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On exact blockers and anti-blockers, CIS-graphs, and related problems [a]

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ON EXACT BLOCKERS AND ANTI-BLOCKERS, CIS-GRAPHS, AND RELATED PROBLEMS [1]

Vladimir Gurvich

Abstract. Let us consider two binary systems of inequalities (i) $Cx \geq e$ and (ii) $Cx \leq e$, where $C \in \{0,1\}^{m \times n}$ is a $m \times n$ $(0,1)$-matrix, $x \in \{0,1\}^n$, and e is the vector of m ones. The set of all support-minimal (support-maximal) solutions x to (i) (to (ii)) is called the **blocker (anti-blocker)**. A blocker $\mathcal{B}$ (anti-blocker $\mathcal{A}$) is called **exact** if $Cx = e$ for every $x \in \mathcal{B}$ (respectively, $x \in \mathcal{A}$). The exact blockers can be completely characterized. They are in one-to-one correspondence with the P_4-free graphs or, equivalently, with the read-once Boolean functions.
Yet, the class of exact anti-blockers is much wider and it is more sophisticated, too. We demonstrate that it is closely related to the family of the so-called CIS-graphs and its extensions.

Key words: blocker, anti-blocker, exact blocker, exact anti-blocker, read-once, CIS-graph, modular decomposition, substitution

1 Graphs and hypergraphs; basic definitions

A hypergraph $\mathcal{H} \subseteq 2^V$ on the vertex-set $V = V(\mathcal{H}) = \{v_1, \ldots, v_n\}$ is a non-empty family of non-empty subsets $H \subseteq V$ called its *edges*, $H \in \mathcal{H}$.

A subset $A \subseteq V$ is an *independent set* of $\mathcal{H}$ if A contains no edge, $H \not\subseteq A$ for all $H \in \mathcal{H}$. An independent set A is called *maximal* if every its proper superset $A' \supset A$ is not independent, $A' \supseteq H$ for some $H \in \mathcal{H}$. A subset $B \subseteq V$ is called *transversal* to $\mathcal{H}$ if B meets all its edges, $B \cap H \neq \emptyset$ for each $H \in \mathcal{H}$. A transversal B is called *minimal* if no its proper subset $B' \subset B$ is a transversal, $B' \cap H = \emptyset$ for some $H \in \mathcal{H}$. Obviously, the complement to a (minimal) transversal is a (maximal) independent set and vice versa.

A hypergraph $\mathcal{H} \subseteq 2^V$ is called a *graph* if every its edge $H \in \mathcal{H}$ consists of two vertices, $|H| = 2$, which are called *adjacent*.

Standardly, we will denote a graph by G (rather than $\mathcal{H}$) and the set of its edges by $E = E(G)$. The complementary graph $\overline{G}$ of G is defined by the same vertex-set, $V(\overline{G}) = V(G)$, and the complementary edge-set, $(v', v'') \in E(\overline{G})$ if and only if $(v', v'') \notin E(G)$ for any two distinct $v', v'' \in V(G)$. A set of pairwise adjacent (non-adjacent) vertices is called a *clique* (respectively, independent or stable set) of G. Obviously, a (maximal) independent set of G is a (maximal) clique in $\overline{G}$ and vice versa. It is also clear that concepts of a stable set for graphs and hypergraphs are in agreement.

To each hypergraph $\mathcal{H} \subseteq 2^V$ let us assign its *co-occurrence graph* $G = G(\mathcal{H})$ with the same vertex-set $V = V(G) = V(\mathcal{H})$ in which two vertices $v', v'' \in V$ are adjacent in G if and only if they are distinct, $v' \neq v''$, and adjacent in $\mathcal{H}$, that is, $v', v'' \in H$ for an edge $H \in \mathcal{H}$. Yet, distinct hypergraphs can have the same co-occurrence graph.

Example 1 *Let us consider the following three pairs of hypergraphs:*

$\mathcal{H}_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$ *and* $\mathcal{H}_2 = \{(v_1, v_2, v_3)\}$

both correspond to the complete graph on the ground set $V = \{v_1, v_2, v_3\}$*;*

$\mathcal{H}_3 = \{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_1)\}$ *and* $\mathcal{H}_4 = \mathcal{H}_3 \cup \{(v_1, v_3, v_5)\}$

both generate the same graph, so-called sun or 3*-anti-comb; finally*

$\mathcal{H}_5 = \{(v_1, v_3, v_6), (v_1, v_4, v_5), (v_1, v_4, v_6), (v_2, v_3, v_5), (v_2, v_3, v_6), (v_2, v_4, v_5)\}$ *and*
$\mathcal{H}_6 = \mathcal{H}_5 \cup \{(v_1, v_3, v_5) \cup \{(v_2, v_4, v_6)\}$

both generate the same complete 3*-partite* $2 \times 2 \rtimes 2$ *graph.*

With a graph G let us associate its *clique-hypergraph* $\mathcal{H}_C = \mathcal{H}_C(G)$ and its *stable-set-hypergraph* $\mathcal{H}_S = \mathcal{H}_S(G)$ as follows: both have the same vertex-set, $V = V(G) = V(\mathcal{H}_C) = V(\mathcal{H}_S)$, while the edges are all maximal cliques and all maximal stable sets of G, respectively.

A hypergraph $\mathcal{H}$ is called *normal* if it is the clique-hypergraph of its own co-occurrence graph, that is, $\mathcal{H} = \mathcal{H}_C(G(\mathcal{H}))$.

In the above example, $\mathcal{H}_2$, $\mathcal{H}_4$, and $\mathcal{H}_6$ are normal, while $\mathcal{H}_1$, $\mathcal{H}_3$, and $\mathcal{H}_5$ are not.

By definition, for any hypergraph $\mathcal{H}$ there is a unique normal hypergraph $\mathcal{H}'$ with the same co-occurrence graph, $G = G(\mathcal{H}) = G(\mathcal{H}')$; obviously, $\mathcal{H}' = \mathcal{H}_C(G(\mathcal{H}))$.

