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On Scarf and Sperner Oiks

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On Scarf and Sperner Oiks

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Abstract. The idea of "Lemke pivoting in a family of oiks (Euler complexes)" generalizes, and abstracts to pure combinatorics, the Lemke-Howson exchange algorithm for finding a Nash equilibrium in bimatrix games, as well as the classical algorithm for finding the properly colored room in Sperner's Lemma. Given a "room-partitioning", this algorithm finds another (distinct) room-partitioning by traversing the exchange graph.
In this paper we show that each family of k oiks $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ can be reduced to a pair of oiks $\mathcal{O}' = \{\mathcal{O}_1 + \ldots + \mathcal{O}_k, \mathcal{O}_0\}$ (one of which, $\mathcal{O}_0$, is a Sperner oik) such that the exchange graphs for $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic. Numerous application of Sperner's Lemma in combinatorial topology are well known.
Also we formulate the famous Scarf Lemma in terms of oiks. This Lemma has two fundamental applications in game and graph theories. In 1967, Scarf derived from it core-solvability of balanced cooperative games. Recently, it was shown that kernel-solvability of poerfect graphs also results from this Lemma.
We show that Scarf's combinatorially defined oiks are in fact realized by polytopes. We also show that the pivoting path between room-partitions can be exponentially long already for two equal d-dimensional Scarf oiks on $2d$ vertices.
Keywords: Euler complex (oik), room, wall, manifold, cyclic polytope, binary matroid; exchange algorithm, pivot; bimatrix game, Nash equilibrium, Lemke-Howson algorithm; Sperner Lemma, Brouwer Fixed Point Theorem, KKM-Theorem; core, core-solvability, Scarf Lemma, balanced games; kernel, kernel-solvability, perfect graph.

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1 Introduction

1.1 Oiks; definition and examples

The concept of an *oik* (short for Euler complex) was recently introduced in [12] as follows.

Given two integers n and d such that $n > d > 1$, a *d-dimensional complex* $\mathcal{O} = (V, \mathcal{R})$ is a uniform hypergraph of edge-size d on the ground set V of cardinality n. Standardly, the elements $v \in V$ are called *vertices*, while the edges $R \in \mathcal{R}$ are called *rooms*; each room consists of d vertices.

Furthermore, given a room R and a vertex $v \in R$, the difference $W = R \setminus \{v\}$ (of cardinality $d-1$) is called a *wall*. A vomplex is called an *oik* if each wall W is contained in a **positive even** number $k(W)$ of rooms.

Two rooms $R, R' \in \mathcal{R}$ are called *adjacent* if their intersection is a wall, or in other words, if their symmetric difference $R \Delta R'$ is a pair of vertices $v \in R$ and $v' \in R'$.

An oik $\mathcal{O}$ will be called 2-*adjacent* if $k(W) \equiv 2$ for every wall W, that is, if each wall is contained in exactly two (adjacent) rooms.

The next four examples of 2-adjacent oiks are borrowed from [12].

Example 1 : Pseudo-manifolds. *A $(d-1)$-dimensional simplicial pseudo-manifold is a d-dimensional oik in which each d vertices are contained in exactly zero or two rooms; in other words, each wall is in exactly two rooms.*

An important special case is a triangulation of a compact manifold M, oriented or not.

In particular, if M is a $(d-1)$-dimensional sphere, the corresponding oik $\mathcal{O}(M)$ is realized by a d-dimensional polytope whose every facet is a simplex with d vertices.

The oiks generated by pseudo-manifolds, manifolds, and polytopes will be called PM-, M-, and P-oiks, respectively. The latter will be also called *polytopal* and represented as follows.

Example 2 : Polytopal oiks. *Let $Ax = b, x \geq 0$ be a tableau, as in the simplex method, that is, A is a $m \times n$ matrix that contains an $m \times m$ identity submatrix and all coordinates of $b \in RR^m$ are strictly positive. Let us also assume that the solution set is bounded and all basic feasible solutions are non-degenerate.*

Let V be the column set of A. By definition, its subset $R \subseteq V$ is a room if and only if $V \setminus R$ is a basis of the tableau. The hypergraph $\mathcal{O} = (V, \mathcal{R})$ of the rooms defines an oik of dimension $d = n-m$. This results from the following exchange property of the bases. Given a basic set of columns in A (the complement to a room), let us add to it an arbitrary "entering" column (thus getting the complement to a wall). Then there exists a unique "leaving" column such that all coefficients of the right-hand-side remain positive.

Combinatorially the above oik is defined by the boundary of an $(n-m)$-dimensional simplicial polytope.

Remark 1 *The boundary (surface) of a simplicial polytope of dimension d is a manifold of dimension $d-1$. Thus, the corresponding oik can be called either d- or $(d-1)$-dimensional. Respectively, there are two options: to call an oik d-dimensional when its rooms are of cardinality d or $d+1$. Here we chose the first option, while the second one is chosen in [12].*

Let us consider two examples of special polytopal oiks.

Example 3 : Gale oiks. *Let us consider Gale's cyclic polytope $P = P(d, n) \subseteq \mathbb{R}^d$ with n vertices. In [15], David Gale proved that the rooms of the corresponding oiks are defined by the* **cyclic** *binary n-vectors $x \in \{0, 1\}^n$ with d ones such that the following* **Gale evenness condition** *holds: If d is even then all sequences of successive ones in x are even. (Let us remark that the first and the last such sequences in x make one sequence s_0, since x is cyclic.) If d is odd then all above sequences are still even, except s_0, which must be odd.*

For more details we refer the reader to [17, 18].

Example 4 : Sperner oiks. *Let the n elements of a set V be colored by d colors, where $d < n$. A subset $R \subset V$ is a room if and only if $V \setminus R$ contains exactly one vertex of each color.*

The defined hypergraph $\mathcal{O} = (V, \mathcal{R})$ is an oik of dimension $n - d$.

Indeed, the complement to a wall, which is colored $\{1, 2, \ldots, d, j\}$, contains exactly two complements to rooms, which are colored $\{1, 2, \ldots, d\}$.

This oik is polytopal. In particular, when V consists of $2d$ vertices and each color appears twice, $\{1, 1, 2, 2, \ldots, d, d\}$, the corresponding polytope is polar to the d-dimensional cube. We leave the proofs to the reader.

Remark 2 *Let us note that in the latter case the complement to a room is also a room. However, in general, such claim does not hold for the above four Examples. See also the next Remark 3.*

In Sections 1.3 and 3, we introduce one more family of 2-adjacent oiks based on the Scarf Lemma [33] and, in Section 4, we prove that these oiks are polytopal.

Now, we borrow from [12] another four examples of oiks related to Euler graphs and binary matroids. Let us note that these oiks might be not 2-adjacent.

Example 5 *An Euler graph, that is, a connected graph $G = (V, E)$ in which each vertex has an even degree is an oik, where E and V are the sets of rooms (edges) and walls (vertices), respectively. Let us remark that a disjoint union of Euler graphs is an oik, too.*

Example 6 *An Euler graph $G = (V, E)$ can be also interpreted as an oik $\mathcal{O} = (E, \mathcal{R})$ in a different way: E is the vertex-set of $\mathcal{O}$, while its rooms are the spanning trees of G. The dimension of this oik is $d = |V| - 1$.*

Example 7 *A connected bipartite graph $G = (V, E)$ defines a $|E| - |V| + 1$-dimensional oik $\mathcal{O} = (E, \mathcal{R})$ whose vertices are the edges of G and rooms are the complements to the spanning trees of G.*

Remark 3 *Thus, a bipartite Euler graph G defines two oiks whose rooms are complementary.*

The following example generalizes the previous two.

