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NASH-SOLVABLE BIDIRECTED CYCLIC TWO-PERSON GAME FORMS

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Nash-solvable bidirected cyclic two-person game forms

Endre Boros, Vladimir Gurvich, Kazuhisa Makino and Wei Shao

Abstract. We consider 2-person positional game forms (with perfect information and without moves of chance) modeled by finite directed graphs that might have directed cycles. Each such cycle, as well as each terminal position, is a separate outcome, and every player $i \in I = \{1, 2\}$ has an arbitrary utility function (or preference) u_i defined on the set of all outcomes. Furthermore both players are restricted to their pure and positional strategies, that is, a move (j, j') in a position j can depend only on this position but not on the preceding positions or moves.
A graph is called bidirected if (j, j') is its arc whenever (j', j) is.
A game form g is called *solvable* if for each payoff $u = (u_1, u_2)$ the obtained game (g, u) has a Nash equilibrium. We suggest necessary and sufficient conditions for solvability of the bidirected positional game forms in pure positional strategies.
Key words: game form, game in normal form, positional game, mean-payoff game, cyclic game, Nash equilibrium, Nash-solvable, tight

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1 Introduction

1.1 On solvability of n-person game forms

Let $A = \{a_1, \dots, a_p\}$ be a set of outcomes, $I = \{1, \dots, n\}$ be a set of players, and X_i be a set of strategies of a player $i \in I$. (In this paper, we always assume that sets X_i are finite. In particular, we do not consider mixed strategies.)

Furthermore, let $X = \prod_{i \in I} X_i$ be the direct product of these n sets. An element of it, that is, an n-tuple $x = (x_i \mid i \in I) \in X$, is called a *strategy profile or situation.* A *game form* $g : X \to A$ is a mapping that assigns an outcome $a \in A$ to each situation $x \in X$.

Typically, mapping g is not injective, that is, the same outcome is assigned to several distinct situations. A two-person ($I = \{1, 2\},\ n = 2$) game form g is conveniently represented by a matrix whose entries are the outcomes of A; see examples in Figures 1 and 2.

In general, an n-person game form g is given by an n-dimensional table.

A *utility function or payoff* is a mapping $u : I \times A \to \mathbb{R}$. Its value $u(i, a)$ is interpreted as the profit of player $i \in I$ in case of outcome $a \in A$.

A utility function u is called *zero-sum* if $\sum_{i \in I} u(i, a) = 0$ for all $a \in A$.

A pair (g, u) is called a *game in normal form.*

Remark 1 *Representation of a normal form game by a pair (g, u) is a convenient approach. Then, game form g is "responsible" for structural properties, which hold for every payoff u.*

In a game (g, u), a situation $x \in X$ is called a *Nash equilibrium* if $u(i, g(x)) \geq u(i, g(x'))$ for all players $i \in I$ and for each situation $x' = (x'_k \mid k \in I)$ that differs from $x = (x_k \mid k \in I)$ only in the ith, coordinate, that is, $x'_k = x_k$ whenever $k \neq i$.

In other words, in situation x no player $i \in I$ can make a profit by replacing the old strategy x_i by a new one x'_i provided all other players keep their old strategies.

In case of zero-sum two-person games Nash equilibria are called *saddle points.*

A game (g, u) is called *Nash-solvable* if it has a Nash equilibrium.

A game form g is called *Nash-solvable* if for each u the obtained game (g, u) is solvable.

A two-person game form g is called *zero-sum-solvable* (± 1 solvable) if for each zero-sum payoff u (that takes only values $+1$ and -1) the obtained game (g, u) is solvable.

It appears that all three properties of a two-person game form g, Nash-, zero-sum-, and ± 1-solvability, are equivalent. Moreover, they are equivalent to the following property.

To each outcome $a \in A$ let us assign a Boolean variable and denote it, for simplicity, by the same symbol a. Let $g : X \to A$ be a two-person game form, where $I = \{1, 2\}$ and $X = X_1 \times X_2$. We introduce two monotone disjunctive normal forms (DNFs)

$$F_1 = F_1(g) = \bigvee_{x_1 \in X_1} \bigwedge_{x_2 \in X_2} g(x_1, x_2); \quad F_2 = F_2(g) = \bigvee_{x_2 \in X_2} \bigwedge_{x_1 \in X_1} g(x_1, x_2). \tag{1}$$

A game form g is called *tight* if these two DNFs define dual monotone Boolean functions, that is, $F_1^d = F_2$. For example, in Figure 1 only the third game form is tight and in Figure 2 the last two game forms are tight, while the first two are not.

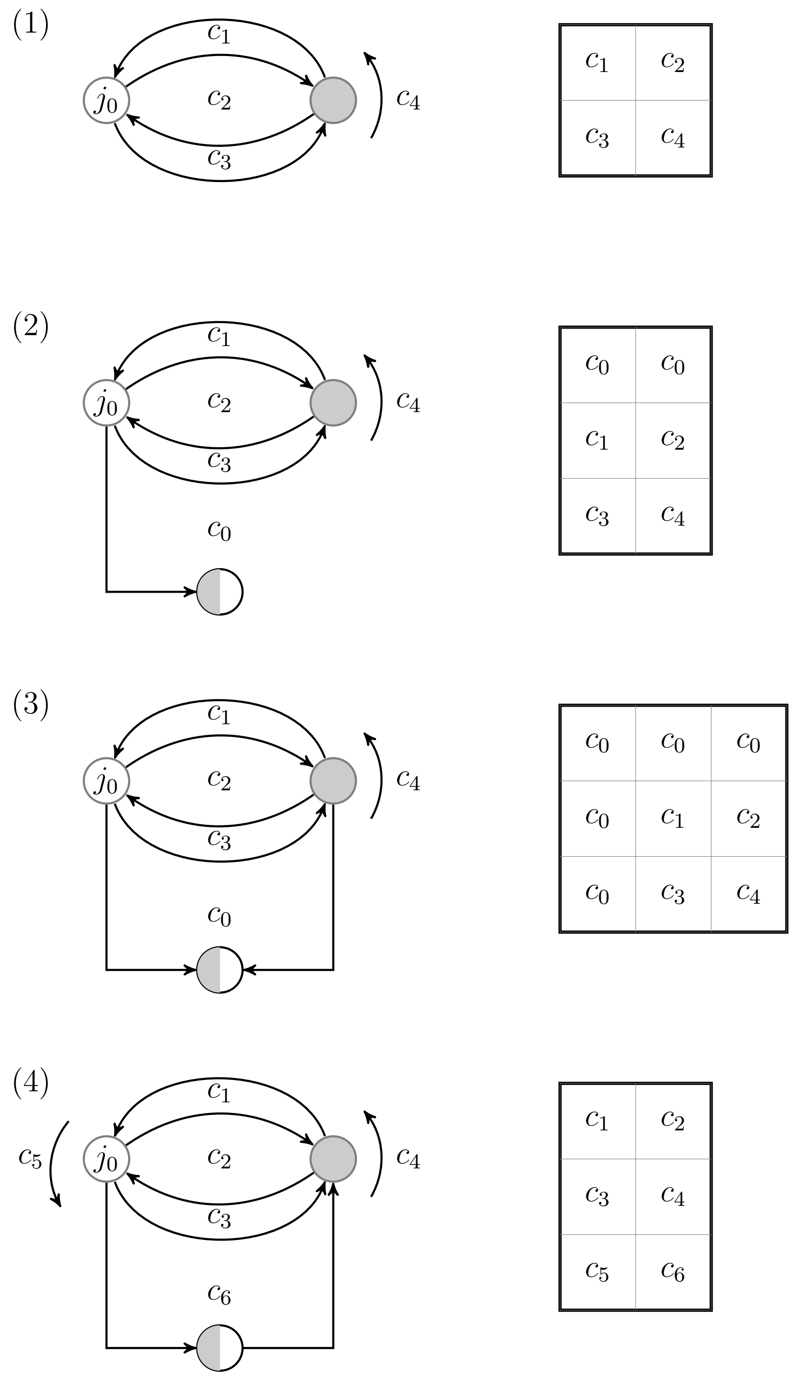


Figure 1: Nash-Solvability is monotone and ergodic
(1) $F_1 = c_1c_2 \vee c_3c_4, \quad F_2 = c_1c_3 \vee c_2c_4, \quad F_1^d \neq F_2.$
(2) $F_1 = c_0 \vee c_1c_2 \vee c_3c_4, \quad F_2 = c_0c_1c_3 \vee c_0c_2c_4, \quad F_1^d \neq F_2.$
(3) $F_1 = c_0 \vee c_0c_1c_2 \vee c_0c_3c_4 \approx c_0, \quad F_2 = c_0 \vee c_0c_1c_3 \vee c_0c_2c_4 \approx c_0, \quad F_1^d = F_2.$
(4) $F_1 = c_1c_2 \vee c_3c_4 \vee c_5c_6, \quad F_2 = c_1c_3c_5 \vee c_2c_4c_6, \quad F_1^d \neq F_2.$