Let us call $\mathcal{H}$ *semi-normal* if $\mathcal{H} \subseteq \mathcal{H}_C(G(\mathcal{H}))$; or in other words, if each edge of $\mathcal{H}$ is a **maximal** clique in $G(\mathcal{H})$; yet, some maximal cliques of $G(\mathcal{H})$ might be missing in $\mathcal{H}$. In the above example, all hypergraphs are semi-normal, except for $\mathcal{H}_1$.

Furthermore, $\mathcal{H}$ is called a *Sperner* hypergraph if its edges are not nested ($H' \subseteq H''$ for no distinct $H', H'' \in \mathcal{H}$) and $V(\mathcal{H}) = \cup_{H \in \mathcal{H}} H$.

Obviously, all six hypergraphs of Example 1 are Sperner.

It is easily seen that, in general, the above three families, of (i) normal, (ii) semi-normal, and (iii) Sperner hypergraphs, are nested, $(i) \subset (ii) \subset (iii)$.

Example 1 shows that both containments are strict.

Given a hypergraph $\mathcal{H}$ with n vertices, $V(\mathcal{H}) = \{v_1, \ldots v_n\}$, and m edges, $\mathcal{H} = \{H_1, \ldots, H_m\}$, its *incidence* matrix $C = C(\mathcal{H})$ is defined as an $m \times n$ $(0,1)$-matrix whose entry $c(i,j)$ is 1 whenever $v_i \in H_j$ and 0 otherwise.

We refer the reader to the monograph [3], by Claude Berge, for more concepts and details.

2 Blockers and anti-blockers

The hypergraph $\mathcal{B} = \mathcal{B}(\mathcal{H})$ of all minimal transversals to $\mathcal{H}$ is called the *blocker* of $\mathcal{H}$.

By definition, $\mathcal{B}$ is a Sperner hypergraph and $\cup_{B \in \mathcal{B}} B = V(\mathcal{B}) \subseteq V$.

If $\mathcal{H}$ is a Sperner hypergraph too then it is obvious and well-known that

(i) $V(\mathcal{B}) = \cup_{B \in \mathcal{B}} B = \cup_{H \in \mathcal{H}} H = V(\mathcal{H})$ and

(ii) $\mathcal{B}$ is a blocker of $\mathcal{H}$ if and only if $\mathcal{H}$ is a blocker of $\mathcal{B}$.

In this case, hypergraphs $\mathcal{H}$ and $\mathcal{B} = \mathcal{B}(\mathcal{H})$ are called dual and notation $\mathcal{B} = \mathcal{H}^d$ or, equivalently, $\mathcal{B}^d = \mathcal{H}$ is used. In other words, the blocker mapping is an involution, that is, $\mathcal{B}(\mathcal{B}(\mathcal{H})) \equiv \mathcal{H}$ for any Sperner hypergraph $\mathcal{H}$.

In general, an arbitrary (not necessarily Sperner) hypergraph $\mathcal{H}$ can be reduced to a Sperner hypergraph $\mathcal{H}'$ by successive elimination of every edge that contains another edge. It is clear that $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}')$.

Given a hypergraph $\mathcal{H} \subseteq 2^V$, a subset $A \subseteq V$ will be called *weakly independent* if A meets each edge of $\mathcal{H}$ in at most one vertex, $|A \cap H| \leq 1$ for all $H \in \mathcal{H}$. Standardly, a weakly independent set A is *maximal* if no its proper superset is weakly intependent, $|A' \cap H| \geq 2$ for an edge $H \in \mathcal{H}$ whenever $A' \supset A$.

The hypergraph $\mathcal{A} = \mathcal{A}(\mathcal{H})$ of all maximal weakly independent sets of $\mathcal{H}$ is called the *anti-blocker* of $\mathcal{H}$. By definition, $\mathcal{A}(\mathcal{H})$ is a **normal** hypergraph and $\cup_{A \in \mathcal{A}} A = V(\mathcal{A}) = V$.

It is also not difficult to verify that $\mathcal{A}(\mathcal{H}_C(G)) = \mathcal{H}_S(G)$.

More generally, $\mathcal{A}(\mathcal{H}) = \mathcal{H}_S(G(\mathcal{H}))$ for each hypergraph $\mathcal{H}$ (Sperner or not). In particular, the anti-blocker $\mathcal{A}(\mathcal{H})$ depends only on the co-occurrence graph of $\mathcal{H}$. In other words, all hypergraphs with the same co-occurrence graph have the same anti-blocker.

Of course, by symmetry, $\mathcal{A}(\mathcal{H}_S(G)) = \mathcal{H}_C(G)$ for any graph G.

In general, $\mathcal{A}(\mathcal{A}(\mathcal{H})) = \mathcal{H}$ if and only if $\mathcal{H}$ is normal, or in other words, if $\mathcal{H} = \mathcal{H}_C(G)$ (or equivalently, $\mathcal{H} = \mathcal{H}_S(G)$) for a graph G. Even more generally, but still obviously, $\mathcal{A}(\mathcal{A}(\mathcal{H})) = H_C(G(\mathcal{H}))$ for all $\mathcal{H}$.

In general, for an arbitrary (not necessarily normal or Sperner) hypergraph $\mathcal{H}$ let us consider the corresponding normal hypergraph $\mathcal{H}' = \mathcal{H}_C(G(\mathcal{H}))$. It is clear that $G(\mathcal{H}) = G(\mathcal{H}')$ and $\mathcal{A}(\mathcal{H}) = \mathcal{A}(\mathcal{H}')$.

It is easy to see that blocker $\mathcal{B}(\mathcal{H})$ (anti-blocker $\mathcal{A}(\mathcal{H})$) can be equivalently redefined as the set of all support-minimal (support-maximal) binary solutions $x \in \{0,1\}^n$ of the binary system $Cx \geq e$ (respectively, $Cx \leq e$), where $C = C(\mathcal{H})$ is the $m \times n$ incidence matrix of $\mathcal{H}$ and e is the vector of m ones.

For applications of blockers and anti-blockers, we refer, for example, to [14].