A binary matroid M is the set of columns of a binary matrix A, $\mod 2$. The bases of M are the linearly independent sets of columns. The co-cycles are the supports of the row vectors generated by the rows of A. The co-circuits are the minimal co-cycles. Matroid M is called *Euler* when each row of A has an even number of ones. See, for example, [25, 34] for more details.

Example 8 *Let M be an Euler binary matroid of rank r in which each co-circuit (or equivalently, each co-cycle) is even. Then M defines an r-dimensional oik whose vertices are the elements and rooms are the bases of M.*

1.2 Finding another room-partition or room-selection of fixed degrees by the exchange algorithm

An *oik-family* is a set of k oiks $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ (of dimensions $d_1, \ldots, d_k$) defined on the same vertex-set V. Some of these oiks may be isomorphic or even identical.

Given an oik-family $\mathcal{O}$, a room-selection is a hypergraph $\mathcal{R} = \{R_1, \ldots, R_k\}$ in which R_i is a room of oik $\mathcal{O}_i$ for all $i \in [k] = \{1, \ldots, k\}$. Standardly, $\delta_{\mathcal{R}}(v)$ denote the degree of a vertex $v \in V$ in $\mathcal{R}$, that is, the number of rooms of $\mathcal{R}$ that contain v.

A room-selection $\mathcal{R}$ is called *a room-partition* if $\delta_{\mathcal{R}}(v) \equiv 1$ for each $v \in V$.

It was shown in [12] that every oik-family has an even number of room-partitions.

Remark 4 *Let us note, however, that this number may be 0. Moreover, it might be NP-hard to verify the existence of a room-partition.*

Furthermore, given a room-partition, an **exchange algorithm** to get another one is suggested in [12]. This algorithm is based on constructing and, then, traversing the *exchange graph.* Given a family of oiks $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ on the common vertex-set ($\mathcal{O}_i = (V, \mathcal{R}_i)$, $i \in [k] = \{1, \ldots, k\}$), let us fix a special vertex $w \in V$ and define the exchange graph $\mathcal{G} = \mathcal{G}(\mathcal{O}, w)$ as follows.

A room-selection $\mathcal{R}$ is called a *skew room-partition* or a *butterfly* if $deg_{\mathcal{R}}(w) = 0$, $deg_{\mathcal{R}}(u) = 2$ for a unique vertex $u \in V$, and $deg_{\mathcal{R}}(v) \equiv 1$ for all other vertices $v \in V \setminus \{u, w\}$. Let $\mathcal{V}$ and $\mathcal{V}_1$ denote the sets of all room-partitions and skew room-partitions, respectively.

Two room-selections $\mathcal{R} = \{R_1, \ldots, R_k\}$ and $\mathcal{R}' = \{R'_1, \ldots, R'_k\}$ are called *adjacent* if their symmetric difference $\mathcal{R} \Delta \mathcal{R}'$ is a pair of **adjacent** rooms (R_i, R'_i) from $\mathcal{O}_i$ for some $i \in [k]$. If also $\mathcal{R}, \mathcal{R}' \in \mathcal{V} \cup \mathcal{V}_1$ then $(\mathcal{R}, \mathcal{R}') \in \mathcal{E}$. Thus, the exchange graph $\mathcal{G}(\mathcal{V} \cup \mathcal{V}_1, \mathcal{E})$ is defined.

(Let us recall that rooms R_i and R'_i are adjacent if their symmetric difference $R_i \Delta R'_i$ is a pair of vertices, or in other words, if their intersection $R_i \cap R'_i$ is a wall W_i in $\mathcal{O}_i$.)

It is easy to list all rooms adjacent to a given room R of a given oik $\mathcal{O}$. To do so, let us select a vertex $v \in R$ and enumerate all rooms of $\mathcal{O}$, except R, that contain the wall $W = R \setminus \{v\}$. By definition of an oik, there is an odd number $k(W) - 1$ of such rooms. We get all rooms of $\mathcal{O}$ adjacent to R just repeating the above procedure for all $v \in R$.

Furthermore, by this procedure, it is also easy to obtain all room-selections adjacent to a given one $\mathcal{R} = \{R_1, \ldots, R_k\}$ in a given oik-family $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$.

Above definitions and observations immediately imply the following properties of the exchange graph $\mathcal{G}(\mathcal{V} \cup \mathcal{V}_1, \mathcal{E})$.

Lemma 1 *Vertices of $\mathcal{V}$ (room-partitions) and $\mathcal{V}_1$ (skew room-partitions) have odd and even degrees in $\mathcal{G}$, respectively. No two vertices of $\mathcal{V}$ are adjacent.* □

Obviously, number $|\mathcal{V}|$ of the room-paritions is even, since in any graph number of the odd degree vertices is even.

Furthermore, given a room-partition $\mathcal{R} \in \mathcal{V}$, let us traverse $\mathcal{G}$ arbitrarily, yet, beginning in $\mathcal{R} \in \mathcal{V}$ and passing no edge twice, until no possible move is left. In other words, we construct an Eulerian path beginning in an odd degree vertex (room-partition) $\mathcal{R} \in \mathcal{V}$. Obviously, any such path ends in another odd degree vertex (room-partition) $\mathcal{R}' \in \mathcal{V}$ distinct from $\mathcal{R}$. Indeed, $\mathcal{R}' \notin \mathcal{V}_1$, since all vertices of $\mathcal{V}_1$ have even degrees. Also $\mathcal{R}' \neq \mathcal{R}$, since vertex $\mathcal{R} \in \mathcal{V}$ is of odd degree. In particular, the following statement follows.

Theorem 1 *Every oik-family $\mathcal{O}$ has an even number of room-partitions.*

Given a vertex $w \in V$ and a room-partition $\mathcal{R}$, we get another room-partition $\mathcal{R}'$ distinct from $\mathcal{R}$ by traversing the exchange graph $\mathcal{G}(\mathcal{O}, w)$ starting in $\mathcal{R}$ and passing no edge twice. □

If $\mathcal{O}$ is a family of 2-adjacent oiks then obviously vertices of $\mathcal{V}$ and $\mathcal{V}_1$ have degrees 1 and 2 respectively. In this case the exchange graph has a very simple structure: it is a disjoint union of simple pathes whose ends form $\mathcal{V}$ and simple cycles whose vertices form the rest of $\mathcal{V}_1$. These pathes uniquely define the traversing procedure, as well as a matching on the set $\mathcal{V}$ of room-partitions.

The above results can be generalized in many ways; for example, as follows.

Let $\delta : V \to \mathbb{Z}_+$ be a mapping of V into set $\mathbb{Z}_+$ of the non-negative integers. A room-selection $\mathcal{R}$ is called a δ-selection if $\delta_{\mathcal{R}}(v) = \delta(v)$ for each $v \in V$. Given $\mathcal{O}$ and δ, let us define $\mathcal{V}$ as the set of all δ-selections and $\mathcal{V}_1$ as follows.

Let us fix a vertex $w \in W$ such that $\delta(w)$ is **odd**. A *skew* $(\delta \pm 1)$-selection (or a dragonfly) is defined as a room-selection $\mathcal{R}'$ such that $\delta_{\mathcal{R}'}(w) = \delta(w) - 1$, there is a vertex $u \in V$ such that $\delta_{\mathcal{R}'}(u) = \delta(u) + 1$, and $\delta_{\mathcal{R}'}(v) = \delta(v)$ for all other vertices $v \in V \setminus \{u, w\}$.

Given w and δ, let $\mathcal{V}_1$ be the set of all *skew* $(\delta \pm 1)$-selections. Finally, the adjacency relation $\mathcal{E}$ on the vertex-set $\mathcal{V} \cup \mathcal{V}_1$ and the exchange graph $\mathcal{G} = \mathcal{G}(\mathcal{O}, w) = \mathcal{G}(\mathcal{V} \cup \mathcal{V}_1, \mathcal{E})$ are defined exactly as before. It is easy to verify that all claims of Lemma 1 and Theorem 1 still hold.