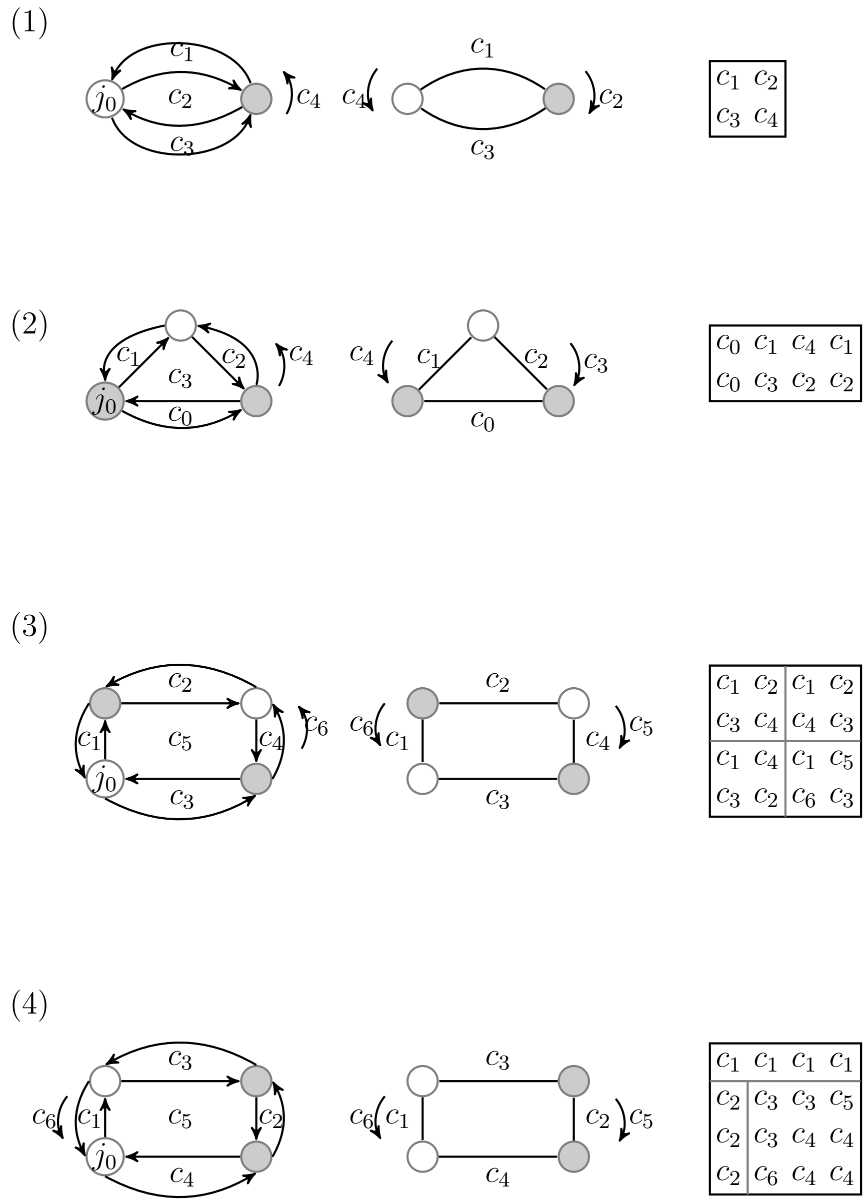


Figure 2: Solvability of cycles
(1) $F_1 = c_1c_2 \vee c_3c_4, \quad F_2 = c_1c_3 \vee c_2c_4, \quad F_1^d \neq F_2.$
(2) $F_1 = c_0c_1c_4 \vee c_0c_2c_3, \quad F_2 = c_0 \vee c_1c_2 \vee c_1c_3 \vee c_2c_4, \quad F_1^d \neq F_2.$
(3) $F_1 = c_1c_2 \vee c_3c_4 \vee c_1c_4c_5 \vee c_2c_3c_6, \quad F_2 = c_1c_3 \vee c_2c_4 \vee c_1c_4c_6 \vee c_2c_3c_5, \quad F_1^d = F_2.$
(4) $F_1 = c_1 \vee c_2c_3c_4 \vee c_2c_3c_5 \vee c_2c_4c_6 = c_1 \vee c_2(c_3c_4 \vee c_3c_5 \vee c_4c_6),$
. $F_2 = c_1c_2 \vee c_1c_3c_4 \vee c_1c_3c_6 \vee c_1c_4c_5 = c_1(c_2 \vee c_3c_4 \vee c_3c_6 \vee c_4c_5), \quad F_1^d = F_2.$

In other words, g is tight if and only if the sets of outcomes in the rows and columns of the corresponding matrix form two dual (transversal) hypergraphs on the ground set A.

Remark 2 *Let us notice the similarity of the DNFs defined by (1) to* max min *and* min max*, respectively. Then, duality* $F_1^d = F_2$ *"indicates" that* max min = min max*, for every payoff.*

The definition of tightness can be reformulated in several equivalent ways. For example, g is tight if and only if $B_1 \cap B_2 \neq \emptyset$ for any two sets of outcomes $B_1, B_2 \subseteq A$ such that B_1 (respectively, B_2) has an outcome in each column (row) of the matrix of g.

More reformulations of tightness in terms of this matrix can be found in [17, 18, 5, 3].

Theorem 1 *The following properties of two-person game forms are equivalent:*
(i) tightness; (ii) Nash-solvability; (iii) zero-sum solvability; (iv) ± 1*-solvability.*

Game forms satisfying these properties will be called *solvable*.

Implications *(ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(iv)* follow immediately from the definitions.

Equivalence of (i), (iii), and (iv) was shown in 1970 by Edmonds and Fulkerson [8] and independently in [16]. The list was extended by statement (ii) in [17]; see also [18, 3].

However, tightness is not necessary and not sufficient for Nash-solvability of n-person game forms when $n > 2$. Examples of Nash-solvable but not tight and tight but not Nash-solvable 3-person game forms were given in [18] and [17, 18], respectively. A short survey can be found Appendix I of [5] or in [3]. The proof of Theorem 1 given there is based on some ideas from [2, 21, 7] and it differs from the original proof in [17, 18].

Remark 3 *A game form* $g : X \rightarrow A$ *is called* rectangular *if*

$g(x') = g(x'') = a \;\Rightarrow\; g(x) = a$, *where*
$x = (x_i \mid i \in I),\ x' = (x'_i \mid i \in I),\ x'' = (x''_i \mid i \in I) \in X = \prod_{i \in I} X_i$ *and*
$(x_i = x'_i$ *or* $x_i = x''_i$ *) for all* $i \in I = \{1, \ldots, n\}$.

An n-person game form is Nash-solvable whenever it is tight and rectangular. Moreover, g is tight and rectangular if and only if g is the normal form of an acyclic positional game form realized by a tree. This result was announced in [17]; see [22] for more details.

1.2 Positional and normal cyclic game forms

The above general criterion of solvability can be applied in several cases; see Section 8.

In this paper we consider solvability of **cyclic** game forms defined as follows.

Given a finite directed graph $\overrightarrow{G}$ in which loops and multiple arcs are allowed; a vertex $j \in V = V(\overrightarrow{G})$ is a *position* and a directed edge $\overrightarrow{e} = (j, j') \in E(\overrightarrow{G})$ is a *move* from j to j'. A position of out-degree 0, or in other words, a position with no moves, is called *terminal*. Let V_T denote the set of all terminal positions. Let us also fix an *initial* position $j_0 \in V$.

Furthermore, let us introduce a set of players $I = \{1, \ldots, n\}$ and a partition
$P : V = V_1 \cup \ldots \cup V_n \cup V_T$, assuming that each player $i \in I$ is in control of all positions of V_i.

Let $C = C(\overrightarrow{G})$ denote the set of all simple directed cycles (dicycles) of digraph $\overrightarrow{G}$.

In particular, a loop $c_j = (j, j)$ is a dicycle of length 1 and a pair of oppositely directed edges $\overrightarrow{e} = (j, j')$ and $\overrightarrow{e}' = (j', j)$ is a dicycle of length 2.

The dicycles and terminal positions form the set of outcomes, $A = C \cup V_T$.

Remark 4 *It will be convenient to get rid of the terminal positions V_T. To do so, let us introduce a new loop $c_j = (j, j)$ for each $j \in V_T$. This loop c_j will replace the outcome j. Clearly, after this transformation, the sets of outcomes A and dicycles $C = C(\overrightarrow{G})$ coincide.*

Furthermore, since the loop (j, j) is a unique move in each position $j \in V_T$, we can assign any player $i \in I$, say, $i = 1$, to make such a move. In the obtained game, which is obviously equivalent to the original one, $V_T = \emptyset$ and $P : V = V_1 \cup \ldots \cup V_n$.

From now on, we shall assume that $V_T = \emptyset$, unless the opposite is mentioned explicitly. In fact, Sections 8.1 and 8.2 will be the only exceptions.

The triplet $(\overrightarrow{G}, P, j_0)$ will be called a *positional cyclic game form.*

To introduce the corresponding normal game form we need the concept of strategies.

A *strategy* x_i of a player $i \in I$ is a mapping that assigns a move $\overrightarrow{e} = (j, j') \in \overrightarrow{E}$ to each position $j \in V_i$; in other words, it is a "general **plan**" of player i for the whole game.

Remark 5 *We assume that the chosen move depends only on the present position (not on the previous positions and/or moves) and also that the choice is deterministic (not random). In other words, we restrict the players to their* pure positional *strategies.*

Let X_i denote the set of all such strategies of a player $i \in I$ and let $X = \prod_{i \in I} X_i$ be the set of all *strategy profiles or situations.* Given $x \in X$, a unique move is defined in each position $j \in V$. Furthermore, these moves determine a unique directed path (dipath) $p = p(x)$ that begins in the initial position j_0 and results in a dicycle $c \in C$. Such a path is called a *play.*

The obtain a mapping $g(\overrightarrow{G}, P, j_0) : X \to C(\overrightarrow{G})$ will be called a *normal cyclic game form.*

Since all players are restricted to their pure positional strategies, for each vertex $j \in p(x)$, there is a unique edge $\overrightarrow{e}(j, x)$ going from j. In particular, $p(x)$ has at most one vertex of in-degree 0 (the initial position j_0) and at most one of in-degree 2, after this position the play will repeat itself. For all other positions of $p(x)$ both the in- and out-degree equal 1.

In other words, every play $p(x)$ consists of an initial dipath (if any) which is followed by an infinitely repeated dicycle $c = c(x)$.

1.3 On solvability of cyclic games

A normal cyclic game form g is extended to a game (g, u) (in normal form) by a utility function $u : I \times C \to \mathbb{R}$ whose value $u(i, c)$ is standardly interpreted as a profit of the player $i \in I$ when the play $p = p(x)$ results in the dicycle $c \in C$.

In this paper we assume that $u : I \times C \to \mathbb{R}$ is an arbitrary function. Several special cases were already considered in the literature; see Section 8 for details.

In accordance with general definitions of Section 1.1, a cyclic game form g is Nash-solvable if for every utility function u the obtained game (g, u) has a Nash equilibrium.

Since the n-person case looks untractable, we restrict ourselves to the case of two players, $I = \{1, 2\}$. Then, by Theorem 1, Nash-solvability is equivalent with ± 1-solvability. For this reason, we shall restrict ourselves to the zero-sum utility functions that take only values $+1$ and -1. These two restrictions are assumed through the whole paper, except for Section 8.

