S_{13} + \frac{6}{NM(N-1)}S_{31} + \frac{4}{NM}S_{22} \\ = \left(S_{11} - \frac{2}{M}S_{12} - \frac{2}{N}S_{21} + \frac{4}{NM}S_{22}\right) \\ - \frac{6}{NM(M-1)}\left[\left(\frac{M-2}{3}S_{12} - S_{13}\right) + \left(\frac{N-2}{3}S_{21} - S_{31}\right)\right] \\ \le S_{11} - \frac{2}{M}S_{12} - \frac{2}{N}S_{21} + \frac{4}{NM}S_{22}. \end{aligned}$$

4 Extensions of the method

If in the inequalities (1.5), beside the at most m-order bivariate binomial moments, some univariate moments of higher order are also allowed, then the Min and Max Algorithms of Mádi-Nagy and Prékopa (2004) can be applied. The connection between multivariate discrete moment problems and multivariate Bonferroni-type bounds is also clarified there. If more than two event sequences are considered, then the Min Algorithm of Mádi-Nagy (2009) can be used.

Another way of constructing more Bonferroni-bounds is the application of the reduction formula of Corollary 2.1 in Galambos and Xu (1995) for the inequalities given by our Algorithm. Lemma 1 in Simonelli (1996) can also be applied for our bounds to get further inequalities. In those generalizations the order of the moments can be higher, however the obtained formula remains simple enough to use in practice.

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