Lemma 2 *Vertices of $\mathcal{V}$ (δ-selections) and $\mathcal{V}_1$ (skew $(\delta \pm 1)$-selection) have odd and even degrees in $\mathcal{G}$, respectively. No two vertices of $\mathcal{V}$ are adjacent in $\mathcal{G}$.* □

Theorem 2 *Every oik-family $\mathcal{O}$ has an even number of δ-selection.*

Given a vertex $w \in V$ of **odd** *$\delta(w)$ and a δ-selection $\mathcal{R}$, we get another room-partition $\mathcal{R}'$ distinct from $\mathcal{R}$ by traversing the exchange graph $\mathcal{G}(\mathcal{O}, w)$ starting in $\mathcal{R}$ and passing no edge twice.* □

1.3 Scarf's oiks; main results

Here we recall one more example of 2-adjacent oiks introduced by Herbert Scarf in [33]; see also [31, 32, 20].

An $m \times (m+n)$ real non-negative matrix A is called a *Scarf* matrix if

(i) $m \geq 2$ and $n \geq 1$;

(ii) $a(i,j) > a(i,m+k) > a(i,i) \geq 0$ for all $i,j \in [m] = \{1,\dots,m\}$, where $i \neq j$, and $k \in [n] = \{1,\dots,n\}$;

(iii) $a(i,m+k) \neq a(i,m+\ell)$ for all $i \in [m]$ and distinct $k,\ell \in [n]$.

Let us consider the following example in which $m = 3$ and $n = 4$.

$0\ M\ M\quad 1\ 2\ 3\ 4$
$M\ 0\ M\quad 3\ 1\ 2\ 4$
$M\ M\ 0\quad 4\ 3\ 2\ 1$

A $m \times (m+n)$ Scarf matrix will be called *canonical* if

(iv) $a(i,i) = 0$ for any $i \in [m]$ and $a(i,j) = M > n$ for any distinct $i,j \in [m]$;

(v) in each row, the last n entries form a permutation of $[n] = \{1,\dots,n\}$.

Obviously, given a constant $M > n$, there are $(n!)^m$ canonical Scarf matrices.
The above example represents one of them.

Let $V = [m+n] = \{1,\dots,m+n\}$ be the set of columns of a Scarf (not necessarily canonical) matrix A. A subset $J \subseteq V$ is called *dominating* if for each column $k \in [m+n]$ there is a row $i \in [m]$ such that $a(i,k) \leq a(i,j)$ for each $j \in J$.

Remark 5 *Names "subordinating" [2] or "primitive" set [20] also appear in the literature. We choose name "dominating" following [1, 23].*

The next four properties (vi - ix) obviously hold whenever J is dominating:

(vi) If $J' \subseteq J$ then J' is dominating, too.

In other words, domination is a hereditary property defined on V.

By definition, each column $j \in J$ is dominated by J, as well as any other column. This immediately results in the following claim.

(vii) Let us take a minimum $a(i, j_i)$ in each row $i \in [m]$ of the $[m] \times J$ submatrix A_J. Each column $j \in J$ contains at least one such minimum.

Indeed, otherwise column j is not dominated by J.

Obviously, a minimum $a(i, j_i)$ can appear in the row i more than once only when $J \subseteq [m]$ and $a(i, j_i) = M$. This observation together with (vii) imply the following two claims:

(viii) $|J| \leq m$.

(ix) If $|J| = m$ then each row $i \in [m]$ of A_J has a unique minimum $a(i, j_i)$ and these m minima form a permutation in A_J.

The following simple observation will play an important role.

Lemma 3 *In a Scarf matrix A, the first m columns, $J = [m]$, do not form a dominating set, while each proper subset $J \subset [m]$ is a dominating set.*

Proof . Each row of the $[m] \times [m]$ submatrix A_J contains 0. Hence, no column $i \notin J$ is dominated by J.

In contrast, if J is a proper subset of $[m]$, say, $J \subseteq [m] \setminus \{i\}$, then

(a) each column $j \notin [m]$ is dominated by J in row i, while

(b) each column $j \in [m]$ is also dominated by J, since j contains a "zero", or more precisely, the absolute minimum in the corresponding row. □

It will be convenient to call $J = [m]$ a *special dominating* set.

It is easy to verify that four properties (vi - ix) still hold after this extension. Moreover, due to it, the following key statement becomes true.

Theorem 3 *(***Scarf's Lemma***). Each dominating set of $m - 1$ columns is contained in exactly two dominating sets of m columns; one of these two sets might be $[m]$.*

Each dominating set of $m-1$ (respectively, m columns, including the special one) will be called a wall (respectively, room). By (vi) and Theorem 3, this structure defines a 2-adjacent oik of dimension $d = m$.

Remark 6 *In fact, Theorem 3 is stronger than the last statement, which would result from the following weaker claim: "each dominating set of cardinality $m - 1$ is contained either in two or in none of the dominating sets of cardinality m. Yet, by Theorem 3, the second option cannot hold.*

Remark 7 *Without convention on the special dominating set* $[m]$*, Theorem 3 and oik structure would fail. Indeed, by Lemma 3, for each* $j \in [m]$ *the column-set* $J = [m] \setminus \{j\}$ *is dominating in a Scarf matrix. By Theorem 3,* J *is contained in exactly two dominating sets one of which is* $[m]$*.*

Remark 8 *Theorem 3 would also fail for* $n = 0$*. Indeed, in this case every set of* $m - 1$ *columns is dominating but there is only one set of cardinality* m*. Thus, condition* $n \geq 1$ *is essential.*

Remark 9 *The definition and properties of the dominating sets are based only on the order of the entries* $a(i, m+j)$ *for* $j \in [n]$ *in each row* $i \in [m]$*. Given these* m *complete orders over* $[n]$*, the real values of the entries are irrelevant. Thus, without any loss of generality, we can fix some* $M > n$ *and restrict ourselves by* $(n!)^m$ *canonical Scarf matrices. In particular, it will be sufficient to prove Theorem 3 only for them.*

The following example shows that properties (i,ii,iii), that define the Scarf $m \times (m + n)$ matrices, can hardly be relaxed; even a slight modification of them might destroy the oik-structure.

Example 9 *Let us consider the following "almost" Scarf matrix.*

$$\begin{matrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{matrix}$$

It is easy to verify that columns 1 *and* 3 *form a dominating set, while* 2 *and* 3 *do not, since for them both row-minima are in the column* 2*. Whether* $\{1, 2\}$ *is a special dominating set or it is not, still the oik properties fail. Indeed, set* $\{3\} = \{1, 3\} \setminus \{1\}$ *should be a wall, yet, it is contained in only one room.*

The first proof of Theorem 3 given in [33] was then simplified in [20] and later in [2]. In Section 3, we will try to make it even simpler.

The rest of the paper is organized as follows.

In Section 4, we prove that the Scarf oiks are polytopal, that is, they can be realized by the construction given in Example 2.

Let us extend list (i - ix) by the following obvious observation:

(x) If $n = m$ then the last m columns of a Scarf oik form a room.

This room and the special room partition the set of all columns $V = [m + n]$.

In Section 5, we construct a sequence of $m \times 2m$ matrices such that the exchange path beginning in the above room-partition ends in another one only in $3 \times 2^m - 1$ steps. This construction shows that the exchange algorithm can be exponential in dimension d already for $2d$ vertices.

1.4 Main applications of oiks

Several classical results can be explained in terms of oiks and exchange algorithms, which, given a room partition find another one. Let us consider the following three examples.