Yet, it still seems too difficult to characterize solvability efficiently, so we will introduce one extra assumption. In this paper we shall characterize solvable cyclic game forms $g(\overrightarrow{G}, P, j_0)$ whose digraphs are *bidirected*, that is, $(j, j') \in \overrightarrow{E}$ whenever $(j', j) \in \overrightarrow{E}$.

The obtained characterization strengthens and completes earlier results of [13, 14, 15], where a criterion of solvability was announced (and its proof sketched) for the **bipartite** bidirected digraphs. In the present paper we get rid of the first assumption (bipartiteness), slightly correct the original statement (see Remark 15) and give an accurate proof.

Yet, since it is quite long, we will have to refer the reader to the Research Report [5] for some technical details.

2 Main properties of the bidirected cyclic game forms

First, we have to define the bidirected digraphs more accurately. Given a digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ and two distinct vertices $j, j' \in V$, let $k(j, j')$ denote the number of edges in $\overrightarrow{E}$ directed from j to j'. A digraph $\overrightarrow{G}$ will be called *bidirected* if $k(j, j') = k(j', j)$ for all distinct $j, j' \in V$.

Remark 6 *Soon we shall see that both game forms $(\overrightarrow{G}, P, j)$ and $(\overrightarrow{G}, P, j')$ are* **not** *solvable whenever $k(j, j') \geq 2$, $k(j', j) \geq 2$ and positions j and j' are controlled by distinct players; see Figures 1 (1) and 2 (1). Furthermore, without loss of generality, we can assume that $k(j, j') \leq 1$ (and $k(j', j) \leq 1$) whenever j and j' are controlled by the same player.*

Both claims will immediately follow from the results of this section. Hence, without loss of generality, we can assume that, in a bidirected solvable game form, $k(j, j')$ takes only values 0 and 1, or in other words, that any two parallel edges are oppositely directed. In particular, digraph $\overrightarrow{G}$ is not bidirected whenever $(j, j') \in \overrightarrow{E}$, while $(j', j) \notin \overrightarrow{E}$ for distinct $j, j' \in V$.

Given a bidirected digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, let us define a (non-directed) graph $G = G(\overrightarrow{G}) = (V, E)$ as follows: E contains k (non-directed) edges between j and j' whenever $k(j, j') = k(j', j) = k$; furthermore, to every directed loop in $\overrightarrow{G}$ we assign a non-directed loop in G.

Obviously, the following three properties of a bidirected graph $\overrightarrow{G}$ are equivalent:

(i) $\overrightarrow{G}$ is strongly connected, (ii) $\overrightarrow{G}$ is connected, (iii) G is connected.

We shall show that solvability of a connected bidirected game form is an ergodic property, that is, it does not depend on the initial position.

Proposition 1 *Given a strongly connected digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ and a partition $P : V = V_1 \cup V_2$ (recall that $n = 2$ and $V_T = \emptyset$), solvability of cyclic game forms $(\overrightarrow{G}, P, j)$ is an ergodic property, i.e., it is solvable either for every $j \in V$ or for no $j \in V$.*

In the first case, pair $(\overrightarrow{G}, P)$ will be called *solvable*. Furthermore, (G, P) will be called *solvable* if and only if $(\overrightarrow{G}, P)$ is solvable, provided $\overrightarrow{G}$ is bidirected and $G = G((\overrightarrow{G})$.

Remark 7 *However, let us note that the values of the games $(\overrightarrow{G}, P, j, u)$ and $(\overrightarrow{G}, P, j', u)$ might differ even for a strongly connected bidirected digraph $\overrightarrow{G}$ and ± 1 zero-sum payoff u.*

A pair $(\overrightarrow{G}, P)$ is called ergodic if it is solvable and, for any fixed zero-sum payoff u, the value of the games $(\overrightarrow{G}, P, j, u)$ does not depend on j. Simple necessary and sufficient for ergodicity conditions were given in [24]. Yet, to verify these conditions is an NP-complete problem, as it was also shown in [24]. See Appendix 2 of [5] for a mini-survey on ergodicity.

Standardly, we say that $G' = (V, E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. In this case, we make use of the notation $G' \leq G$; furthermore, we write that $G' < G$ if at least one of the above two containments is strict.

Furthermore, we say that a pair (G', P') is majorized by (G, P) if $G' = (V', E')$ is a subgraph of $G = (V, E)$ and partition $P' = V' : V_1' \cup V_2'$ is a subpartition of $P : V = V_1 \cup V_2$ induced by the subset $V' \subseteq V$, that is, $P' : V' = (V_1 \cap V') \cup (V_2 \cap V')$. Respectively, we use notation $(G', P') \leq (G, P)$ when $G' \leq G$ and $(G', P') < (G, P)$ when $G' < G$.

In particular, we can talk about connected and 2-connected components of a pair (G, P).

A graph G and a corresponding pairs (G, P) are called 2-*connected* if G is connected and it remains connected after deleting a vertex.

In Section 2, we will prove that solvability is a hereditary property.

Proposition 2 *If (G, P) is solvable and $(G', P') \leq (G, P)$ then (G', P') is solvable too.*

It is clear that pair (G, P) is solvable if and only if every its connected component is solvable. Hence, without loss of generality, we can assume that graph G is connected. Moreover, we can also assume that G is 2-connected, due to the following statement.

Proposition 3 *A pair (G, P) is solvable if and only if every its 2-connected component is.*

Thus, it is sufficient to characterize all maximal 2-connected solvable pairs (G, P).

In particular, pair (G, P) is solvable whenever it has no 2-connected components.

Corollary 1 *A pair (G, P) is solvable whenever G is a forest.* □

We will prove the above three Propositions in Section 4

3 Main result: necessary and sufficient conditions of solvability for bidirected cyclic game forms

Here, we introduce a list $\mathcal{L}$ of solvable 2-connected pairs. Our main theorem will claim that an arbitrary pair (G, P) is solvable if and only if each its 2-connected component is in $\mathcal{L}$.

3.1 Solvability of simple cycles

Proposition 4 *Given a simple cycle G, a pair (G, P) is solvable if and only if*

$(V_1 = V,\ V_2 = \emptyset)$, *or* $(V_2 = V,\ V_1 = \emptyset)$, *or* $(|V_1| > 1$ *and* $|V_2| > 1)$.

In other words, we can say that (G, P) is **not** solvable if and only if

$(|V_1| = 1$ and $V_2 \neq \emptyset)$ or $(|V_2| = 1$ and $V_1 \neq \emptyset)$.

A pair (G, P) satisfying the above condition will be called a 1-cycle.

By Proposition 4, a cycle (G, P) is solvable unless it is a 1-cycle.

Four examples in Figure 2 illustrate this statement. Its proof will be given in Section 5.3.

Remark 8 *Cycles (G, P) are 2-connected and solvable (except for 1-cycles), yet, they are not maximal solvable pairs; see Remark 12 below.*

3.2 Reducing (G, P) to $(\mathcal{G}, \mathcal{P})$)

To characterize solvable 2-connected pairs (G, P) we will need one more transformation.

To each pair (G, P) with $G = (V, E)$ and $P : V = V_1 \cup V_2$ let us assign another pair $(\mathcal{G}', \mathcal{P})$ with $\mathcal{G}' = (\mathcal{V}, \mathcal{E})$ and $\mathcal{P} : \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 = (V_1 \cap \mathcal{V}) \cup (V_2 \cap \mathcal{V})$ defined as follows.

Let $\mathcal{V} \subseteq V$ be the set of all vertices of degree at least 3 in G, which will be called *nodes*.

Given two nodes $j, j' \in \mathcal{V}$ and a simple path $p = p(j, j')$ between them that contains no other node, let us assign to p an edge $e(p)$ between j and j'. Furthermore, let $\mathcal{E}$ denote the set of all such edges. We will call p a 0-path (respectively, 1-path) if all vertices of p (respectively, all but exactly one), including j and j', are controlled by one player; all other paths will be called *regular*. The corresponding edge $e(p) \in \mathcal{E}$ will be called a 0-edge, 1-edge, or a regular edge, respectively. Obviously, when $j = j'$, a path turns into a cycle and we obtain the corresponding concepts of 0-, 1-, and a regular cycles in G.

Let us also note that if G is a simple cycle then $\mathcal{G}'$ is empty, that is, $\mathcal{V} = \emptyset$; yet, if G is any other 2-connected graph then $\mathcal{G}'$ is not empty and 2-connected too.

Let $\mathcal{G} = (\mathcal{V}; \mathcal{E}, \mathcal{E}_0, \mathcal{E}_1)$ denote a graph $\mathcal{G}'$ together with the lists of its 0- and 1-edges.

A pair $(\mathcal{G}, \mathcal{P})$ is called *solvable* if every corresponding pair (G, P) is solvable.