3 Exact blockers and anti-blockers and P_4-free graphs

A blocker $\mathcal{B} = \mathcal{B}(\mathcal{H})$ (anti-blocker $\mathcal{A} = \mathcal{A}(\mathcal{H})$ is called *exact* if every its minimal transversal $B \in \mathcal{B}$ (maximal weakly independent set $A \in \mathcal{A}$) and each edge $H \in \mathcal{H}$ have exactly one vertex in common; $|B \cap H| = 1$ (respectively, $|A \cap H| = 1$) for all $H \in \mathcal{H}$.

Equivalently, in terms of the incidence matrix $C = C(\mathcal{H})$, a blocker (anti-blocker) is exact if and only if $Cx \equiv e$ whenever x is a support-minimal (support-maximal) solution to $Cx \geq e$ (respectively, $Cx \leq e$).

Graph P_4 is defined on four vertices $\{v_1, v_2, v_3, v_4\}$ by three edges
$P_4 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$. It is self-complementary, that is, the complementary graph $\overline{P}_4 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$ is obviously isomorphic to P_4. Standardly, a graph G is called P_4-free if it contains no induced P_4.

A hypergraph $\mathcal{H}$ will be called *B-exact (A-exact)* if its blocker $\mathcal{B}(\mathcal{H})$ (anti-blocker $\mathcal{A}(\mathcal{H})$) is exact.

The B-exact hypergraphs are completely characterized by the following statement.

Theorem 1 *([17], see also [21, 16, 19]).*

The next five properties of a hypergraph $\mathcal{H}$ are equivalent:

(i) the blocker $\mathcal{B} = \mathcal{B}(\mathcal{H})$ to $\mathcal{H}$ is exact, that is, $\mathcal{H}$ is B-exact;

(ii) hypergraph $\mathcal{H}$ is normal and its graph $G(\mathcal{H})$ is P_4-free;

(iii) $|B \cap H| = 1$ for all $B \in \mathcal{B}(\mathcal{H})$ and $H \in \mathcal{H}$;

(iv) the co-occurrence graphs $G(\mathcal{H})$ and $G(\mathcal{B}(\mathcal{H}))$ are edge-disjoint;

(v) the co-occurrence graphs $G(\mathcal{H})$ and $G(\mathcal{B}(\mathcal{H}))$ are complementary. □

Remark 1 *It is also shown in [17, 18, 21, 19, 16] that P_4-free graphs are in one-to-one correspondence with the so-called read-once Boolean functions.*

Remark 2 *To show that normality is essential in (ii), let us consider hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_5$ from Example 1. They are not normal and not B-exact, although their co-occurrence graphs are P_4-free: $G(\mathcal{H}_1) = K_3$ and $G(\mathcal{H}_5)$ is the complete 3-partite $2 \times 2 \times 2$ graph.*

In contrast, hypergraph $\mathcal{H}_4$ is normal but not B-exact, since $G(\mathcal{H}_4)$ contains a P_4.

Moreover, the clique hypergraphs $\mathcal{H}_C(G)$ of a P_4-free graph G is not only B-exact, it is A-exact too. Indeed, as we already know, if $\mathcal{H} = \mathcal{H}_C(G)$ then $\mathcal{A} = \mathcal{H}_S(G)$ is the anti-blocker of $\mathcal{H}$. Furthermore, if G is a P_4-free graph then this anti-blocker is exact, by (iii). Thus, both the anti-blocker $\mathcal{A}(\mathcal{H})$ and blocker $\mathcal{B}(\mathcal{H})$ are exact whenever $\mathcal{H}$ satisfies (ii).

By Theorem 1, (ii) is also **necessary** for B-exactness. Yet, not for A-exactness. In the next Section, we will see that an A-exact hypergraph $\mathcal{H}$ might be not normal, although it must be semi-normal, and its co-occurrence graph $G(\mathcal{H})$ might contain induced P_4.

Characterizing A-exact hypergraphs is one of the goals of the present paper.

4 On A-exact and semi-normal hypergraphs

Semi-normality is a necessary condition for A-exactness.

Proposition 1 *A hypergraph $\mathcal{H}$ is semi-normal whenever it is A-exact.*

Proof Let us assume indirectly that $\mathcal{H}$ is not semi-normal; in other words, it has an edge $H_0 \in \mathcal{H}$ and its co-occurrence graph $G = G(\mathcal{H})$ has a (maximal) clique C_0 such that $H_0 \subset C_0$ and containment is strict, i.e., there is a vertex $v \in C_0 \setminus H_0$. Let S_0 be a maximal stable set in G that contains v. Then obviously, S_0 is weakly independent, $|S_0 \cap H| \leq 1$ for any $H \in \mathcal{H}$ and $S_0 \cap C_0 = \emptyset$. Thus, $\mathcal{H}$ is **not** A-exact. □

We call G a *CIS graph* (or say that G has the CIS property) if $C \cap S \neq \emptyset$ for every maximal clique C and every maximal stable set S in G. Each P_4-free graph is a CIS graph, yet, there are many others; see Section 5 and also [1] for more details.

The following condition is sufficient for A-exactness.

Proposition 2 *A hypergraph $\mathcal{H}$ is A-exact whenever it is semi-normal and its co-occurrence graph $G(\mathcal{H})$ is a CIS graph.*

Proof . As we already know, a maximal weakly independent set S of $\mathcal{H}$ is a maximal stable set of $G(\mathcal{H})$. Hence, $H \cap S \neq \emptyset$, since H is semi-normal and $G(\mathcal{H})$ is a CIS graph. □

However, $\mathcal{H}$ might be A-exact when $G(\mathcal{H})$ is not a CIS graph.