The Lemke-Howson algorithm [27] (of finding a Nash equilibrium in mixed strategies in a bimatrix game) can be interpreted as the exchange algorithm for two polytopal oiks. In [39], it was reformulated as an exchange algorithm for three oiks: two polytopal and one Sperner. More details can be found in [26, 28, 37].

The famous Sperner Lemma can be interpreted as Lemma 1 and Theorem 1 for an oik-family which consists of two oiks: a polytopal and Sperner one. In this case, given a malty-colored simplicial facet of a polytope, the exchange algorithm finds another one. This result has fundamental applications in combinatorial topology: Brouwer's Fixed Point Theorem [9] and KKM-Theorem [22]; see [3, 38] and also the next subsection for more details.

Similarly, the Scarf Lemma [33] can be interpreted as Lemma 1 and Theorem 1 for two oiks: a polytopal and Scarf oiks. In this case, the exchange algorithm begins with the origin and come to a dominating vertex of a given polytope; see Sections 3, 4, and 5 for more details.

This result has important applications in cooperative game theory. In [33], Scarf derived from his lemma existence of a non-empty core in every **balanced** game with non-transferrable utility (balanced NTU-game); see also [31, 32, 20, 36, 37, 10, 11, 21, **?**].

Interestingly, core-solvability of the NTU-games appears to be equivalent with kernel-solvability of perfect graphs [7, 2]; see also [1, 8, 23, 24]. The similarity between the Scarf and Sperner lemmas is discussed in [1, 23, 24].

Fractional versions of cores and kernels were considered in [2, 1, 23, 24]. In these papers, fractional core-solvability of all (not only balanced) NTU-games and fractional kernel-solvability of all (not only perfect) graphs were derived from Scarf's Lemma.

Remark 10 *Let us note that all applications mentioned above are related with partitioning of V in* **two** *rooms in a family which consists of two (distinct) oiks. It would be interesting to find an application of a partitioning (or δ-selection) in at least three rooms.*

2 Every oik-family can be reduced to a pair of oiks one of which is a Sperner oik

Given a d-dimensional polytope (or, more generally, a $(d-1)$-dimensional manifold) P whose n vertices are colored by d colors $[d] = \{1, \dots, d\}$, we also assume that P is *simplicial*, that is, every facet of P contains only d vertices. A facet is called *multi-colored* if its d vertices are colored by d distinct colors. The classical Sperner's Lemma claims that the number of the multi-colored facets is even; moreover, given one of them, another one is uniquely determined by the exchange algorithm. This claim can be generalized in many ways [3, 38]. In particular, Lemma 1 and Theorem 1 generalize it to an arbitrary oik as follows.

Let $\mathcal{O}_1 = (V, \mathcal{R}_1)$ be an oik whose n vertices are colored by d colors, $c : V \to [d]$. A room $R_1 \in \mathcal{R}_1$ is *multi-colored* if $c(R_1) = [d]$. By Theorem 1, the number of multi-colored rooms is even; moreover, given one of them, another one can be obtained by the exchange algorithm.

To see that Theorem 1 is applicable let us add to the oik $\mathcal{O}_1$ a $(n-d)$-dimensional Sperner oik $\mathcal{O}_2 = (V, \mathcal{R}_2)$ defined on the same vertex-set V by the coloring c as follows. A set $R_2 \subseteq V$ is a room of oik $\mathcal{O}_2$ if and only if $|R_2| = n - d$ and the **complementary** set $V \setminus R_2$ of cardinality d is multi-colored; see Example 4. By this definition, a room $R_1 \in \mathcal{R}_1$ is multi-colored in oik $\mathcal{O}_1$ if and only if its complement $R_2 = V \setminus R_1$ is a room of $\mathcal{O}_2$, or in other words, sets R_1 and R_2 form a room-partition in the oik-pair $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)$.

Thus, Theorem 1 is applicable; in particular, it results in the standard "geometrical" Sperner Lemmas when $\mathcal{O}_1$ is a PM-, M-, or P-oik; see Example 1. . Yet, in general, this approach is purely combinatorial and geometry is ignored. Moreover, oik $\mathcal{O}_1$ might be not 2-adjacent. In this case, given a room-partition, another one, defined in Theorem 1, is not necessarily unique.

Now let $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)$ be an arbitrary oik-pair defined on a common vertex-set. Then, Theorem 1 results in the

(a) Sperner Lemma when $\mathcal{O}_1$ is a polytopal oik, while $\mathcal{O}_2$ is a Sperner oik;

(b) Scarf Theorem [33] when $\mathcal{O}_1$ is a polytopal oik, while $\mathcal{O}_2$ is a Scarf oik;

(c) Lemke-Howson exchange algorithm [27] when oiks $\mathcal{O}_1$ and $\mathcal{O}_2$ are polytopal.

Somewhat surprisingly, an arbitrary oik-family $\mathcal{O} = (\mathcal{O}_1, \dots, \mathcal{O}_k)$ is equivalent with an oik-pair $\mathcal{O}' = (\mathcal{O}_{k+1}, \mathcal{O}_0)$, where $\mathcal{O}_{k+1} = \mathcal{O}_1 + \ldots + \mathcal{O}_k$ is a sum, which will be defined below, and $\mathcal{O}_0$ is a Sperner oik, that is, the exchange graphs of $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic. Hence, one can execute the exchange algorithm for $\mathcal{O}'$ rather than for $\mathcal{O}$.

Remark 11 *In particular, due to this reduction, the Scarf theorem [33] can be derived from the Sperner Lemma as well as from the Scarf Lemma. The last observation is the main result of the recent paper by Kiraly and Pap [24].*

The reduction is simple. Let $\mathcal{O} = (\mathcal{O}_1, \dots, \mathcal{O}_k)$ be an arbitrary oik-family in which $\mathcal{O}_i = (V, \mathcal{R}_i)$ is a d_i-dimensional oik for $i \in [k] = \{1, \dots, k\}$ and $\sum_{i=1}^k d_i = n = |V|$. First, let us define the sum $\mathcal{O}_{k+1} = \sum_{i=1}^k \mathcal{O}_i$ as follows: $\mathcal{O}_{k+1} = (kV, \mathcal{R}_{k+1})$, where kV consists of k pairwise disjoint copies $V_1, \dots, V_k$ of V and $R \in \mathcal{R}_{k+1}$ if and only if $R \cap V_i$ is a room of the oik $\mathcal{O}_i = (V_i, \mathcal{R}_i)$ (which is a copy of $\mathcal{O}_i = (V, \mathcal{R}_i)$) for all $i \in [k] = \{1, \dots, k\}$. In particular, $|kV| = kn$ and $d_{k+1} = \sum_{i=1}^k d_i = n$ are the size and dimension of the oik $\mathcal{O}_{k+1}$.

Let us color n vertices of V by n pairwise distinct colors and then copy this coloring in every V_i, $i \in [k]$, thus, coloring kn vertices of the set kV in n colors. This coloring standardly defines the Sperner oik $\mathcal{O}_0 = (kV, \mathcal{R}_0)$ in which $R \in \mathcal{R}_0$ if and only if $kV \setminus R$ is multi-colored. Thus, the oik-pair $\mathcal{O}' = (\mathcal{O}_{k+1}, \mathcal{O}')$ is defined. Let us choose two vertices: $w \in V$ and $w' \in kV$.

Theorem 4 *Two exchange graphs $\mathcal{G} = \mathcal{G}(\mathcal{O}, w)$ and $\mathcal{G}' = \mathcal{G}(\mathcal{O}', w')$ are isomorphic whenever vertices w and w' are of the same color.*

Proof . We will make use of the standard notation: $V = \{v_1, \ldots, v_n\}$, $V_i = \{v^i_1, \ldots, v^i_n\}$ for all $i \in [k]$ and $\mathcal{G} = (\mathcal{V} \cup \mathcal{V}_1, \mathcal{E})$, $\mathcal{G}' = (\mathcal{V}' \cup \mathcal{V}'_1, \mathcal{E}')$, as in Section 1.2. Without any loss of generality, we can assume that vertices v_j and v^i_j are of color j for all $j \in [n], i \in [k]$ and also that $w = v_1$ and $w' = v^1_1$.