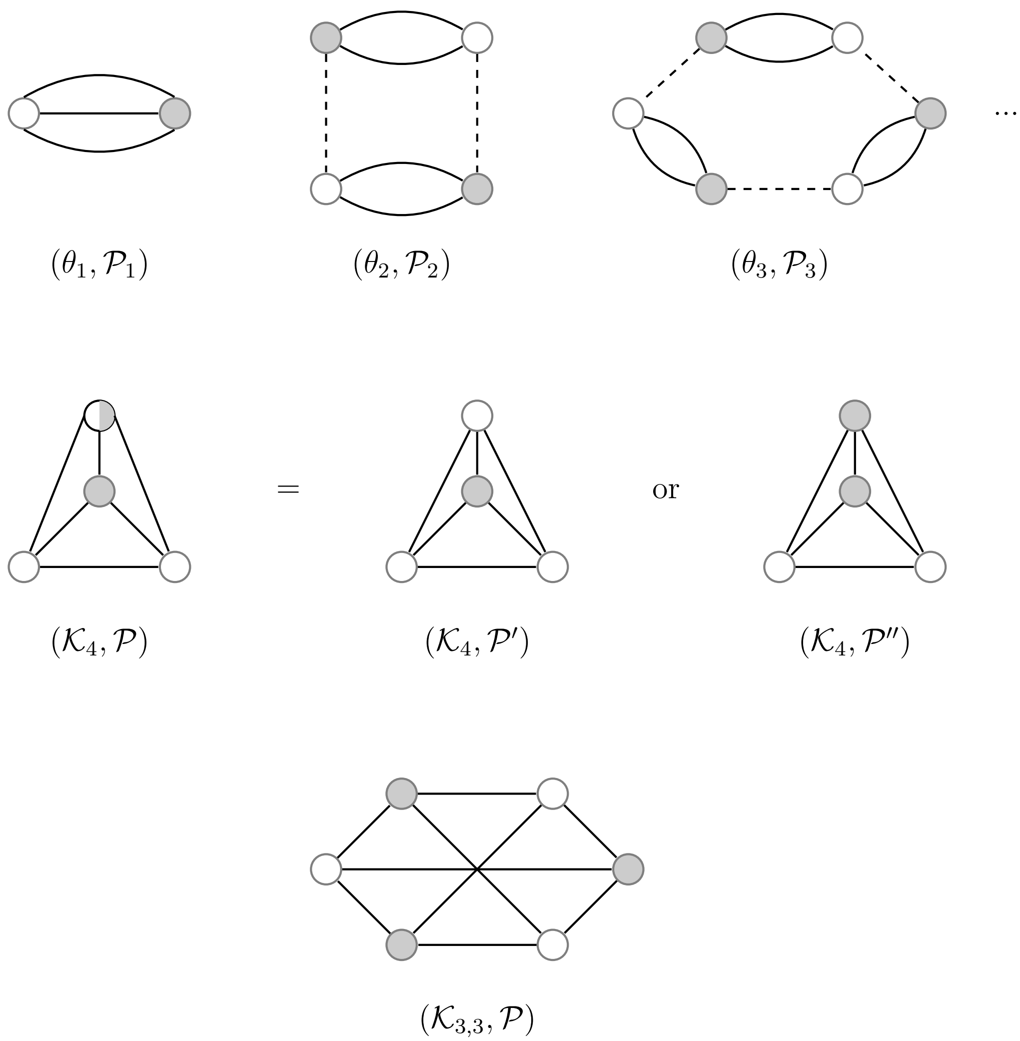


Figure 3: Solvable 2-connected graphs; dashed lines might correspond to 1-edges but solid lines cannot correspond to 0- or 1-edges.

Remark 9 *It is sufficient for solvability of* $(\mathcal{G}, \mathcal{P})$ *if at least* **one** *corresponding pair* (G, P) *is solvable. However, the lists of* 0- *and* 1-*edges can matter. Both claims will follow from the results of this section; see Propositions 5-8 and Theorem 2 below.*

A pair $(\mathcal{G}, \mathcal{P})$ is called *bipartite* if $\mathcal{G}$ is a bipartite graph, $\mathcal{G} = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{E})$, and $\mathcal{P} : \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is the corresponding partition.

In the Figures, we color positions of players 1 and 2 by white and black, respectively.

By double, black-and-white, coloring we will denote 'uncertain" positions, that is, the case when both options, $j \in \mathcal{V}_1$ or $j \in \mathcal{V}_2$, are possible. More precisely, a partition $\mathcal{P}$ with ℓ black-and-white positions represents 2^ℓ distinct partitions rather than one.

Remark 10 *For example, each terminal position* $j \in V_T$ *is uncertain, since the corresponding loop is a unique move in* j *and it does not matter which player makes this move.*

3.3 Solvable θ-pairs

Let us consider the family of bipartite pairs $(\theta_k, \mathcal{P}_k)$ given in Figure 3, where $k = 1, 2, \ldots$.

Graphs θ_k contain two types of edges: simple (type 1) and parallel (type 2); for example, θ_1 consists of three parallel edges; see Figure 3.

Proposition 5 *Pair* $(\theta_k, \mathcal{P}_k)$ *is solvable unless it contains a* 1-*edge of type* 2.

Remark 11 *Let us underline that* 1-*edges of type* 1 *do not contradict solvability. Furthermore, since graphs* $(\theta_k, \mathcal{P}_k)$ *are bipartite for all* k, *they cannot contain* 0-*edges.*

Remark 12 *Each graph* θ_k *contains a simple cycle as a proper (not induced) subgraph.*

Furthermore, $(\theta_{k+1}, \mathcal{P}_{k+1}) > (\theta_k, \mathcal{P}_k)$ *for all* k *and, hence,* $(\theta_k, \mathcal{P}_k)$ *is an infinite chain of nested solvable* 2-*connected pairs. This chain has no maximal element.*

3.4 Solvable pairs $(\mathcal{K}_4, \mathcal{P})$ and $(\mathcal{K}_{3,3}, \mathcal{P})$

Next, let $\mathcal{G} = \mathcal{K}_4$ and $\mathcal{P}$ consist of two white, one black, and one black-and-white positions. In fact, this case represents two subcases: $(\mathcal{K}_4, \mathcal{P}')$ and $(\mathcal{K}_4, \mathcal{P}'')$; see Figure 3.

Proposition 6 *Pair* $(\mathcal{K}_4, \mathcal{P})$ *is solvable unless it contains a* 0- *or* 1-*edge.*

Remark 13 *Let us notice that both pairs* $(\mathcal{K}_4, \mathcal{P}')$ *and* $(\mathcal{K}_4, \mathcal{P}'')$ *majorize* $(\theta_1, \mathcal{P}_1)$.

Now, let us consider the bipartite pair $(\mathcal{K}_{3,3}, \mathcal{P})$ in Figure 3.

Proposition 7 *Pair* $(\mathcal{K}_{3,3}, \mathcal{P})$ *is solvable unless it contains a* 1-*edge.*

Let us recall that bipartite pairs cannot contain 0-edges.

Remark 14 *It is easily seen that* $(\mathcal{K}_{3,3}, \mathcal{P}) > (\mathcal{K}_4, \mathcal{P}'')$. *Indeed,* $(\mathcal{K}_{3,3} - e, \mathcal{P})$ *contains two simple paths of length* 2*; let us substitute two edges for them and get* $(\mathcal{K}_4, \mathcal{P}'')$.

On the other hand, $(\mathcal{K}_{3,3}, \mathcal{P}) \not\geq (\mathcal{K}_4, \mathcal{P}')$.

3.5 Solvable monochromatic pairs

A pair $(\mathcal{G}, \mathcal{P})$ is called *monochromatic* if ($\mathcal{V} = \mathcal{V}_1$ and $\mathcal{V}_2 = \emptyset$) or ($\mathcal{V} = \mathcal{V}_2$ and $\mathcal{V}_1 = \emptyset$).

To characterize solvable monochromatic pairs, we will need one more transformation. Given a pair (G, P), let us consider the corresponding pair $(\mathcal{G}, \mathcal{P})$, duplicate every 0-edge in it, and denote the obtained pair by $(\mathcal{G}_0, \mathcal{P})$.

Proposition 8 *A monochromatic pair $(\mathcal{G}, \mathcal{P})$ is solvable unless $(\mathcal{G}_0, \mathcal{P})$ contains a 1-edge and two more edge-disjoint simple paths between its ends.*

Examples of solvable (Y) and not solvable (N) monochromatic pairs are given in Figure 4.

Remark 15 *In [13, 14, 15], the monochromatic case was* **not** *considered accurately. It was claimed that a monochromatic pair $(\mathcal{G}, \mathcal{P})$ is solvable unless it contains a 1-edge.*

3.6 Main results

In Sections 6 and 7, we will obtain the following criterion that summarizing the above five propositions and also showing that there are no other solvable cyclic bidirected game forms.

Let $\mathcal{L}$ denote the set of all 2-connected solvable bidirected pairs $(\mathcal{G}, \mathcal{P})$ from Propositions 4 - 8; they are shown in Figures 3 and 4.

Remark 16 *Together with a pair $(\mathcal{G}, \mathcal{P})$, by symmetry, $\mathcal{L}$ must contain the pair $(\mathcal{G}, \overline{\mathcal{P}})$, where partitions $\mathcal{P}$ and $\overline{\mathcal{P}}$ are complementary, i.e., the black and white vertices are switched in them. Yet, let us note that all pairs in $\mathcal{L}$ are self-complementary, except for $(\mathcal{K}_4, \mathcal{P}')$. Thus, we have to add only one pair: $(\mathcal{K}_4, \overline{\mathcal{P}'})$ with the cental white and three black vertices.*

Theorem 2 *A pair $(\mathcal{G}, \mathcal{P})$ is solvable if and only if all its 2-connected components are in $\mathcal{L}$.*

Remark 17 *In particular, pair (G, P) is solvable when graph G contains no 2-connected subgraphs, or when it is a simple cycle, yet, not 1-cycle.*

Thus, we obtain necessary and sufficient conditions for solvability of the bidirected (but not necessarily bipartite) cyclic game forms. Yet, in general, a characterization of solvability of (not necessarily bidirected) cyclic game forms still remains an open problem.

By Theorem 2 we can efficiently verify solvability of a bidirected cyclic game form $(\overrightarrow{G}, P, j_0))$ as follows. First, let us check whether digraph $\overrightarrow{G}$ is bidirected. If it is not then Theorem 2 is not applicable. If it is then let us construct the corresponding pair (G, P); then, $(\mathcal{G}, \mathcal{P})$; then, the 2-connected components of the latter, and verify whether they all belong to $\mathcal{L}$. If they do then the considered game form is solvable, otherwise it is not.