Example 2 *Recall hypergraph $\mathcal{H}_3 = \{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_1)\}$ from Example 1. Clearly, its co-occurrence graph $G(\mathcal{H})$ is not a CIS graph, since $C \cap S = \emptyset$ for $C = \{(v_1, v_3, v_5\}$ and $S = \{(v_2, v_4, v_6\}$. Yet, the anti-blocker $\mathcal{A}(\mathcal{H}_3) = \{(v_1, v_4), (v_2, v_5), (v_3, v_6), (v_2, v_4, v_6)\}$ is exact. Let us note, however, that the hypergraph $\mathcal{A}(\mathcal{H}_3)$ is not A-exact, although it is the exact anti-blocker to $\mathcal{H}$. Indeed, $S = (v_2, v_4, v_6) \in \mathcal{A}(\mathcal{H})$, while it is easy to check that $C = (v_1, v_3, v_5) \in \mathcal{A}(\mathcal{A}(\mathcal{H}))$. Furthermore, for the same reason, the normal hypergraph $\mathcal{H}_4 = \mathcal{H}_3 \cup \{(v_1, v_3, v_5)\}$ is not A-exact, either.*

5 Basic facts related to CIS graphs

By definition, CIS-graphs are closed under complementation.

It is also easy to show that they are *exactly closed* under substitution [1].

In other words, let $G = G'(v \to G'')$ mean that graph G is obtained from G' by substituting graph G'', as a module, for a fixed vertex v in G'. Then, G is a CIS graph if and only if G' and G'' are CIS graphs.

However, the family of CIS-graphs is not hereditary. For example, the bull (or A-graph) is a CIS graph; yet, it contains an induced P_4, which is not a CIS graph.

For this reason, CIS graphs cannot be characterized in terms of forbidden subgraphs. In fact, every graph G' is an induced subgraph of a CIS graph G. Given G', to get G it is sufficient to add a simplicial vertex to each maximal clique of G'. Let us notice, however, that G might be exponential in the size of G'. See [1] for more details.

Perhaps, for the same reason, no efficient characterization or recognition algorithm for CIS graphs is known. Yet, some necessary but not sufficient and sufficient but not necessary conditions are known.

For an integer $k \geq 2$, we define a *k-comb* G_k as a graph on $2k$ vertices $\{v_1, \ldots, v_k; v'_1, \ldots, v'_k\}$ with $k(k+1)/2$ edges which form the clique on $\{v_1, \ldots, v_k\}$ and perfect matching (v_i, v'_i), $i \in [k] = \{1, \ldots, k\}$.

The complementary graph $\overline{G}_k$ is called a k-anti-comb. For example, 2-comb, 2-anti-comb, and P_4 are isomorphic graphs. It is easy to see that a k-comb contains k induced $(k-1)$-combs for each $k \geq 3$.

It is also clear that k-combs G_k and k-anti-combs $\overline{G}_k$ are not CIS-graphs. Indeed, two disjoint sets $\{v_1, \ldots, v_k\}$ and $\{v'_1, \ldots, v'_k\}$ induce a maximal clique and maximal stable set in G_k, and vice versa in $\overline{G}_k$.

In 80s, Claude Berge noticed that in a CIS graph G every induced P_4 must be contained in an induced bull-graph; see [26]. More generally, for each $k \geq 2$, in a CIS graph G, every induced k-comb G_k (respectively, k-anti-comb $\overline{G}_k$) must be settled, that is, G must contain a vertex v_0 adjacent to every v_i and not adjacent to every v'_i for all $i \in [k] = \{1, \ldots, k\}$ (respectively, vice versa) [1]. Berge's necessary conditions correspond to the case $k = 2$. However, even for all $k \geq 2$, the above conditions do not imply the CIS property. The corresponding example was constructed by Ron Holzman in 1994; see [1].

By Theorem 1, G is a CIS graph whenever it is P_4-free. In this case, G contains no induced combs and anti-combs. In fact, the following relaxation still implies the CIS property.

Theorem 2 *Graph G is a CIS graph whenever it contains no induced 3-combs and 3-anti-combs and every induced 2-comb is settled in G.*

This was conjectured in by Vasek Chvatal in 90s. His RUTCOR student Wenan Zang published partial results in 1995 [26]; Finally, Theorem was proved by Deng, Li, and Zang [10, 11], and independently in [1].

Graph G is called an *almost CIS* graph if every its maximal clique C and maximal stable set S intersect, except a unique pair. In contrast to CIS graphs, the family of almost CIS graphs admits a simple (although not-trivial) characterization.

Theorem 3 *Graph G is an almost CIS graph if and only if G is a split graph with a unique split-partition.*

It was conjectured in [1]. Partial results were obtained in [5]. Finally, Theorem was recently proved by Wu, Zang and Zhang [25].

6 On ℓ-normal hypergraphs and (ℓ, ℓ')-CIS graphs

A hypergraph $\mathcal{H} \subseteq 2^V$ will be called *ℓ-normal* if in its co-occurrence graph $G(\mathcal{H})$ every clique of cardinality ℓ is contained by an edge of H.

Without any loss of generality, we will assume that $1 \le \ell \le n = |V|$.

Indeed, n-normality obviously implies n'-normality for all n'.

In fact, a hypergraph $\mathcal{H}$ is n-normal if and only if it is normal.

Yet, an ℓ-normal hypergraph might be not semi-normal when $\ell < n$.

In general, ℓ-normality obviously implies ℓ'-normality whenever $\ell \ge \ell'$.

Given positive integer ℓ and ℓ', a graph G will be called an *(ℓ, ℓ')-CIS graph* if there exist ℓ- and ℓ'-normal hypergraphs $\mathcal{H}$ and $\mathcal{H}'$ whose co-occurrence graphs are G and $\overline{G}$, $G(\mathcal{H}) = G$, respectively, $G(\mathcal{H}') = \overline{G}$, and whose edges pairwise intersect, $H \cap H' \neq \emptyset$ for all $H \in \mathcal{H}$, $H' \in \mathcal{H}'$.

Proposition 3 *Hypergraphs $\mathcal{H}$ and $\mathcal{H}'$ are semi-normal whenever G is an (ℓ, ℓ')-CIS graph.*

Proof can be copied from the proof of Proposition 1. □

Above definitions obviously result in the following characterization of A-exactness.

Theorem 4 *A hypergraph $\mathcal{H}$ is A-exact if and only if it is 2-normal and its co-occurrence graph $G(\mathcal{H})$ is a $(2, n)$-CIS graph.* □

Let us note that Proposition 1 immediately results from Proposition 3 and Theorem 4.