Then $\mathcal{V}$ is the set of all room-partitions $\mathcal{R} = (R_1, \ldots, R_k)$ of V, that is, rooms $R_i \in \mathcal{R}_i,\ i \in [k]$ are pairwise disjoint and $\cup^k_{i=1} R_i = V$.

Respectively, $\mathcal{V}'$ consists of the set-families $\mathcal{R}' = (R'_1, \ldots, R'_k)$ such that R'_i is a room of $\mathcal{O}_i = (V, \mathcal{R}_i)$ for all $i \in [k]$ and the union $R' = \cup^k_{i=1} R'_i \subseteq kV$ is a multi-colored set of cardinality n. Indeed, by definition, in this and only in this case the complementary set $kV \setminus R'$ is a room of the Sperner oik $\mathcal{O}_0 = (kV, \mathcal{R}_0)$.

There is an obvious one-to-one correspondence f between the sets $\mathcal{V}$ and $\mathcal{V}'$.

Furthermore, $\mathcal{V}_1$ is the set of all skew room-partitions (or butterflies) $\mathcal{R}_1 = (R^1_1, \ldots, R^1_k)$ in $\mathcal{O}$, that is, $\cup^k_{i=1} R_i = V \setminus \{w\}$ and the rooms $R_i \in \mathcal{R}_i,\ i \in [k]$ are pairwise disjoint, except for a unique pair, which has a unique common vertex $u \in V$ distinct from w.

Respectively, $\mathcal{V}'_1$ consists of the set-families $\mathcal{R}' = (R'_1, \ldots, R'_k)$ in $\mathcal{O}' = (kV, \mathcal{R}')$ such that R'_i is a room of $\mathcal{O}_i = (V, \mathcal{R}_i)$ for all $i \in [k]$ and the union $R' = \cup^k_{i=1} R'_i \subseteq kV$ is an "almost" multi-colored set of cardinality n, that is, in its coloring 1 does not appear, while some other color appears twice.

Again, there is an obvious one-to-one correspondence f_1 between the sets $\mathcal{V}_1$ and $\mathcal{V}'_1$. Moreover, the obtained two mappings f and f_1 realize an isomorphism between the exchange graphs $\mathcal{G} = \mathcal{G}(\mathcal{O}, w)$ and $\mathcal{G}' = \mathcal{G}(\mathcal{O}', w')$. This is not difficult to derive just from the definitions of Section 1.2. □

It is important to notice that the obtained reduction is exponential in k but it is polynomial in size of $\mathcal{O}$. Hence, it is polynomial when k is a constant.

Thus, the room-partitions of an arbitrary oik-family $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ are in one-to-one correspondence with the multi-colored rooms of the sum $\mathcal{O}_1 + \ldots + \mathcal{O}_k$. In particular, instead of looking for another room-partition, one can search for another multi-colored room. This shows a sort of universality of the Sperner Lemma, cf. [24].

3 One more proof of the Scarf Lemma

By Remark 9, it is enough to prove Theorem 3 for the canonical Scarf matrices.

Let $J \subseteq [m+n]$ be a dominating set of $m-1$ columns. In the corresponding $[m] \times J$ submatrix A_J, let us choose a minimum entry $a(i, j_i) = \min_{j \in J} a(i, j)$ in every row $i \in [m]$. Each column $j \in J$ contains at least one of them, since otherwise j is not dominated by J; see (vii) in Section 1.3.

If $J \subseteq [m]$, or in other words, $J = [m] \setminus \{j\}$ for a column $j \in [m]$ then row j in matrix A_j consists of $m-1$ equal entries M. This case we will consider a little later.

In case $J \not\subseteq [m]$, it is easy to verify that in each row $i \in [m]$ the minimum $a(i, j_i)$ is unique. Since there are m rows and $|J| = m-1$ columns, there is a (unique) column $j^* \in J$

that contains exactly two row-minima, say (without loss of generality), $a(1, j^*)$ and $a(2, j^*)$ of rows 1 and 2.

Clearly, two equalities $a(1, j^*) = a(2, j^*) = 0$ cannot hold, since no column of A contains two zeros.

Let us assume that $a(2, j^*) = 0$. Then all other entries of column j^* are equal to M. In particular, $a(1, j^*) = \min_{j \in J} a(1, j) = M$. Hence, $a(1, j) = M$ for all $j \in J$ and, thus, $J = [m] \setminus \{1\} = \{2, \dots, m\}$.

Let us show that in this case there are exactly two extensions j_1, j_2 of J to a dominating set J_0 of m columns: $J_0 = J \cup \{j_1\}$ and $J_0 = J \cup \{j_2\}$. Indeed, $j_2 = 1$ is one of them, since by convention, $[m]$ is a special dominating set.

Then, j_1 is uniquely defined as follows: $a(1, j_1) = \max_{j \in [n]}(a(1, m + j)$; in other words, $a(1, j_1)$ is a unique largest entry, distinct from M, in the first row. Indeed, in this case $J_0 = J \cup \{j_1\}$ dominates every column $j \in [m + n]$:

if $j > m$ then $a(1, j)$ is dominated by J_0 in the first row;

if $j \leq m$, column j contains 0 and is dominated by J_0 in the corresponding row.

Furthermore, one can substitute no other column j_1' for j_1, because then column j_1 is not dominated by $J_0' = J \cup \{j_1'\}$.

Finally, let us note that the above arguments are based on assumption $n \geq 1$, or in other words, $[n] \neq \emptyset$. Indeed, otherwise j_1 fails to exist; see Remark 7.

Since case $a(2, j^*) = 0$ is fully similar to the just considered one, let us consider the last case, $a(1, j^*) > 0,\ a(2, j^*) > 0$, and prove again that there are exactly two extensions j_1, j_2 of J to a dominating set J_0 of m columns.

For $k = 1, 2$ let J_k denote the set of columns such that

$$a(k, j^*) > a(k, j^k), \quad \text{while} \quad a(i, j_i) < a(i, j^k) \quad \text{for each} \quad j^k \in J_k, \ \ i \neq k. \tag{3.1}$$

In other words, if we extend J by a column $j^k \in J_k$ and obtain $J_0 = J \cup \{j^k\}$ then in every row $i \in [m] \setminus \{k\}$ the minimum $a(i, j_i) = \min_{j \in J} a(i, j) = \min_{j \in J_0} a(i, j)$ in J_0 remains the same as in J, while in the row k the minimum $a(k, j^k) = \min_{j \in J_0} a(k, j)$ becomes updated: it moves from column j_i to j^k.

Lemma 4 *Column-sets J_1, J_2, and J are not empty and pairwise disjoint.*

Proof . First, $|J| = m - 1 > 0$, by definition; hence, $J \neq \emptyset$.

Then, $J_k \neq \emptyset$ too; in particular, $k \in J_k$ for $k \in \{1, 2\}$. Indeed, (3.1) holds, since $a(k, k) = 0$ and $a(i, k) = M$ whenever $i \neq k$.

Furthermore, $J_k \cap J = \emptyset$ for $k = 1, 2$, by (3.1).