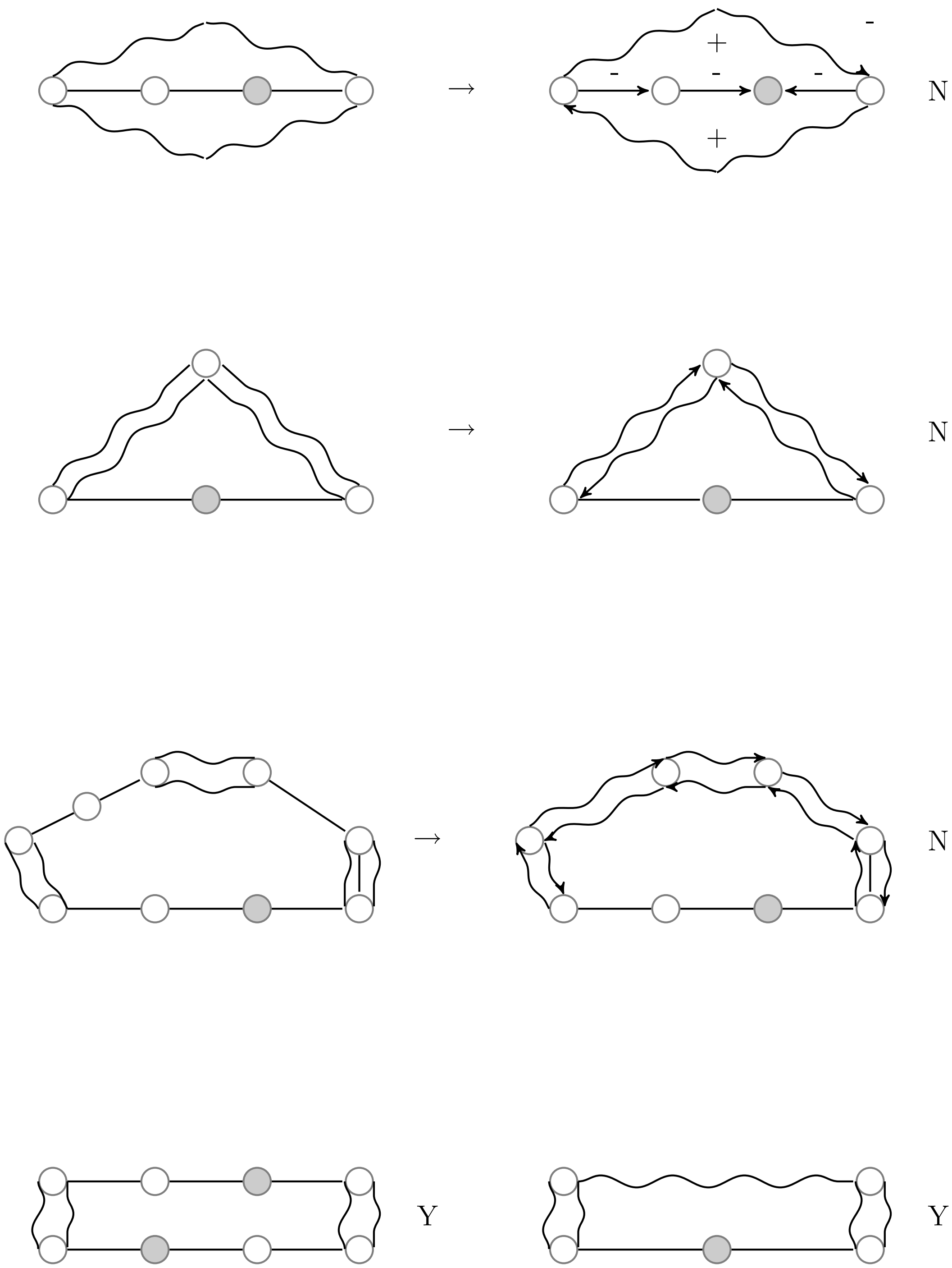


Figure 4: Solvability in monochromatic case.
Straight lines are edges; waved lines are simple paths.

4 Proofs of Propositions 1, 2, and 3

4.1 Maxmin and minmax might depend on the initial position

Given a cyclic game $(\overrightarrow{G}, P, j, u)$ with a zero-sum utility function u, although we cannot guarantee that the value $v = v(\overrightarrow{G}, P, j, u)$ exists, yet, in every game there exist maxmin $v_1 = v_1(\overrightarrow{G}, P, j, u)$ and minmax $v_2 = v_2(\overrightarrow{G}, P, j, u)$. They represent the values that players 1 and 2, respectively, can guarantee. Of course, inequality $v_1 \leq v_2$ always holds.

Furthermore, given $\overrightarrow{G}, P$, and u, maxmin and minmax depend on the initial position $j \in V$ (that is, $v_1 = v_1(j)$ and $v_2 = v_2(j)$) and satisfy the following inequalities.

Lemma 1 *For each move from j to j' we have:*

(i) $v_1(j') \leq v_1(j)$ *and* $v_2(j') \leq v_2(j)$ *whenever* $j \in V_1$ *and*

(ii) $v_1(j') \geq v_1(j)$ *and* $v_2(j') \geq v_2(j)$ *whenever* $j \in V_2$.

Moreover, for each position $j \in V = V_1 \cup V_2$ there are moves (j, j') and (j, j'') such that
(i') $v_1(j') = v_1(j)$ *and* *(ii')* $v_2(j'') = v_2(j)$.

Proof Indeed, by definition of maxmin and minmax, we have:

$$v_1(j) = \max\{v_1(j') \mid (j, j') \in \overrightarrow{E}\},\ v_2(j) = \max\{v_2(j'') \mid (j, j'') \in \overrightarrow{E}\} \text{ for } j \in V_1;$$

$$v_1(j) = \min\{v_1(j') \mid (j, j') \in \overrightarrow{E}\},\ v_2(j) = \min\{v_2(j'') \mid (j, j'') \in \overrightarrow{E}\} \text{ for } j \in V_2.$$

These equalities imply (i,i') and (ii,ii'), respectively. □

In other words, both maxmin and minmax do not increase (respectively, decrease) with any move of player 1 (respectively, 2). Moreover, in each position there are two moves that keep unchanged maxmin and minmax, respectively. However, these two moves can be distinct, since we cannot guarantee that (i') and (ii') hold simultaneously. For example, let $j \in V_1$ and there are exactly two moves (j, j') and (j, j'') from j which result in positions j' and j'' such that $v_1(j') = v_2(j') = 0$, $v_1(j'') = -1$, $v_2(j'') = 1$. Then $v_1(j) = 0$ and $v_2(j) = 1$.

Let us recall Theorem 1 and from now on restrict ourselves to the zero-sum ± 1 payoffs.

Then maxmin $v_1(j)$ and minmax $v_2(j)$ take only values ± 1, too. Since $v_1(j) \leq v_2(j)$ for all $j \in V$, pairs $(v_1(j), v_2(j))$ can take only three pairs of values: $(-1, -1)$, $(1, 1)$, and $(-1, 1)$ that define the partition $V = V_- \cup V_+ \cup V_\pm$. Let us consider three induced subgraphs $\overrightarrow{G}_- = \overrightarrow{G}[V_-]$, $\overrightarrow{G}_+ = \overrightarrow{G}[V_+]$, $\overrightarrow{G}_\pm = \overrightarrow{G}[V_\pm]$ and three partitions $P_-, P_+, P_\pm$ induced on sets $V_-, V_+, V_\pm$ by the original partition $P : V = V_1 \cup V_2$.

For zero-sum ± 1 games we can reformulate Lemma 1 as follows.

Lemma 2 *Digraphs* $\overrightarrow{G}_- = \overrightarrow{G}[V_-]$, $\overrightarrow{G}_+ = \overrightarrow{G}[V_+]$, *and* $\overrightarrow{G}_\pm = \overrightarrow{G}[V_\pm]$ *have no dead-ends.*

Furthermore, player 1 has no moves from V_- to $V_\pm$, from $V_\pm$ to V_+, and from V_- to V_+, while player 2 has no moves from V_+ to $V_\pm$, from $V_\pm$ to V_-, and from V_+ to V_-. □

In other words, in the sequence $V_-, V_\pm, V_+$ player 1 can move only from right to left, while player 2 only from left to right, and each player can always stay in the same set.

These observations easily imply the following two claims.

Corollary 2 *Triplets* $(\overrightarrow{G}_-, P_-, j)$, $(\overrightarrow{G}_+, P_+, j)$, *and* $(\overrightarrow{G}_\pm, P_\pm, j)$ *form cyclic game forms if the initial position* j *belongs to* V_-, V_+, *and* $V_\pm$, *respectively.* □

4.2 Uniformly non-solvable pairs

By definition and Theorem 2, cyclic game form $(\overrightarrow{G}, P, j)$ is not solvable if and only if there is a utility functions u such that $j \in V_\pm$ in the obtained game $(\overrightarrow{G}, P, j, u)$. We can strengthen this condition as follows. A pair $(\overrightarrow{G}, P)$ will be called *uniformly non-solvable* if there is a utility function u such that for every initial position $j \in V$ the obtained game $(\overrightarrow{G}, P, j, u)$ is not solvable, that is, $-1 = v_1(j) < v_2(j) = 1$ for each $j \in V$, or in other words, $V_\pm = V$.

Lemma 3 *Pair* $(\overrightarrow{G}_\pm, P_\pm)$ *is uniformly non-solvable.*

Proof Suppose that $V_\pm \neq \emptyset$ for some payoff u and consider the subgame $(\overrightarrow{G}_\pm, P_\pm, j, u_\pm)$, where $u_\pm : C(\overrightarrow{G}_\pm) \to \{-1, +1\}$ is the restriction of u to the dicycles of digraph $\overrightarrow{G}_\pm$.

By definition of $V_\pm$, we have $-1 = maxmin = v_1(j) < v_2(j) = minmax = 1$ for every $j \in V_\pm$. In other words, game form $(\overrightarrow{G}_\pm, P_\pm, j)$ is not solvable for each j, that is, pair $(\overrightarrow{G}_\pm, P_\pm)$ is uniformly non-solvable. □

Lemma 4 *Furthermore, let us choose two arbitrary vertices* $j', j'' \in V_\pm$ *and add the extra arc* $e = (j', j'')$ *to digraph* $\overrightarrow{G}_\pm$. *The obtained pair* $(\overrightarrow{G}'_\pm, P_\pm)$ *is uniformly non-solvable, too.*

Proof Without loss of generality we can assume that $j' \in V_1$. In this case let us extend $u_\pm$ to $u'_\pm : C(\overrightarrow{G}'_\pm) \to \{-1, +1\}$ by setting $u'_\pm(c) = -1$ for every dicycle c that contains the new arc e. Let us consider the obtained game $(\overrightarrow{G}'_\pm, P_\pm, j, u'_\pm)$ and show that $-1 = v'_1(j) < v'_2(j) = 1$ for every initial position j. The second equality is obvious, since the only new move $e = (j', j'')$ belongs to player 1, while the set of strategies of player 2 remains the same.