In Section 9 we will extend the above (ℓ, ℓ')-CIS property from graphs to d-graphs. This generalization is of independent interest, although it is not directly related to exact anti-blockers. In the next Section, we extend the standard CIS property from graphs to d-graphs.

7 Basic facts related to CIS d-graphs

A d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ is a complete graph on the vertex-set $V = \{v_1, \ldots, v_n\}$ whose $\binom{n}{2}$ edges are partitioned into d subsets (colored by d colors) some of which might be empty. We say that $\mathcal{G}$ is ℓ-colored if ℓ is the number of its non-empty chromatic components $E_i \neq \emptyset$ for $i \in [d] = \{1, \ldots, d\}$. Obviously, $\ell = 0$ if and only if $\mathcal{G}$ consists of a unique vertex, $|V| = 1$. Such d-graph is called *trivial*.

In case $d = 2$ a d-graph is just a graph, or more precisely, a pair that consists of a graph and its complement. Thus, d-graphs can be viewed as a generalization of graphs.

Given a d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$, let G_i denote its ith chromatic component, that is, the graph on the vertex-set V with the edge-set E_i; furthermore, let $S_i \subseteq V$ be a **maximal** stable set in G_i, where $i \in [d] = \{1, \ldots, d\}$; finally, let $\mathcal{S} = \{S_i \mid i \in [d]\}$ be a collection of d such sets and let $S = \cap_{i=1}^{d} S_i$. Obviously, $|S| \leq 1$ for every collection $\mathcal{S}$, since $v, v' \in S$ implies that edge (v, v') has no color in $\mathcal{G}$.

We call $\mathcal{G}$ a *CIS d-graph*, or say that it has CIS d-property, if $S \neq \emptyset$ for each collection $\mathcal{S}$ defined above.

It is easy to verify that the family of CIS d-graphs is exactly closed under substitution [1]. Yet, as we already know, it is not hereditary already for $d = 2$.

Two d-graphs Π and Δ shown in Figure 1 will play an important role:

Π is defined for any $d \geq 2$ by $V = \{v_1, v_2, v_3, v_4\}$;
$E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$, $E_2 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$, and $E_i = \emptyset$ whenever $i > 2$;

Δ is defined for any $d \geq 3$ by $V = \{v_1, v_2, v_3\}$,
$E_1 = \{(v_1, v_2)\}$, $E_2 = \{(v_2, v_3)\}$, $E_3 = \{(v_3, v_1)\}$, and $E_i = \emptyset$ whenever $i > 3$.

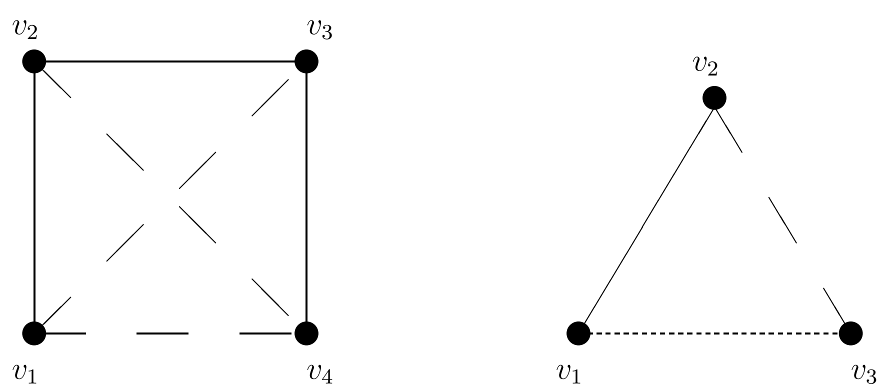

Figure 1: 2- and 3-colored d-graphs Π and Δ.

Clearly, Π and Δ are respectively 2- and 3-colored d-graphs; both non-empty chromatic components of Π are isomorphic to P_4 and Δ is a three-colored triangle.

Both d-graphs Π and Δ were introduced by Gallai (1967) in his seminal paper [12]; Δ-free d-graphs are frequently referred to as Gallai graphs; we will call them Gallai d-graphs, which is more accurate. It is not difficult to verify that the Gallai d-graphs are exactly closed under substitution and that the Π- and Δ-free d-graphs have the CIS d-property [1].

Although many CIS d-graphs contain Πs, yet, it seems that they cannot contain Δs.

Δ-Conjecture ([18] page 71). Each CIS d-graph is a Gallai d-graph; in other words, no CIS d-graph contains Δ.

Several partial results in this direction are obtained in [1]; in particular, Δ-conjecture for an arbitrary d is reduced to the case $d = 3$.

It is also shown in [1] (Sections 1.6, 1.7, and 4) that, modulo Δ-conjecture, the problem of characterizing the CIS d-graphs is reduced to the case $d = 2$, that is, to characterization of the CIS graphs. This reduction is based on a general modular decomposition of the Gallai d-graphs outlined in the next Section. Let us remark, however, that case $d = 2$ is still very difficult; see [26, 10, 11, 1, 5, 25, 4] for partial results and more details.

Obviously, Π and Δ are not CIS d-graphs. Moreover, they are minimal, that is, each sub-d-graph of Π or Δ is a CIS d-graph. Recently, in [4], it was shown that Π and Δ are the only (locally) minimal non-CIS d-graphs, that is, every non-CIS d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ distinct from Π and Δ contains a vertex $v \in V$ such that the reduced d-graph $\mathcal{G}[V \setminus \{v\}]$ is still not CIS. This result easily follows from Δ-conjecture, yet, in [4], it was also proven independently.

8 Modular decomposition of the Gallai d-graphs and its applications

Gallai d-graphs are well studied [1, 2, 6, 7, 8, 13, 20, 22, 23]. In particular, it is well-known that they are closed under substitution. Let us substitute a d-graph $\mathcal{G}''$ (a module) for a vertex v of a d-graph $\mathcal{G}'$ and denote the obtained d-graph by $\mathcal{G} = \mathcal{G}'(v \to \mathcal{G}'')$.