Finally, $J_1 \cap J_2 = \emptyset$. Indeed, if $j_0 \in J_1 \cap J_2$ then column-set $J_0 = J \cup \{j_0\}$ is **not** dominating, since both minima in rows 1 and 2 come to the same column j_0, in contradiction with (iii) of Section 1.3. □

Also by (vii), containment $j_0 \in J_1 \cup J_2$ must hold for every dominating extension $J_0 = J \cup \{j_0\}$. Since $J_1 \cap J_2 = \emptyset$, let us consider two cases: $j_0 = j^1 \in J_1$ and $j_0 = j^2 \in J_2$. Furthermore, $a(k, j^k) = max_{j \in J_k} a(k, j)$ must hold for $k = 1, 2$, too. Indeed, otherwise, for any j_0 distinct from j^k, column j^k is **not** dominated by $J_0 = J \cup \{j_0\}$.

Finally, if both above conditions $j^k \in J_k$ and $a(k, j^k) = max_{j \in J_k} a(k, j)$ hold then extension $J_0 = J \cup \{j^k\}$ **is** a dominating set, by (3.1).

Thus, we obtain exactly two extensions $j_k = j^k$ for $k = 1$ and $k = 2$. □

4 Scarf's oiks are polytopal

Given a $m \times (m+n)$ Scarf matrix A (not necessarily in canonical form) defined by formulae (i,ii,iii) of Section 1.3, introduce a $n \times (m+n)$ matrix B as follows:

(i) $b(i, m+j) = \delta_i^j$ for $i, j \in [n]$,

where standardly $\delta_i^j = 1$ if $i = j$ and $\delta_i^j = 0$ if $i \neq j$; in other words, the last n columns of B form the $n \times n$ identity matrix;

(ii) $b(j, i) = 1 - a(i, m+j)^{-K}$; $i \in [m]$, $j \in [n]$;

in other words, the last n columns of A form a $m \times n$ matrix C and the first m columns of B form a $n \times m$ matrix D such that

(iii) $d(i, j) = 1 - c(j, i)^{-K}$; $i \in [n]$, $j \in [m]$, where $K > 0$ is a large constant.

Transformation $f(a) = 1 - a^{-K}$ was suggested by Scarf in [31]. Obviously,

(iv) f is a monotone increasing function for any fixed positive K;

(v) $f(a) = 1 - 1/a^K > 0$ when $a > 1$ and $f(a) \leq 0$ when $0 < a \leq 1$.

Due to (v), it will be convenient to assume that

(vi) $a(i, m+j) > 1$ for all $i \in [m], j \in [n]$, in the considered Scarf matrix A.

It is clear that (vi) can be assumed without any loss of generality, since the Scarf oik $\mathcal{O}_A$ remains the same if we add a constant to these entries of A.

Furthermore, by construction,

(vii) matrices A and B have the common column-set $V = [m+n]$.

Let us consider system of n equations $Bx = e_n$ of $m+n$ real variables $x \in \mathbb{R}^{m+n}$, where $e_n \in \mathbb{R}^n$ is the vector of n ones. Standardly, a set of columns $J \subseteq V$ is called *basic* if $Bx = e_n$ for a non-negative x such that $x_j = 0$ whenever $j \notin J$. [Conversely, $x_j > 0$ for $j \in J$, because matrix B is not degenerate for any Scarf matrix A and sufficiently large K.] Obviously, set $J_0 = \{m+j \mid j \in [n]\}$ of the last n columns in B is basic. Moreover, it is well known that each basic set J can be obtained from J_0 by a sequence of exchanges produced by simplex-method; see Example 2. In that example, we assigned the room $V \setminus J$ to each basic set J in B thus getting an m-dimensional oik $\mathcal{O}_B$. The next theorem shows that A and B generate the same oik, $\mathcal{O}_A = \mathcal{O}_B$. In other words, that the Scarf oik $\mathcal{O}_A$ is polytopal.

Theorem 5 *A column-set J is basic in B if and only if the complementary set $V \setminus J$ is dominating in A, provided $K > 0$ is sufficiently large.*

Proof . It is easily seen that the original basis J_0 in B is complementary to the special dominating set $[m]$ in A.

As an exercise, let us exchange one column in J_0, say, $m+1$ by m. We have to show that column-set $J = (J_0 \setminus \{m+1\}) \cup \{m\}$ is basic in B if and only if $V \setminus J = [m-1] \cup \{m+1\}$ is dominating in A. As we know, J is basic iff the following system of equations has a (strictly) positive solution:

$$b(1,m)x_m = 1; \;\; b(i,m)x_m + x_{m+i} = 1 \;\text{ for }\; i = 2,\dots,n. \tag{4.2}$$

Let us substitute $b(i,m) = 1 - a(m, m+i)^{-K}$ for $i \in [n]$ and obtain

$$x_m/a(m, m+1)^K = x_m - 1; \tag{4.3}$$

$$x_m/a(m, m+i)^K - x_{m+i} = x_m - 1 \;\text{ for }\; i = 2,\dots,n. \tag{4.4}$$

It is not difficult to verify that

(viii) $x_m = (1 - a(m, m+1)^{-K})^{-1} > 1$, since $a(m, m+1) > 1$, by (vi).

(ix) Right-hand-side of system (4.3, 4.4) is a positive constant $x_m - 1$.

(x) for $i = 2,\dots,n$: $\quad x_{m+i} > 0$ if and only if $a(m, m+1) > a(m, m+i)$.

Proposition 1 *The following four statements are equivalent*

(xi) $a(m, m+1) > a(m, m+i)$ for $i = 2,\dots n$;

(xii) column-set $V \setminus J = \{1,\dots,m-1, m+1\}$ is dominating in A;

(xiii) system (4.3-4.4) has a (strictly) positive solution;

(xiv) column-set $J = \{m, m+2,\dots,m+n\}$ is basic in B.

Proof follows easily from properties (viii), (ix), and (x). □

Now let us consider the general case. Without any loss of generality, we can assume that J consists of the first ℓ and last $n-\ell$ columns of B, that is, $J = \{1,\dots,\ell, m+\ell+1,\dots,m+n\}$. The corresponding system of equation is

$$b(i,1)x_1 + \ldots + b(i,\ell)x_\ell = 1 \;\text{ for }\; i \in [\ell] = \{1,\dots,\ell\}; \tag{4.5}$$

$$b(\ell+j,1)x_1 + \ldots + b(\ell+j,\ell)x_\ell + x_{m+\ell+j} = 1 \;\text{ for }\; j \in [n-\ell] = \{1,\dots,n-\ell\}. \tag{4.6}$$

Substituting $b(j,i) = 1 - a(i, m+j)^{-K}$ for each entry of B we get

$$x_1/a(1, m+i)^K + \ldots + x_\ell/a(\ell, m+i)^K = (x_1 + \ldots + x_\ell) - 1 \text{ for } i \in [\ell]; \tag{4.7}$$

$$x_1/a(1,m+\ell+j)^K+\ldots+x_\ell/a(\ell,m+\ell+j)^K-x_{m+\ell+j} = (x_1+\ldots+x_\ell)-1 \text{ for } j \in [n-\ell]. \quad (4.8)$$

Let us notice that the right-hand-side of system (4.7, 4.8) is a constant $(x_1+\ldots+x_\ell)-1$. We will show that this constant is positive.

Let us recall that, by (vi), $a(i,j) > 1$ for all entries of the Scarf matrix A; also in each row of A, the last n entries are pairwise distinct. Hence, in each column of (4.7, 4.8) the coefficient $a(i,j)^{-K}$ corresponding to the smallest entry $a(i,j)$ is much larger than all others, provided $K > 0$ is very large.

First, let us consider ℓ equations of (4.7) and choose the (unique) largest coefficient for each variable, or in other words, the (unique) minimum entry in every column. Then let us scale positive variables $x_i, i \in [\ell]$ to make all these largest coefficients equal. Let us show that they form a permutation in the considered $\ell \times \ell$ matrix.

(xv) If system (4.7) has a positive solution for a sufficiently large K then each equation of (4.7) contains (exactly) one of these largest coefficients, or equivalently, all columns $\{m + j \mid j \in [\ell]$ are dominated by $V \setminus J$ in A.