Yet, we have to show that $v'_1(j) = -1$, that is, player 1 cannot use the new move e in a strategy that will guarantee the result $+1$. This is not fully obvious. Although, by definition, $u'_\pm(c) = -1$ for every dicycle c that contains e, yet, perhaps, e could create a vital "communication" for player 1. However, this cannot happen either. Indeed, obviously, $v'_1(j'') = -1$, since if $j = j''$ is the initial position then player 1 will immediately lose after move $e = (j', j'')$. Furthermore, since $v'_1(j'') = -1$ and $e = (j', j'')$ results in j'', this move is useless as a communication. Thus, $v'_1(j) = -1$, while $v'_2(j) = 1$ for all $j \in V_\pm$. □

Similarly to the non-directed case, we will say that pair $(\overrightarrow{G}, P)$ is majorized by $(\overrightarrow{G}', P')$ if the digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ is a subgraph of $\overrightarrow{G}' = (V', \overrightarrow{E}')$ (that is, $V \subseteq V'$, $\overrightarrow{E} \subseteq \overrightarrow{E}'$)

and partition $P : V = V_1 \cup V_2$ is a subpartition of $P' = V' : V_1' \cup V_2'$ induced by the subset $V \subseteq V'$ (that is, $P : V = (V_1' \cap V) \cup (V_2' \cap V)$). Standardly, we use the notation $\overrightarrow{G} \leq \overrightarrow{G}'$ and $(\overrightarrow{G}, P) \leq (\overrightarrow{G}', P')$.

Theorem 3 *If a bidirected cyclic game form* $(\overrightarrow{G}, P, j_0)$ *is*
(i) not solvable, (ii) $(\overrightarrow{G}, P) \leq (\overrightarrow{G}', P')$*, and (iii) for every position* $j' \in V'$ *in* $\overrightarrow{G}'$ *there is a directed path from* j' *to* j_0*, then (iv)* $(\overrightarrow{G}', P')$ *is uniformly non-solvable.*

Proof Since $(\overrightarrow{G}, P, j_0)$ is not solvable, there is a utility function $u : C(\overrightarrow{G}) \to \{-1, +1\}$ such that the corresponding game $(\overrightarrow{G}, P, j_0, u)$ is not solvable either, that is, $-1 = maxmin = v_1(j_0) < v_2(j_0) = minmax = 1$. In other words, $V_\pm \neq \emptyset$; in particular, $j_0 \in V_\pm$. Then, by Lemma 3, the induced pair $(\overrightarrow{G}_\pm, P_\pm)$ is uniformly non-solvable, that is, game form $(\overrightarrow{G}_\pm, P_\pm, j)$ is not solvable for every j. In other words, $-1 = maxmin = v_1(j) < v_2(j) = minmax = 1$ for every initial position $j \in V_\pm$ of the game $(\overrightarrow{G}_\pm, P_\pm, j, u_\pm)$, where $u_\pm : C(\overrightarrow{G}_\pm) \to \{-1, +1\}$ is the restriction of u to $C(\overrightarrow{G}_\pm)$.

To prove that pair $(\overrightarrow{G}', P')$ is uniformly non-solvable we will define a similar utility function $u' : C(\overrightarrow{G}') \to \{-1, +1\}$ such that $-1 = maxmin = v_1'(j) < v_2'(j) = minmax = 1$ for every $j \in V'$. To get u' we will extend $u_\pm$ from $C(\overrightarrow{G}_\pm)$ to $C(\overrightarrow{G}')$. To do so, we set $u'(c) = u_\pm(c)$ for each dicycle c in $C(\overrightarrow{G}_\pm)$. Now we will extend, step by step, digraph $\overrightarrow{G}_\pm$ to $\overrightarrow{G}'$, so that in each step we obtain a uniformly non-solvable pair. Thus, to get u' from u, first, we reduce u from $C(\overrightarrow{G}_)$ to $C(\overrightarrow{G}_\pm)$ getting $u_\pm$ and then extend it to $C(\overrightarrow{G}')$.

Step A1. Let us add to $\overrightarrow{G}_\pm$ all arcs $(j', j'') \in \overrightarrow{E}'$ such that $j', j'' \in V_\pm$. The obtained pair $(\overrightarrow{G}_1, P_1)$ is uniformly non-solvable, by Lemma 4.

Step B1. By condition (iii) of the Theorem, since $j_0 \in V_\pm$, there is an arc (j', j'') in $\overrightarrow{G}'$ such that $j'' \in V_\pm$, while $j' \notin V_\pm$. Let us add such an arc (together with vertex j') to $\overrightarrow{G}_1$. Obviously, the obtained pair $(\overrightarrow{G}_2, P_2)$ is uniformly non-solvable, since the move in position j' is forced.

Step A2. Now let us add to digraph $\overrightarrow{G}_2$ the arcs $(j, j') \in \overrightarrow{E}'$ for all $j \in V_\pm$ to j'. By Lemma 4, the obtained pair $(\overrightarrow{G}_2', P_2)$ is uniformly non-solvable.

By this, we extend u' from $C(\overrightarrow{G}_1) = C(\overrightarrow{G}_2)$ to $C(\overrightarrow{G}_2')$ as follows. If C contains (j, j') then $u'(C) = -1$ for $j \in V_1$ and $u'(C) = +1$ for $j \in V_2$.

Now we can proceed with Step B2, etc., until we obtain the final digraph $C(\overrightarrow{G}')$ and show that pair $(\overrightarrow{G}', P')$ is uniformly non-solvable. □

Example 1 *As an illustration, let us consider four pairs* $(\overrightarrow{G}_k, P_k)$*,* $k \leq 4$*, in Figure1.*

The first and the last pairs are uniformly non-solvable; $(\overrightarrow{G}_3, P_3)$ *is uniformly solvable, moreover, it is ergodic; finally,* $(\overrightarrow{G}_2, P_2)$ *is* **not solvable, yet, not uniformly**. *Let us notice that* $(\overrightarrow{G}_1, P_1) < (\overrightarrow{G}_2, P_2) < (\overrightarrow{G}_3, P_3)$.

This example shows that condition (iii) of Theorem 3 is essential. Indeed, both extensions $(\overrightarrow{G}_1, P_1)$ *to* $(\overrightarrow{G}_2, P_2)$ *and* $(\overrightarrow{G}_2, P_2)$ *to* $(\overrightarrow{G}_3, P_3)$ *satisfy (i) and (ii) but (iii) and (iv) fail.*

Let us also note that the first pair is bidirected, while three other are not.

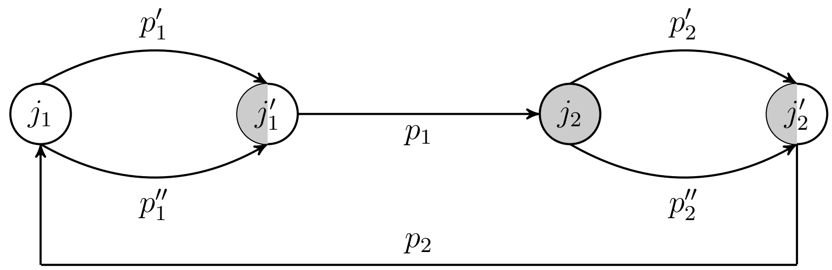


Figure 5: Uniformly non-solvable pair

Example 2 *The simplest (bidirected) uniformly non-solvable pair is given in Figure 1 (1).*

Another (not bidirected) example, with the same 2×2 *normal form, is given in Figure 5. Here* p'_i, p''_i, *and* p_i, *for* $i = 1, 2$, *are directed paths. It is essential that positions* j_1 *and* j_2 *are controlled by distinct players; all other positions can be assigned arbitrarily.*

Let us note that "topology" is not essential. What matters is that, first, one player has an alternative to choose, then, the other. In this sense, the above two examples are equivalent.

These two (and many other equivalent) constructions are instrumental in proofs of non-solvability; see Section 7 and also Section 7 of [5].

Now, let us recall four pairs on Figure 2. As we already know, the last two are solvable, while the first two are not. Since the considered four graphs are bidirected and connected, both solvability and non-solvability are uniform for these examples.

Yet, let us repeat again that the set of Nash equilibria or even the value, in the zero-sum case, might depend on the initial position; see [5] Appendix 2 for more details.

Given a bidirected graph $\overrightarrow{G}$, the pairs (G, P) and $(\mathcal{G}, \mathcal{P})$ are called *solvable or (uniformly) non-solvable* whenever the corresponding pair $(\overrightarrow{G}, P)$ has the corresponding property.

4.3 Propositions 1, 2, and 3 result from Theorem 3

Let us notice, however, that condition (iii) automatically holds if digraph $\overrightarrow{G}'$ is strongly connected, or in particular, if $\overrightarrow{G}'$ is bidirected and the corresponding graph G' is connected. Hence, Theorem 3 implies Proposition 2.

Furthermore, Proposition 1 follows from Theorem 3, too. Indeed, conditions (ii) and (iii) of the theorem hold whenever $\overrightarrow{G} = \overrightarrow{G}'$ is a strongly connected digraph. Hence, in this case solvability does not depend on the initial position.

Moreover, Proposition 1 can be strengthened as follows.

Proposition 9 *If digraph $\overrightarrow{G}$ is strongly connected then a pair $(\overrightarrow{G}, P)$ is either solvable or uniformly non-solvable.* □

Finally, let us derive Proposition 3 from Theorem 3. Again, we prove a stronger claim.

Given k digraphs $\overrightarrow{G_\ell} = (V_\ell, \overrightarrow{E}_\ell)$, $\ell \in [k] = \{1, \ldots, k\}$ that are pairwise disjoint, except for a unique common vertex j_0, that is, $V_\ell \cap V_{\ell'} = \{j_0\}$ for each two distinct $\ell, \ell' \in [k]$; then, obviously, their k arc-sets are pairwise disjoint. Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be the union of these k digraphs, $V = \cup_{\ell=1}^k V_\ell$ and $\overrightarrow{E} = \cup_{\ell=1}^k \overrightarrow{E}_\ell$.