It is easy to see that $\mathcal{G}$ contains both $\mathcal{G}'$ and $\mathcal{G}''$ as sub-d-graphs.

We say that a family $\mathcal{F}$ of d-graphs is *(exactly) closed under substitution* if $\mathcal{G} \in F$ whenever (respectively, if and only if) $\mathcal{G}' \in \mathcal{F}$ and $\mathcal{G}'' \in \mathcal{F}$. This definition is applicable to graphs too; in this case $d = 2$.

Proposition 4 *Families of Gallai and CIS d-graphs are exactly closed under substitution.*

For the second family a long but simple case analysis is needed; see [1]. For the first family it is enough to show that a d-graph $\mathcal{G} = \mathcal{G}'(v \to \mathcal{G}'')$ contains a Δ if and only if $\mathcal{G}'$ or $\mathcal{G}''$ contains it.

Proof . Let $\mathcal{G}$ contain a Δ. Clearly, this Δ cannot have exactly one edge in $\mathcal{G}''$, since then two remaining edges are of the same color. If two edges of this Δ are in $\mathcal{G}''$ then the third one is in $\mathcal{G}''$ too and, hence, G'' contains a Δ. Finally, if all three edges are in $\mathcal{G}'$ then G' contains a Δ. Conversely, if $\mathcal{G}'$ or $\mathcal{G}''$ contains a Δ then $\mathcal{G}$ contains it too, since $\mathcal{G}'$ and $\mathcal{G}''$ are sub-d-graphs of $\mathcal{G}$. □

Every Gallai d-graph can be obtained from 2-colored d-graphs by successive substitutions.

Theorem 5 *(Cameron and Edmonds [6]; Gyárfás and Simonyi [20]).*
Let $\mathcal{G}$ be a Gallai d-graph with at least three non-trivial chromatic components. Then $\mathcal{G} = \mathcal{G}'(v \to \mathcal{G}'')$, where $\mathcal{G}'$ and $\mathcal{G}''$ are non-trivial Gallai d-graphs.

Clearly, we can proceed with this decomposition until there are at least three non-trivial chromatic components in $\mathcal{G}'$ or in $\mathcal{G}''$, since both these d-graphs are still Δ-free.

Thus, decomposing recursively, we will represent $\mathcal{G}$ by a binary tree $T(\mathcal{G})$ whose leaves correspond to 2-colored d-graphs.

Some nice properties of the Gallai colorings follow from Theorem 5.

Corollary 1 *A Gallai d-graph with n vertices contains at most $n-1$ non-trivial chromatic components.*

As it was mentioned in [20], this result by Erdős, Simonovits, and Sós [13] immediately follows from Theorem 5 by induction.

Corollary 2 *If all but one chromatic components of a Gallai d-graph are perfect graphs then the remaining one is a perfect graph too.*

This claim was proved by Cameron, Edmonds, and Lovász [7]. (Clearly, it turns into Lovász' Perfect Graph Theorem if $d = 2$.) Later, Cameron and Edmonds [6] strengthened this claim showing that the same statement holds not only for perfect graphs but, in fact, for any family of graphs that is closed under:

(i) substitution, (ii) complementation, and (iii) taking induced subgraphs.

In [1], this claim was further strengthened, as follows:

Theorem 6 *([1]). Let $\mathcal{F}$ be a family of graphs closed under complementation and exactly closed under substitution and let $\mathcal{G} = (V; E_1, \ldots, E_d)$ be a Gallai d-graph such that at least $d-1$ of its chromatic components, say $G_i = (V, E_i)$ for $i = 1, \ldots, d-1$, belong to $\mathcal{F}$. Then*

(a) the last component $G_d = (V, E_d)$ is in $\mathcal{F}$ too, and moreover,

(b) all 2^d graphs $G_J = (V, \bigcup_{j \in J} E_j)$ are in $\mathcal{F}$, for every subset $J \subseteq [d] = \{1, \ldots, d\}$.

Proof . **Part (a)**. By Theorem 5, $\mathcal{G}$ can be obtained from 2-colored d-graphs by substitutions. Such a decomposition of $\mathcal{G}$ is given by a tree $T(\mathcal{G})$ whose leaves correspond to 2-colored d-graphs. It is easy to see that by construction each chromatic component of $\mathcal{G}$ is decomposed by the same tree $T(\mathcal{G})$. Hence, all we have to prove is that both chromatic components of every 2-colored d-graph belong to $\mathcal{F}$. For colors $1, \ldots, d-1$ this holds, since $\mathcal{F}$ is exactly closed under substitution, and for the color d it holds, too, since $\mathcal{F}$ is also closed under complementation.

Part (b). It follows easily from part (a). Given a $(d+1)$-graph $\mathcal{G} = (V; E_1, \ldots, E_d, E_{d+1})$, let us identify the last two colors d and $d+1$ and consider the d-graph $\mathcal{G}' = (V; E_1, \ldots, E_{d-1}, E_{\mathbf{d}})$, where $E_{\mathbf{d}} = E_d \cup E_{d+1}$. We assume that $\mathcal{G}$ is Δ-free and that $G_i = (V, E_i) \in \mathcal{F}$ for

$i = 1, \ldots, d-1$. Then $\mathcal{G}'$ is Δ-free too and it follows from part (a) that $G_{\mathbf{d}} = (V, E_{\mathbf{d}})$ is also in $\mathcal{F}$. Hence, the union of any two colors is in $\mathcal{F}$. From this by induction we derive that the union of any set of colors is in $\mathcal{F}$. □

Cameron-Edmonds' Theorem results from Theorem 6 and the following simple fact.

Proposition 5 *Let $\mathcal{F}$ be a family of graphs closed under substitution and taking induced subgraphs then $\mathcal{F}$ is exactly closed under substitution.*

Proof . Both G' and G'' are induced subgraphs of $G = G'(v \to G'')$. □

As we know, CIS graphs are closed under complementation and exactly closed under substitution. Yet, an induced subgraph of a CIS graph might be not a CIS graph. Thus, Theorem 6 is applicable to the the CIS graphs, while Cameron-Edmonds' Theorem is not.

