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UNIFORM PARTITIONS AND ERDÖS-KO-RADO THEOREM [a]

Vladimir Gurvich [b]

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RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

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[b]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ 08854-8003; email: gurvich@rutcor.rutgers.edu

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Uniform partitions and Erdös-Ko-Rado Theorem [1]

Vladimir Gurvich

Abstract. Let n, d, and k be positive integers such that $k \geq 2$, $d \geq 1$, and $n = kd$. Furthermore, let $N = \{1, \ldots, n\}$ be the ground set and $B(n, d) = \{S \subseteq N : |S| = d\}$ denote the family of all subsets of cardinality d in N; obviously, $|B(n, d)| = \binom{n}{d}$. A subfamily $P(n, d) \subseteq B(n, d)$ that consists of k pairwise disjoint subsets is called a **partition**. Obviously, all sets of $B(n, d)$ participate the same number of partitions. A subfamily $T(n, d) \subseteq B(n, d)$ is called an **(exact) transversal** if it has a (unique) common set with each partition. Clearly, a transversal is exact if and only if every two of its sets intersect. From the above claims, it is not difficult to derive that $|T(n, d)| \geq \binom{n-1}{d-1}$ for every transversal and the equality holds if and only if $T(n, d)$ is exact. Thus, we obtain a new short proof of the famous Erdös-Ko-Rado Theorem in case $n = kd$, as well as the following application. Let $F \subseteq B(n, d)$ be a set-family in which ℓ partitions are given explicitly, while the remaining sets of F are given by a membership oracle, to verify whether F contains another partition $\binom{n-1}{d-1} - \ell$ tests might be needed.
Key words Erdös-Ko-Rado Theorem, set-packing, set-cover, set-partition, intersecting set-families, uniform hypergraph, exact transversal.

1 On the size of a minimum transversal

A *hypergraph* $\mathcal{H} \subseteq 2^V$ is defined as a family of subsets of the vertex-set V. These subsets $H \in \mathcal{H}$ are called the *edges* of $\mathcal{H}$.

A subset $T \subseteq V$ is called *transversal* to $\mathcal{H}$ if $T \cap H \neq \emptyset$ for every $H \in \mathcal{H}$; furthermore, T is called *exact* if $|T \cap H| = 1$ for every $H \in \mathcal{H}$.

We call $\mathcal{H}$ *edge-uniform* if all its edges contain the same number of vertices, $|H| = d$ for all $H \in \mathcal{H}$.

Respectively, $\mathcal{H}$ is called *vertex-uniform* if all its vertices are contained in the same number of edges, $deg(v) = \#\{H \subseteq \mathcal{H} \mid v \in H\} = \delta$ for all $v \in V$.

Proposition 1 *Each exact transversal (if any) of a vertex-uniform hypergraph $\mathcal{H}$ is of size $|\mathcal{H}|/\delta$, while any other transversal to $\mathcal{H}$ is of a strictly larger size.*

Proof is straightforward. □

Remark 1 *Restriction to the vertex-uniform hypergraphs is essential. For example, hypergraph $\mathcal{H} = \{(1,2),(1,3)\}$ is not vertex-uniform and it has two exact transversals $\{1\}$ and $\{2,3\}$ of different sizes.*

Even a vertex- and edge-uniform hypergraph might have no exact transversal; see, for example, $\mathcal{H} = \{(1,2),(1,3),(2,3)\}$.

2 On transversals to partition hypergraphs

Let n, d, and k be positive integers such that $k \geq 2$, $d \geq 1$, and $n = kd$. Furthermore, let $N = \{1, \ldots, n\}$ be the ground set and $B(n,d) = \{S \subseteq N : |S| = d\}$ denote the family of all subsets of cardinality d in N; obviously, $|B(n,d)| = \binom{n}{d}$. A subfamily $P = P(n,d) \subseteq B(n,d)$ that consists of k pairwise disjoint subsets is called a **partition**. Let us set $V = B(n,d)$ and define the *partition hypergraph* $\mathcal{P} = \mathcal{P}(n,d) \subseteq 2^V$ as the family of all partitions.

Proposition 2 *Hypergraph $\mathcal{P}$ is edge- and vertex-uniform.*

Proof is straightforward, by symmetry. □

Standardly, a subfamily $T = T(n,d) \subseteq B(n,d)$ is called an (exact) transversal to $\mathcal{P}(n,d)$ if T has a (unique) common set with each partition $P(n,d) \in \mathcal{P}(n,d)$.

Proposition 3 *A transversal is exact if and only if each two its sets intersect.*

Proof Obviously, two sets of $B(n,d)$ can be extended to a partition of N if and only if they are disjoint. □

Remark 2 *More generally, a subset $I \subseteq B(n,d)$ and each partition $P(n,d) \in \mathcal{P}(n,d)$ intersect in at most one d-set if and only if I is an intersecting family.*

Another way to generalize Proposition 3 is to give up the assumption that $n = kd$, consider $n = kd + r$, where $0 \leq r < d$ instead, and redefine $P(n,d) \subseteq B(n,d)$ as a maximal set-packing*, that is a collection of k pairwise disjoint sets of $B(n,d)$. Clearly, $P(n,d)$ becomes a partition if and only if $r = 0$. However, statement of Proposition 3 holds for any $r \in [0,d)$.*

For an arbitrarily fixed $i \in N$, let $F(i,n,d)$ and $F'(i,n,d)$ denote the family of all sets in $B(n,d)$ that contain and do not contain i, respectively.