Lemma 5 *Digraph $\overrightarrow{G}$ is strongly connected if and only if digraphs $\overrightarrow{G_\ell}$ are strongly connected for all $\ell \in [k]$.*

Proof Each of the above digraphs is strongly connected if and only if every its vertex can be reached by a directed path from j_0 and j_0 can be reached by a directed path from each vertex. Clearly, this property holds for $\overrightarrow{G}$ if and only if it holds for $\overrightarrow{G_\ell}$ for all $\ell \in [k]$. □

In the rest of this subsection, we will assume that digraph $\overrightarrow{G}$ is strongly connected.

Furthermore, let $P_\ell : V_\ell = V_1^\ell \cup V_2^\ell$ be k partitions such that the initial position j_0 belongs to the same player, $i = 1$ or $i = 2$, in all of them. Let $P : V = V_1 \cup V_2$ be the union of these partitions, that is, $V_1 = \cup_{\ell=1}^k V_1^\ell$ and $V_2 = \cup_{\ell=1}^k V_2^\ell$.

Lemma 6 *Cyclic game form $(\overrightarrow{G}, P, j_0)$ is solvable if and only if game forms $(\overrightarrow{G}_\ell, P_\ell, j_0)$ are solvable for all $\ell \in [k]$.*

Proof Given an arbitrary zero-sum payoff u, the following formulas, obviously, hold

$$v_i = \max(v_i^\ell \mid \ell \in [k]) \text{ if } j_0 \in V_1 \quad \text{and} \quad v_i = \min(v_i^\ell \mid \ell \in [k]) \text{ if } j_0 \in V_2 \quad \text{for } i \in I = \{1, 2\},$$

where $v_i = v_i(\overrightarrow{G}, P, j_0)$ and $v_i^\ell = v_i^\ell(\overrightarrow{G}_k, P_k, j_0)$ for $\ell \in [k]$ are maxmin for $i = 1$ and minmax for $i = 2$. It is easy to see that maxmin and minmax are equal for all u in game $(\overrightarrow{G}, P, j_0, u)$ if and only if they are equal for all u in all games $(\overrightarrow{G}_\ell, P_k, j_0, u)$ for $\ell \in [k]$. □

Let us recall that, by Proposition 9, solvability is ergodic, that is, it does not depend on the initial position. Hence, we can strengthen Lemma 6 as follows.

Lemma 7 *Pair $(\overrightarrow{G}, P)$ is solvable if the pairs $(\overrightarrow{G}_\ell, P_\ell)$ are solvable for all $\ell \in [k]$, otherwise pair $(\overrightarrow{G}, P)$ is uniformly non-solvable.* □

Thus, we have defined an operation of the union of pairs with a unique common position and proved that the resulting pair is solvable (and strongly connected) if and only if all involved pairs are solvable (and strongly connected). Obviously, this operation can be applied several times successively and the same conclusion can be proved by induction.

Let us apply this construction to the corresponding non-directed graphs and pairs. It is easy to see that each connected graph can be obtained in this way from its 2-connected components and some edges. Obviously, a pair corresponding to a single edge is solvable. This and Lemma 7 imply Proposition 3 and, in particular, Corollary 1. □

5 Passing through simple paths and cycles

5.1 Basic notation, assumptions, and method

Let us consider a pair (G, P), where G is a 2-connected graph. By Propositions 1 and 9, (G, P) is either solvable or uniformly non-solvable. Let us consider a simple path Q of length k between two nodes j_0, j_k in G and introduce the following standard notation: $Q = Q_k = (V, E)$, where $V = \{j_0, j_1, \ldots, j_k\}$ and $E = \{(j_{\ell-1}, j_\ell) \mid \ell \in [k] = \{1, \ldots, k\}\}$.

Let us recall that, except for j_0 and j_k, path Q contains no other nodes (vertices of degree at least 3) of G. The corresponding digraph $\overrightarrow{Q} = (V, \overrightarrow{E})$ consists of k short dicycles, $\mathcal{C}_k = \{c_\ell = ((j_{\ell-1}, j_\ell), (j_\ell, j_{\ell-1})) \mid \ell \in [k]\}$ that are in one-to-one correspondence with E.

Given a game (G, P, j, u), we would like to analyze when an optimal play might pass Q, from j_0 to j_k or from j_k to j_0. Let us consider the following few cases.

Case 1: A player can enter Q from j_0 or j_k and win. Then, clearly, pair (G, P) is solvable. To see this, it is enough to choose the initial position accordingly, in j_0 or in j_k.

Case 2: The opponent wins within Q whenever a player enters Q from j_0 or j_k.

Case 2.1: Positions j_0 and j_k are controlled by two distinct players. Then, let us reduce pair (G, P) by eliminating path Q, except for its ends j_0 and j_k. Obviously, the obtained pair (G', P') is solvable if and only if the original pair (G, P) is solvable.

For convenience, "draw path Q horizontally" with the left end in j_0 and right one in j_k.

Case 2.2: Positions j_0 and j_k are controlled by the same player, say 1, that is, vertices j_0 and j_k are both White. By the assumption of Case 2, whenever player 1 enters Q from j_0 (from j_k), player 2 has a strategy x_2^L (respectively, x_2^R) winning within Q. Yet, these two strategies might be distinct and player 2, being restricted to the positional strategies, "does not know" which one to apply. Thus, an extra analysis is required in this case.

5.2 Uniformly winning strategies for simple paths

Proposition 10 *In case 2.2, player 2 has a strategy x_2 that wins within Q uniformly, whether 1 enters Q from j_0 or from j_k, provided Q is a regular path (i.e., it contains at least two Black positions). If Q is a 1-path (i.e., it contains only one Black position) then there exist exactly two payoffs such that uniformly winning strategies fail to exist.*

Proof Clearly, the winning strategies x_2^L and x_2^R instruct player 2 as follows: go to the right (left) until a position j_L (respectively, j_R) appears; then turn left (right) to win in one move.

By assumption, positions j_0 and j_k are White, while j_L and j_R are Black. Yet, j_L and j_R might coincide. The following case analysis proves the Proposition.

Case $j_L \neq j_R$. Then, let us define x_2 as follows: being between j_R and j_L, play arbitrarily, otherwise apply x_2^L between j_0 and j_R and x_2^R between j_L and j_k.

Remark 18 *It might be convenient, to partition this case into two: j_L is strictly to the left or to the right from j_R. Yet, strategy x_2 can be defined similarly, as above, in both subcases.*

The remaining case, $j_L = j_R$, can be partitioned into two subcases E0 and E1 as follows.

E0: path Q is regular, that is, except for $j = j_L = j_R$, there is at least one more Black position j' in Q. By assumption, j' is distinct from j_0 and j_k. If j' is to the left (right) from j then, obviously, x_2^L (respectively, x_2^R) prescribes to go right (left) from j'.

In this case, let us define x_2 as follows: apply x_2^R (respectively, x_2^L) being strictly to the right (left) from j and x_2^L otherwise.

E1: path Q is a 1-path, that is, $j = j_L = j_R$ is a unique Black position in Q.

In this case, let us consider the following two payoff functions: u' (respectively, u''): player 2 wins in all short dicycles of Q (respectively, in all but two, incident to position j.

Furthermore, we assume that none of two players wins when the play passes through Q, from j_0 to j_k or conversely. It is convenient to define that the game is a draw, in both cases.

Remark 19 *If path Q is passed through, we will replaced it in (G, P) by a directed edge (or sometimes by the pair of oppositely directed edges) between the nodes j_0 and j_k and proceed with our analysis of solvability of the pair (G, P); see Sections 5.6.*

Let us add to Q loop $c_L = (j_0, j_0)$ at j_0 or $c_R = (j_k, j_k)$ at j_k and denote the obtained extended graphs by Q_R and Q_L, respectively. In both cases let us set $u(c_L) = u(c_R) = 0$ and consider two pairs of games (Q_R, P, j_0, u) and (Q_L, P, j_k, u), where $u = u'$ or $u = u''$.

If $u = u'$ (respectively, $u = u''$) then player 2, Black, can win (make a draw) in both games of the pair, yet, there is no uniformly winning strategy. If the guess of the entrance position, j_0 or j_k, is wrong then player 2 makes a draw (lose).

Obviously, except u' and u'' there are no other "ambiguous" payoff functions. □

5.3 On solvability of simple cycles; proof of Proposition 4

The above analysis of case 2.2 results in Proposition 4. Let us identify two (White) vertices j_0 and j_k of Q_k and obtain a simple cycle C_k. The corresponding digraph $\overrightarrow{C_k}$ contains k short dicycles $\mathcal{C}_k = \{c_\ell \mid \ell \in [k]\}$ and two long ones, clockwise c_R and counterclockwise c_L. They are similar to the loops c_L and c_R of the previous section. Yet, now we cannot set $u(c_L) = u(c_R) = 0$; instead, we have to extend the zero-sum ± 1 payoff to these two dicycles.

A game form (C_k, P, j_0) is solvable whenever it is regular, that is, players 1 and 2 control at least two positions each. Indeed, either player 1 can start, clockwise or counterclockwise, and win, or player 2 can win in both cases. Yet, then (s)he wins uniformly, unless $V_1 = 1$ or $V_2 = 1$, or more precisely, unless C_k is a 1-cycle.

Of course, 0-cycles are solvable, yet, 1-cycles are not.

The simplest non-solvable pair is (C_2, P) in Figure 1. Graph C_2 consists of two vertices and two edges; players, 1 and 2, control one position each. The corresponding pair $(\overrightarrow{C_2}, P)$ is given in Figure 1 (1). Digraph $\overrightarrow{C_2}$ contains four dicycles. Each player has two strategies. Thus, all four outcomes of the corresponding 2×2 normal game form are distinct. Hence, it is not solvable.