9 On ℓ-CIS d-graphs

Let us extend the concept of CIS d-graph as follows. Let $\ell = (\ell_1, \ldots, \ell_d)$ be a positive integer vector. A d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ will be called an ℓ-CIS d-graph if for each $i \in [d] = \{1, \ldots, d\}$ there is an ℓ_i-normal hypergraph $\mathcal{H}_i$ whose co-occurence graph is $\overline{G}_i$ (hence, without any loss of generality, we can assume that $\ell_i \geq 2$) and such that $\cap_{i=1}^{d} H_i \neq \emptyset$ for every edge-selection $\{H_i \in \mathcal{H}_i \mid i \in [d]\}$.

Obviously, the ℓ-CIS d-graphs turn into the standard CIS d-graphs when $\ell = (n, \ldots, n)$ and $n = |V|$. In this case, all $\mathcal{H}_i$ are normal hypergraphs. In general, it is not difficult to demonstrate (just by copying the proof of Proposition 1) that all $\mathcal{H}_i$ are semi-normal hypergraphs whenever $\mathcal{G}$ is an ℓ-CIS d-graph.

Furthermore, copying the case analysis from [1], it is also easy to verify that

ℓ-CIS d-graphs are exactly closed under substitution.

Hence, the Δ-free (Gallai) ℓ-CIS d-graphs can be reduced to ℓ-CIS 2-graphs (i.e., graphs) by modular decomposition, see Section 7.

Yet, Δ-conjecture does not extend to the case $d = 3$ and $\ell = (2, 2, 2)$ (or even $(2, 2, 5)$).

The next example was constructed by Andrey Gol'berg (1954 - 1985) in 1984.

Example 3 *Let us consider the 3-graph $\mathcal{G}$ on nine vertices $V = \{v_0, v_1, \ldots, v_8\}$ in Figure 2, where solid (dotted) lines are colored by color 1 (respectively, 2), and each edge between $\{v_1, v_2, v_3, v_4\}$ and $\{v_5, v_6, v_7, v_8\}$ is of color 3. It is easy to verify that $\mathcal{G}$ contains eight Δs induced by the vertex-sets*

$(v_0, v_1, v_6), (v_0, v_1, v_7), (v_0, v_4, v_6), (v_0, v_4, v_7), (v_0, v_2, v_5), (v_0, v_2, v_8), (v_0, v_3, v_5), (v_0, v_3, v_8)$.

Let us consider the following three hypergraphs: $\mathcal{H}_1 = \{(v_0, v_1, v_2, v_3, v_4), (v_0, v_5, v_6, v_7, v_8)\}$;
$\mathcal{H}_2 = \{(v_0, v_2, v_3, v_6, v_7), (v_1, v_2, v_5, v_6), (v_1, v_2, v_7, v_8), (v_3, v_4, v_5, v_6), (v_3, v_4, v_7, v_8)\}$;
$\mathcal{H}_3 = \{(v_0, v_1, v_4, v_5, v_8), (v_1, v_3, v_5, v_7), (v_1, v_3, v_6, v_8), (v_2, v_4, v_5, v_7), (v_2, v_4, v_6, v_8)\}$.

It is easy to verify that: (a) their co-occurrence graphs are $\overline{G}_1$, $\overline{G}_2$, and $\overline{G}_3$, respectively;

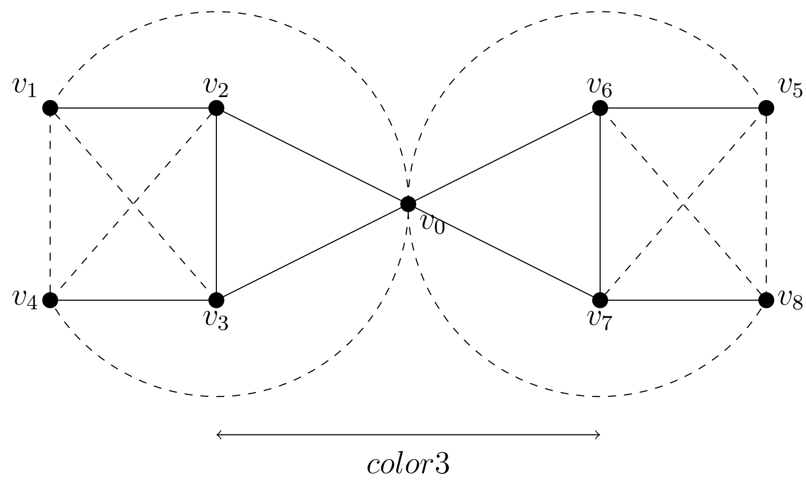

Figure 2: A $(2,2,5)$-CIS 3-graph that contains eight Δs

(b) $\mathcal{H}_1$ *is normal, while* $\mathcal{H}_2$ *and* $\mathcal{H}_3$ *are* 2*-normal, but not* 3*-normal;*
(c) $H_1 \cap H_2 \cap H_3 \neq \emptyset$ *(in fact,* $|H_1 \cap H_2 \cap H_3| = 1$*) for every* $H_1 \in \mathcal{H}_1, H_2 \in \mathcal{H}_2, H_3 \in \mathcal{H}_3$*.*
The corresponding $2 \times 5 \times 5$ *intersection table is given below.*

v_4	v_4	v_2	v_2	v_2		v_6	v_8	v_6	v_6	v_8
v_4	v_4	v_2	v_2	v_2		v_5	v_7	v_7	v_5	v_7
v_4	v_4	v_0	v_1	v_1		v_5	v_8	v_0	v_5	v_8
v_3	v_3	v_3	v_1	v_1		v_6	v_8	v_6	v_6	v_8
v_3	v_3	v_3	v_1	v_1		v_5	v_7	v_7	v_5	v_7

This table represents a 3*-dimensional box-partition with many interesting properties [24].*
Let us repeat that hypergraphs $\mathcal{H}_2$ *and* $\mathcal{H}_3$ *are semi-normal but not normal.*
The corresponding normal hypergraphs are

$\mathcal{H}'_2 = \mathcal{H}_2 \cup \{(v_1, v_2, v_6, v_7), (v_3, v_4, v_6, v_7), (v_2, v_3, v_5, v_6), (v_2, v_3, v_7, v_8)\}$ *and*

$\mathcal{H}'_3 = \mathcal{H}_3 \cup \{(v_1, v_4, v_5, v_7), (v_1, v_4, v_6, v_8), (v_1, v_3, v_5, v_8), (v_2, v_4, v_5, v_8)\}$.