Indeed, let i, i' be two equations of (4.7) such that i contains a largest coefficient, while i' does not. Then, obviously, the left-hand-sides of i and i' cannot be equal, while the right-hand-sides are equal, which is a contradiction.

Thus, without loss of generality, we can assume that $a = a(i, m+i) \le a(j, m+s)$ for all $i, j, s \in [\ell]$ and equality holds only when $j = s$; in other words, the left-hand-side matrix of (4.7) has a dominating constant main diagonal $1/a$.

Let us substitute for (4.7) the following approximating system

$$x_i/a^K = (x_1 + \ldots + x_\ell) - 1; \quad i \in [\ell]. \quad (4.9)$$

Obviously, (4.9) has a unique constant solution $x^0(K) = (x_1, \ldots, x_\ell)$, where $x_i = (\ell - a^{-K})^{-1} > 1/\ell$ for all $i \in [\ell]$. In particular, $(x_1 + \ldots + x_\ell) - 1 > 0$ for each $K > 0$ and $\lim_{K\to+\infty} x^0(K) = x^0 = (1/\ell, \ldots, 1/\ell)$. It is also clear that

(xvi) for any sufficiently large positive K system (4.7) has a unique solution $x(K) = (x_1, \ldots, x_\ell)$, where $x_i > 1/\ell$ for all $i \in [\ell]$; in particular, (4.7, 4.8) has a **positive** constant right-hand-side $(x_1 + \ldots + x_\ell) - 1$; moreover, $x(K)$ tends to the same constant vector $x^0 = (1/\ell, \ldots, 1/\ell)$, as $K \to +\infty$.

Now let us add to (4.7) the $n - \ell$ equations of (4.8).

(xvii) Equation $j \in [n - \ell]$ of (4.8) has a (strictly) positive solution if and only if the corresponding column is dominated by $V \setminus J$ in A.

Proof . If this column is dominated then, by definition, j has a much larger coefficient than any one in (4.7). Hence, by (xvi),

$$x_1/a(1, m + \ell + j)^K + \ldots + x_\ell/a(\ell, m + \ell + j)^K > (x_1 + \ldots + x_\ell)) - 1$$

and there is a (unique) **positive** $x_{m+\ell+j}$ which settles the equation j in (4.8).

Conversely, if the considered column is not dominated by $V \setminus j$ in A then each equation of (4.7) has a much larger coefficient than all coefficients of the equation j of (4.8). Hence, $x_{m+\ell+j}$ must be negative and the set of columns J is not basic in B. □

This concludes the proof of Theorem 5.

5 Exchange paths of exponential lengths

The exchange path between two room-partitions may be exponential:

(i) in the number of vertices $n = |V|$ already for dimension $d = 3$ and

(ii) in dimension d already for $2d$ vertices, or in other words, for only 2 rooms in a partition.

A construction for (i) was given in [13]. For each $k = 3, 4 \ldots$, there is a 3-dimensional oik $\mathcal{O}_k$ defined by $12(k-2)$ rooms (triangles) on $n = 3k$ vertices. (Hence, each room-partition consists of k rooms.) This oik has an exchange path of length $7 \times 2^{k-1} - 5$ between two room-partitions.

Here, for each $d = 2, 3, \ldots$ we shall construct a d-dimensional Scarf oik $\mathcal{O}_d$ with $n = 2d$ vertices. (Hence, each room-partition consists of only two rooms.) This oik has an exchange path of length $3 \times 2^d - 1$ between two room-partitions.

Oiks $\mathcal{O}_d$ for $d = 2, 3, \ldots$ are given by the following $d \times 2d$ Scarf's tables T_d, where, as usual, M is a very large number:

$$\begin{array}{cc|cc} 0 & M & 1 & 2 \\ M & 0 & 2 & 1 \end{array} \quad \text{for } d = 2;$$

$$\begin{array}{ccc|ccc} 0 & M & M & 1 & 2 & 3 \\ M & 0 & M & 3 & 1 & 2 \\ M & M & 0 & 3 & 2 & 1 \end{array} \quad \text{for } d = 3;$$

$$\begin{array}{cccc|cccc} 0 & M & M & M & 1 & 2 & 3 & 4 \\ M & 0 & M & M & 4 & 1 & 2 & 3 \\ M & M & 0 & M & 4 & 3 & 1 & 2 \\ M & M & M & 0 & 4 & 3 & 2 & 1 \end{array} \quad \text{for } d = 4;$$

$$\begin{array}{ccccc|ccccc} 0 & M & M & M & M & 1 & 2 & 3 & 4 & 5 \\ M & 0 & M & M & M & 5 & 1 & 2 & 3 & 4 \\ M & M & 0 & M & M & 5 & 4 & 1 & 2 & 3 \\ M & M & M & 0 & M & 5 & 4 & 3 & 1 & 2 \\ M & M & M & M & 0 & 5 & 4 & 3 & 2 & 1 \end{array} \quad \text{for } d = 5;$$

...

$$\begin{array}{cccccc|cccccc} 0 & M & M & \ldots & M & M & 1 & 2 & 3 & \ldots & d-1 & d \\ M & 0 & M & \ldots & M & M & d & 1 & 2 & \ldots & d-2 & d-1 \\ M & M & 0 & \ldots & M & M & d & d-1 & 1 & \ldots & d-3 & d-2 \\ \ldots & & & & & & & & & & & \\ M & M & M & \ldots & 0 & M & d & d-1 & d-2 & \ldots & 1 & 2 \\ M & M & M & \ldots & M & 0 & d & d-1 & d-2 & \ldots & 2 & 1 \end{array}$$

for an arbitrary $d \geq 2$.

As before, $V_d = \{1, \dots, d, d+1, \dots, 2d\}$ is the set of columns of table T_d. We begin with the following partition of V_d into two rooms:

$$V_d = R_1^0 \cup R_2^0 = \{1, \dots, d\} \cup \{d+1, \dots, 2d\}$$

and eliminate column 1 from room R_1^0 getting the wall $W_1^0 = R_1^0 \setminus \{1\} = \{2, \dots .d\}$. It is not difficult to verify that the entering column is $2d$ and, hence, the adjacent room is $R_1^1 = \{2, \dots, d, 2d\}$.

Since rooms R_1^1 and R_2^0 form a butterfly with the intersection $2d$, next, we eliminate $2d$ from R_2^0 getting the wall $W_2^0 = \{d+1, \dots, 2d-1\}$. It is not difficult to verify that the entering column is $d-1$ and, hence, the adjacent room is $R_2^1 = \{d-1, d+1 \dots, 2d-1\}$.

Since rooms R_1^1 and R_2^1 form a butterfly with the intersection $d-1$, next, we eliminate $d-1$ from R_1^1 getting the wall $W_1^1 = \{2, \dots, d-1, 2d\}$. It is not difficult to verify that the entering column is $d-1$, etc... until we obtain another room partition.