Proposition 4 *Set-family $F(i,n,d)$ is : (i) intersecting; (ii) an exact transversal to $\mathcal{P} = \mathcal{P}(n,d)$; (iii) of cardinality $\binom{n-1}{d-1}$; while set-family $F'(i,n,d)$ has each of the above three properties if and only if $n = 2d$.*

Proof is straightforward. □

Remark 3 *To generalize Proposition 3, one can consider $n = kd + r$, where $0 \leq r < d$, and redefine $P(n,d) \subseteq B(n,d)$ as a* minimal set-cover*. It is easy to verify that properties (i, ii, iii) still hold for $F(i,n,d)$.*

Our main result immediately follows from the above four Propositions.

Theorem 1 *The next five properties of a set-family $F \subseteq B(n,d)$ are equivalent:*

(i) F is an exact transversal to $\mathcal{P}(n,d)$;

(ii) F is a transversal to $\mathcal{P}(n,d)$ of cardinality $\binom{n-1}{d-1}$;

(iii) F is a minimum transversal to $\mathcal{P}(n,d)$;

(iv) F is a maximum intersecting family of sets in $B(n,d)$;

(v) F is an intersecting family of cardinality $\binom{n-1}{d-1}$. □

Equivalence of (iv) and (v) is a classical result of the extremal set theory.

3 Erdös-Ko-Rado Theorem

Given positive integers n and d such that $n \geq 2d$, let $k = \lfloor n/d \rfloor$; then $n = kd + r$, where $k \geq 2$, $d \geq 1$, and $0 \leq r < d$. As before, let $N = \{1, \dots, n\}$ be the ground set and $\mathcal{B}(n,d) = \{S \subseteq N \ : \ |S| = d\}$ denote the family of all $\binom{n}{d}$ subsets of cardinality d in N. A subfamily $\mathcal{I} = \mathcal{I}(n,d) \subseteq \mathcal{B}(n,d)$ is called *intersecting* if every two of its sets intersect.

Theorem 2 *(Erdös-Ko-Rado [3]) The number of d-sets of an intersecting family is at most $\binom{n-1}{d-1}$.* □

Two different and straightforward constructions for which the equality holds are known; namely, $F(i,n,d)$ (respectively, $F'(i,n,d)$) is an intersecting family of cardinality $\binom{n-1}{d-1}$ whenever $n \geq 2d$ (respectively, $n = 2d$).

According to Erdös (1987) [2], the theorem was proved in 1938 but was not published until 1961 [3]. In 1972 Katona [5] gave a short proof. An alternative one, for the case $n = kd, r = 0$, was given above.

4 Non-uniform case; Open problems

Corollary 1 *The number of sets of an intersecting (not necessarily edge-uniform) family is at most* 2^{n-1}.

Proof . By Theorem 2, the number of pairwise intersecting d-sets is at most $\binom{n-1}{d-1}$, for each $d = 1, \ldots, n$. Thus, the total number of pairwise intersecting sets is at most $\sum_{i=1}^{n}\binom{n-1}{d-1} = \sum_{i=0}^{n-1}\binom{n-1}{d} = 2^{n-1}$. □

The equality, obviously, holds for the set-family $G(i,n)$ that consists of all subsets of N containing a fixed element $i \in N$.

It is also clear that $G(i,n)$ is an exact transversal to the hypergraph $\mathcal{Q}(n)$ whose vertices are $2^n - 1$ non-empty subsets of N and edges are all (not only edge-uniform) partitions. Yet, the following questions still remain open:

(i) whether $G(i,n)$ is a minimum transversal to $\mathcal{Q}(n)$?

(ii) if yes, is it unique?

(iii) whether $G(i,n)$ is a unique maximum intersecting set-family on N?

(iv) Except for $F(i,n,d)$ and $F'(i,n,d)$), whether another minimum transversal to $\mathcal{P}(n,d)$ exist?

5 Extending a list of partitions is exponential

Already for explicitly given edge-uniform hypergraphs of dimension 3 it is NP-complete to verify the existence of a partition; see problems "perfect 3-matching" and "partition by 3-sets" in [4].

Such a verification can require exponential time in case when an edge-uniform hypergraph is given by a membership oracle.

Theorem 3 *Let set-family $F \subseteq B(n,d)$ be given by a membership oracle. To verifying, whether F contains a partition, $\binom{n-1}{d-1}$ tests might be needed.*

More generally, when ℓ partitions in F are given explicitly, while the remaining d-sets are given by a membership oracle, $\binom{n-1}{d-1} - \ell$ tests might be needed to verify whether F contains more than ℓ partition.

Proof . Suppose that only negative answers are given by the oracle. Then, by Theorem 1, we need $\binom{n-1}{d-1}$ of them to conclude that there is no partition in F.

By the same arguments, we conclude that $\binom{n-1}{d-1} - \ell$ negative answers might be needed to conclude that F contains no other partition, except for ℓ given. □

An oik (Euler complex) contains an even number of room-partitions; see [1] for the definitions and more details. In case an oik or a complex is given by a room-oracle, Theorem 3 readily implies that an exponential number of tests might be required to find a new room-partition or show that it fails to exist.

References

[1] J. Edmonds, S. Gaubert, and V. Gurvich, On Sperner and Scarf oiks, Rutcor Research Report RRR-2009, Rutgers University; to appear.

[2] P. Erdös, My joint work with Richard Rado, Surveys in Combinatorics, London Math. Soc. Lecture Note Series, **123** (1987) 53–80.

[3] P. Erdös, C. Ko, R. Rado, Intersection theorem for systems of finite sets, Quart. J. of Math., Oxford Series, Ser. 2, **12** (1961) 313–320.

[4] M. R. Garey and D. S. Johnson, Computers and intractability; A guide to the theory of NP-completeness, W.H. Freeman and Co., San Francisco.

[5] G.O.H. Katona, A simple proof of the Erdös, Chao Ko-Rado Theorem, J. of Combinatorial Theory, Ser. B, **13** (1972) 183–184.