However, for the triplet $\mathcal{H}'_1$*,* $\mathcal{H}'_2$*, and* $\mathcal{H}'_3$ *the intersection property fails. For example,*

$\{v_0, v_5, v_6, v_7, v_8\} \cap \{v_1, v_2, v_6, v_7\} \cap \{v_1, v_3, v_5, v_8\} = \emptyset$.

Thus, there is no contradiction with the "normal" (n, n, n)*-CIS* Δ*-conjecture.*

Thus, a $(2,2,5)$-CIS 3-graph can contain a Δ, while an (n, n, n)-one cannot if the Δ-conjecture holds.

More generally, one can ask for which ℓ, if any, the ℓ-CIS d-graphs contain no Δ.

10 On Gallai d-graphs and complete, normal, and solid d-dimensional box-partitions

The obtained intersection table $g : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \{v_0, v_1, \dots, v_8\}$ represents a box-partition of the total $2 \times 5 \times 5$ box $\mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$ into nine boxes $\{v_0, v_1, \dots, v_8\}$.

Let us notice that the first five boxes in this box-partition are *solid*, that is, the corresponding edges got successive numbers in the given edge-enumeration of hypergraphs $\mathcal{H}_1$. $\mathcal{H}_2$, and $\mathcal{H}_3$, while the last four boxes are **not** solid.

It is easy to verify that there is no edge-enumeration of these three hypergraphs such that **all** boxes are solid. This observation is, in fact, a general geometric property of the solid d-dimensional box partitions: they generate either $(d+1)$-graphs or **Gallai** d-graphs.

An accurate statement follows.

A collection of d hypergraphs $\mathcal{H} = \{\mathcal{H}_i \subseteq 2^V \mid i \in [d] = \{1, \dots, d\}\}$ defined on a common ground-set V will be called a CIS-collection (or we will say that it has the CIS property) if

$$\left| \bigcap_{i=1}^{d} H_i \right| = 1 \text{ for all edge-selections } \mathbf{H} = \{H_i \in \mathcal{H}_i \mid i \in [d]\}.$$

Without any loss of generality, we will assume each vertex $v \in V$ is realized as the esge-intersection $v = \cap_{i=1}^{d} H_i$ of such an edge-selection.

Let us consider a mapping $g : \mathcal{H} \to V$ that assigns the intersection-vertex $v = v(\mathbf{H}) = \cap_{i=1}^{d} H_i$ to every edge-selection $\mathbf{H} = \{H_i \in \mathcal{H}_i \mid i \in [d]\}$ of a CIS collection $\mathcal{H}$ of hypergraphs.

Alternatively, this mapping g can be interpreted as a *box-partition* in which every vertex $v \in V$ is a box.

To each such box-partition g we will assign a $(d+1)$-graph $\mathcal{G} = \mathcal{G}(g)$ as follows. Given a pair of vertices $v, v' \in V$, let us consider the subset $d(v, v') \subseteq [d]$ defined by the condition:

$$v, v' \in H_i \text{ for an edge } H_i \in \mathcal{H}_i \text{ if and only if } i \in d(v, v').$$

Geometrically, $i \in [d] \setminus d(v, v')$ if and only if projections **of the interiors** of boxes v and v' in the direction i intersect. It is easy to see that there is at most one such direction, $|[d] \setminus d(v, v')| \le 1$. In particular, $d(v, v') = [d]$ if and only if there is no such direction.
It is also clear boxes v and v' would intersect whenever there are two or more such directions. Let us color each pair $v, v' \in V$ by the unique color $i \in [d] \setminus d(v, v')$ whenever $|[d] \setminus d(v, v')| = 1$ and by a new color $d+1$ whenever $[d] \setminus d(v, v') = \emptyset$.

By this rule, to each box-partition $g : \mathcal{H} \to V$ a $(d+1)$-graph $\mathcal{G}(g) = (V; E_1, \dots, E_d, E_{d+1})$ is assigned. It is easy to see that $\mathcal{G}(g)$ contains a Δ whenever $E_{d+1} \ne \emptyset$.

Furthermore, a box-partition g will be called:

(i) *complete* if $E_{d+1} = \emptyset$, or in other words, if for each $v, v' \in V$ there is a direction $i \in [d]$ such that projections of the interiors of boxes v and v' in this direction intersect;

(ii) *normal* if it is complete and all d hypergraphs of the corresponding family $\mathcal{H}$ are normal; or in other words, if for every direction $i \in [d]$ the following Helly property holds: projections of the interiors of a family of boxes intersect whenever they are pairwise intersect;

(iii) a *Gallai* box-partition (or we will say that g has the Gallai property) if g is complete and the corresponding d-graph $\mathcal{G}(g)$ is Δ-free.

(iv) *solid* if there is an edge-enumeration of all d hypergraphs of $\mathcal{H}$ such that all boxes of the box-partition g are solid.

As we know, the box-partition g from Example 3 is complete. Yet, it is not normal, not solid, and it does not have the Gallai property, instead, the 3-graph $\mathcal{G}(g)$ has eight Δs.

Obviously, the Δ-conjecture can be reformulated as follows:

a normal box-partition has the Gallai property.

Furthermore, it appears that a complete and solid box-partition has this property, too.

Theorem 7 *If g is complete and solid then its d-graph $\mathcal{G}(g)$ contains no Δ.* □

This result was announced in [15], a proof can be found in [24].
It has a natural geometric interpretation:

no three solid boxes that induce a Δ can be extended to a complete solid box partition.

Yet, Example 3 shows that this statement does not extend to the case of non-solid boxes.

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