Since each room-partition consists of only two rooms, the exchange path is uniquely defined by the sequence S_d of the leaving (or entering) columns. (Obviously, they leave (and enter) the first and second rooms alternatingly.) One can verify by induction that sequences S_d can be conveniently represented by the following tables:

$$\begin{array}{lll} 1 & & \\ 6 & 2 & 4 \\ 6 & & \end{array} \qquad \text{for } d = 4;$$

resulting after 5 steps in the partition $\{1, 3, 4\} \cup \{2, 5, 6\}$;

$$\begin{array}{lllll} 1 & & & & \\ 8 & 3 & 6 & & \\ 8 & & & 5 & 2 \\ 8 & 6 & 3 & & \\ 8 & & & & \end{array} \qquad \text{for } d = 4;$$

resulting after 11 steps in the partition $\{1, 3, 4, 5\} \cup \{2, 6, 7, 8\}$;

$$\begin{array}{lllllll} 1 & & & & & & \\ 10 & 4 & 8 & & & & \\ 10 & & & 7 & 3 & & \\ 10 & 8 & 4 & & & & \\ 10 & & & & & 6 & 2 \\ 10 & 4 & 8 & & & & \\ 10 & & & 3 & 7 & & \\ 10 & 8 & 4 & & & & \\ 10 & & & & & & \end{array} \qquad \text{for } d = 5;$$

resulting after 23 steps in the partition $\{1, 3, 4, 5, 6\} \cup \{2, 7, 8, 9, 10\}$;

1									
12	5	10							
12			9	4					
12	10	5							
12					8	3			
12	5	10							
12			4	9					
12	10	5							
12							7	2	for $d = 6$;
12	5	10							
12			9	4					
12	10	5							
12					3	8			
12	5	10							
12			4	9					
12	10	5							
12									

resulting after 47 steps in the partition $\{1,3,4,5,6,7\} \cup \{2,8,9,10,11,12\}$;

1								
$2d$	$d-1$	$2d-2$						
$2d$			$2d-3$	$d-2$				
$2d$	$2d-2$	$d-1$						
$2d$					$2d-4$	$d-3$		
$2d$	$d-1$	$2d-2$						
$2d$			$d-2$	$2d-3$				
$2d$	$2d-2$	$d-1$						
$\dots$								
$2d$							$d+1$	2
$\dots$								
$2d$	$d-1$	$2d-2$						
$2d$			$2d-3$	$d-2$				
$2d$	$2d-2$	$d-1$						
$2d$					$2d-4$	$d-3$		
$2d$	$d-1$	$2d-2$						
$2d$			$d-2$	$2d-3$				
$2d$	$2d-2$	$d-1$						
$2d$								

for an arbitrary $d \geq 2$; resulting after $3 \times 2^d - 1$ steps in the partition

$\{1, 3, 4, \dots d+1\} \cup \{2, d+2, \dots 2d\}$

which can be obtained from the original partition by the exchange of 2 and $d + 1$.

It is not difficult to verify that for each d the table S_{d+1} is built of two copies of slightly modified S_d.

Remark 12 *Similar exponential exchange paths were constructed by Morris [28] and von Stengel [39]; see also [29, 30]. However, at least two distinct oiks are involved in both cases: Sperner and Gale oiks in [28], Sperner and two polytopal oiks (which might be isomorphic) in [39]. In our construction each room-partition consists of two rooms of a single Scarf oik.*

Remark 13 *Let us also remark that no exponential in d example can exist for a pair of d-dimensional Sperner oiks with $2d$ vertices in each. Morris [28] proved that in this case any exchange path between two room-partitions is of length at most $2d$. Moreover, it is not difficult to verify that there are exactly $(d-1)!$ paths of length $2d$. They connect the following room-partitions: Given $2d$ vertices, let us color them $\{1, 2, \dots, d; 1, 2, \dots, d\}$ in the first Sperner oik and $\{1, 2, \dots, d; \sigma(1), \sigma(2), \dots, \sigma(d)\}$ in the second one, where σ is a d-permutation. Let us consider two rooms induced by the first and second d vertices in the first and second oiks, respectively. It is easily seen that an exchange path beginning in this room-partition is of length $2d$ whenever permutation σ is prime (i.e., formed by a single cycle) and of length $< 2d$ otherwise.*

6 Odd walls and open problems

The concepts of a omplex, oik, room, and wall were defined in Section 1.1. A wall of a complex is called *odd* if it is contained in an odd number of rooms. By definition, a complex is an oik if and only if it has no odd walls.

Given a complex $\mathcal{C}$, let us consider the following three decision problems:

Problem 1. Whether $\mathcal{C}$ is an oik or it contains an odd wall?

Problem 2. Whether $\mathcal{C}$ contains a room-partition?

Problem 3. Given a room-partition in $\mathcal{C}$, whether $\mathcal{C}$ contains another one ?

and the corresponding search problems: find out $(Q1)$ an odd wall; $(Q2)$ a room-partition; $(Q3)$ another room-partition.

Complexity of these problems depend on **how** the complex is specified.

First, let us assume that $\mathcal{C}$ is given explicitly. Then, obviously, Problem 1 is linear. In contrast, Problems 2 and 3 are NP-complete; see problems "perfect 3-matching" and "partition by 3-sets" in [16].

Secondly, let us assume that $\mathcal{C}$ is given by an abstract membership oracle, which, for any subset $V' \subseteq V$, answers whether V' is a room of $\mathcal{C}$. Let us assume for simplicity that dimension d of $\mathcal{C}$ is known, that is, we can restrict ourselves by testing only sets of cardinality d. Obviously, by testing all $\binom{n}{d}$ such sets we specify $\mathcal{C}$ explicitly. Thus, if d is bounded by a constant then all three problems can be solved in polynomial, n^{d+1}, time. However, this time becomes exponential when dimension d is a part of the input. Moreover, in this case, each of the above three problems might require exponential in $n = |V|$ number of tests, in the worst case.

Proposition 2 *All $\binom{n}{d}$ tests might be needed in case of Problem 1.*

Proof . Let $n - d$ be odd. Then complex $\mathcal{C}$ is (not) an oik of dimension d whenever every d-subset of V (but one) is a room. Hence, to make a decision in Problem 1, we have to verify all $\binom{n}{d}$ d-subsets of V, when only positive answers are given by the oracle. □

Proposition 3 *For Problems 2 and 3 respectively $\binom{n-1}{d-1}$ and $\binom{n-1}{d-1} - 1$ tests might be needed.*

Proof . Let $\mathcal{P} = \mathcal{P}(n, d)$ be the collection of all partitions of $|V|$ by d-sets. A family $\mathcal{T}$ of d-subsets of V is called *transversal* to $\mathcal{P}$ if every partition of $\mathcal{P}$ contains a set from $\mathcal{T}$. It is shown in [19] that, by the Erdös Ko-Rado Theorem [14], each minimum transversal consists of $\binom{n-1}{d-1}$ d-sets. For example, the family of all d-sets that contain a fixed vertex $v_0 \in V$ form such a minimal transversal. Hence, when only the negative answers are given by the oracle, at least $\binom{n-1}{d-1}$ (respectively, $\binom{n-1}{d-1} - 1$ tests are needed to exclude the existence of any (respectively, of another) room-partition. □

Now, let us combine Problems 1 and 3 as follows.

Problem 4. Given a complex $\mathcal{C}$ with a room-partition, find in $\mathcal{C}$:
(a) another room-partition or (b) an odd wall.

by Theorem 1, (a) or (b) (or both) always exist. Moreover, the exchange algorithm obviously generalizes from oiks to complexes and in the latter case finds either (a) or (b) by traversing the exchange graph. A general step is as follows: given a wall W, test by the membership oracle $n - d + 1$ d-sets that contain W and, choose between them a room R such that the pair (W, R) did not appear before. Clearly, an exchange path P constructed in this way can terminate in a skew-partition only if the last wall in P is odd, otherwise P terminates in a partition distinct from the original one.

As we know (see Section 5), P can be exponential in d already for $n = 2d$. Yet, complexity of Problem 4 remains open.

Finally, let us remark that the reduction of Section 2 naturally extends from the oiks to complexes. Namely, an arbitrary complex-family $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ is reduced to the complex-pair $\mathcal{O}' = \{\mathcal{O}_1 + \ldots + \mathcal{O}_k, \mathcal{O}_0\}$ in which sum of complexes and the Sperner oik $\mathcal{O}_0$ are defined as in Section 2. This reduction is exponential in k but it is polynomial in the size of $\mathcal{O}$.

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