More generally, let (C_k, P) be a 1-cycle, say, $V_1 = \{j_0\}$, $V_2 = V \setminus \{j_0\}$, and $|V| = k \geq 2$. We will show that (C_2, P) is not solvable. The corresponding digraph $\overrightarrow{C_2}$ contains $2k + 2$ dicycles: k *short* dicyclers, of length 2 each, and two *long* dicycles, of length k each, the clockwise c_L and counter-clockwise c_R dicycles. Let us introduce a payoff u as follows: $u(c) = -1$ if dicycle c is long and $u(c) = +1$ if c is short. Player 1 controls only the initial position j_0 and has two strategies: to begin clockwise or counter-clockwise. Player 2 has 2^{k-1} strategies, yet, all of them, but two, are definitely losing, since they always result in a short cycle. (Recall that player 2 is the minimizer.) Only two strategies, the clockwise and counter-clockwise ones, can be winning. Yet, there is no guarantee. Each of these strategies of player 2 wins only if player 1 begins correspondingly. In these two cases two long cycles appear; otherwise (if player 1 begins clockwise and 2 proceeds counter-clockwise, or vice versa) a short cycle appears and player 1 wins. Thus, removing all dominated strategies of player 2, we reduce the original normal form to the 2×2 matrix with $a_{1,1} = a_{2,2} = +1$ and $a_{1,2} = a_{2,1} = -1$. This matrix has no saddle point in pure strategies.

Remark 20 *It is important to notice that both players are restricted to their positional strategies, that is, the move in a position can depend only on this position but not on the preceding positions or moves. By this assumption, player* 2 *is not aware of the move of player* 1 *in* j_0*. In other words, both players choose their (positional) strategies simultaneously.*

5.4 Two special zero-sum ± 1 payoffs assigned to a simple path

Given a simple path Q_k of length k defined in Sections 5.1,

Lemma 8 *(i, ii). There is a unique payoff* u_L *(respectively,* u_R*) such that the obtained game* (Q_L, P, j_0, u_L) *(respectively,* (Q_R, P, j_k, u_R)*) is a draw, that is, it results in* c_L *(respectively, in* c_R*). Payoffs* u_L *and* u_R *are defined respectively by the following equations:*

$$u_L(c_\ell) = (-1)^i \text{ whenever } j_\ell \in V_i, \text{ where } i \in I = \{1, 2\} \text{ and } \ell \in [k] = \{1, \ldots, k\} \quad (2)$$

$$u_R(c_\ell) = (-1)^i \text{ whenever } j_{\ell-1} \in V_i, \text{ where } i \in I = \{1, 2\} \text{ and } \ell - 1 \in [k] = \{1, \ldots, k\} \quad (3)$$

(iii) There is a payoff $u = u_L = u_R$ *satisfying both (2) and (3) if and only if*

(a) $V_1 = V, V_2 = \emptyset,\ u(c_\ell) = -1 \;\forall\, \ell \in [k]$ *or (b)* $V_2 = V, V_1 = \emptyset.\ u(c_\ell) = -1 \;\forall\, \ell \in [k]$.

In both cases, pair (Q_L, P) *or, respectively,* (Q_R, P) *form a* 0*-path.*

Proof At first, let us consider (i) and (ii). Obviously, (Q_L, P, j_0, u_L) (and respectively, (Q_R, P, j_K, u_R)) is a positional game with perfect information whose tree is a caterpillar. Indeed, in each position $j_\ell \in V_i$ player i has two options: either to return to $j_{\ell-1}$ (respectively, to $j_{\ell+1}$) and, by this, to finish the game in c_ℓ (respectively, in $c_{\ell+1}$), or to proceed with $j_{\ell+1}$ (respectively, with $j_{\ell-1}$). It is also easy to see that the first options is winning whenever (2) (respectively, (3)) does not hold. Thus, the play results in c_L (respectively, in c_R), and the game results in a draw, if and only if (2) (respectively, (3)) holds for all ℓ.

Let us proceed with (iii) and notice that in both cases (a) and (b) equations (2) and (3) hold and, moreover, $u_L(c_\ell) = u_R(c_\ell)$ for $\ell \in [k]$.

Furthermore, equations (2), (3), and $u_L(c_k) = u_R(c_k)$ for all $\ell \in [k]$ imply that

$$u_L(c_1) = u_R(c_2) = u_L(c_2) = u_R(c_3) = \ldots = u_L(c_{\ell-1}) = u_R(c_\ell) = \ldots = u_L(c_{k-1}) = u_R(c_k).$$

Hence, (2) and (3) imply that all positions $j_0, j_1, \ldots, j_k$ are controlled by the same player $i \in I = \{1, 2\}$ and, moreover, that $u_L(c_k) = u_R(c_k) = u(c_k) = (-1)^i$ for all $\ell \in [k]$.

It is easy to see that cases (a) and (b) appear for $i = 1$ and $i = 2$, respectively. □

Remark 21 *If $j_0 = j_k$ then path Q turns into a simple cycle C. Obviously, Lemma 8 can be extended to this case without any changes.*

5.5 Solvability in presence of 1-paths

Let us consider a pair (G, P) whose graph G consists of a 1-path Q and two more paths Q' and Q'' between two vertices j_0 and j_k.

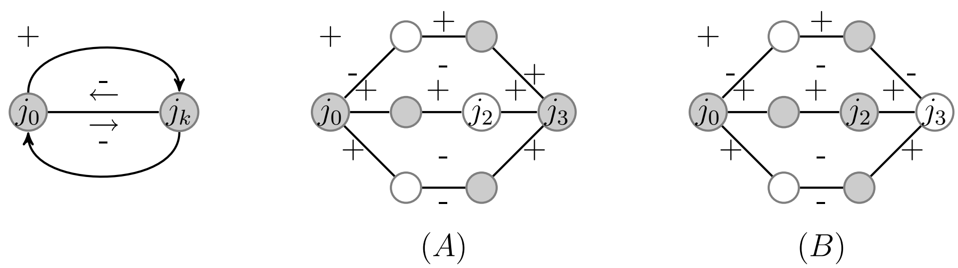


Figure 6: Non-solvability due to 1-paths.

Two examples, A and B, with $k = 3$, are given in Figure 6. In these examples, Q', Q'' and Q have no common vertices, except for j_0 and j_k.

Standardly, let $j_0, j_1, \ldots, j_k$ denote the $k+1$ vertices of Q and let j_{ℓ_0} be a unique vertex of player 1 (White) among them. We say that Q is of type A if $0 < \ell_0 < k$ and B if $\ell_0 = 0$ or $\ell_0 = k$; see Figure 6.

Let us introduce a payoff u as follows. For the short dicycles of Q' and Q'' u is defined in a (unique) way such that Q' and Q'' will be passed clockwise, in accordance with Section 5.4, and let player 1 wins in the obtained long dicycle c, that is, $u(c) = +1$.

Furthermore, let $u(c_\ell) = +1$ for all short dicycles c_ℓ, $\ell \in [k] = \{1, \ldots, k\}$ of path Q.

Then, as it was mentioned in Section 5.3, path Q might be passed through both ways.

Respectively, two more clockwise long dicycles c' and c'', formed by (Q', Q) and (Q'', Q), appear. We set $u(c') = u(c'') = -1$, that is, player 2 wins in both cases.

Let us consider two examples, of types A and B, in Figure 6.

(A): $\ell_0 = 2$. The play can come to position j_2 in two ways: from the left j_1 or right j_3. In both cases player 1 (White) could win. Indeed, it is enough for her just to return from j_2

to the position from which the play came from, since $u(c_2) = u(c_3) = +1$. Yet, she cannot guarantee such a victory, since, by our basic assumption, the players are restricted to their positional strategies and the move in j_2 cannot depend on where the play came to j_2 from.

Thus, player 2 should not surrender. Instead, he should try to enter Q either from j_0 or from j_3 and then approach j_2. Although, in both cases the opponent could win, yet, if her guess is wrong then a long dicycle c' or c'' appears and player 2 wins, since $u(c') = u(c'') = -1$.

(B): $j_0 = 3$. Again, the play can come to position $j_{\ell_0} = j_3$ in two ways: from j_2 or by Q'. Respectively, player 1 should return to j_2 or switch to Q''. In both cases he wins, since $u(c) = u(c_3) = +1$. Yet, he cannot guarantee such a victory, because he is not aware of opponent's strategy. Hence, player 2 should not surrender. Instead, in position j_0 she can try either to enter (and then pass through) path Q or switch to Q'. Although, player 2 can win in both cases, yet, if his guess is wrong then a long dicycle c' or c'' appears and player 2 wins, since $u(c') = u(c'') = -1$.

It is easily seen that both games are uniformly not solvable. These two examples can be generalized as follows.

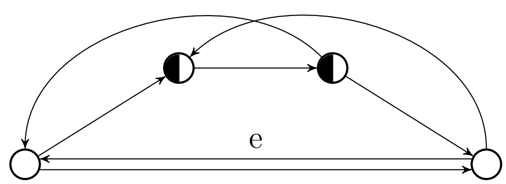


Figure 7: Case $m = 1$.

Lemma 9 *Given a non-directed graph $G = (V, E)$ and an edge $e = (j_0, j_k) \in E$, the next two properties are equivalent:*

(i) Except e, there are at least two edge-disjoint (simple) pathes between j_0 and j_k.

(ii) Let us orient e in both directions; then there is an orientation of the remaining edges $E \setminus \{e\}$ such that there exist two simple dicycles passing e in opposite directions.

Proof If (i) holds, let us orient one of the two **edge-disjoint** pathes from j_0 to j_k, while the other from j_k to j_0. Then (ii) holds too.

Now, let (ii) hold. Let us start witrh e, follow dicycle c', and mark common arcs with c''. In general, these arcs form m vertex-disjoint dipaths. (By convention, e does not count.)

If $m = 0$ then $c' \setminus \{e\}$ and $c'' \setminus \{e\}$ form two edge-disjoint dipathes Q' and Q'' between j_0 and j_k. Case $m = 1$ is shown in Figure 7. Obviously, after elimination of the unique common dipath, one gets two edge disjoint (althugh not directed) pathes. Hence, (i) holds. In general, it is easy, to derive (i), by induction on m. □

An edge $e \in E$ of a non-directed graph $G = (V, E)$ will be called a θ-edge if it satisfies the equivalent properties (i) and/or (ii).

Let us recall quadruple $\mathcal{G} = (\mathcal{V}; \mathcal{E}, \mathcal{E}_0, \mathcal{E}_1)$ and pair $(\mathcal{G}, \mathcal{P})$ from Section 3.2. Furthermore, let us replace each 0-edge in $\mathcal{G}$ by two parallel edges and denote the obtained graph by $\mathcal{G}_0$.

Proposition 11 *A pair $(\mathcal{G}, \mathcal{P})$ is* **not** *solvable whenever $\mathcal{G}_0$ contains a θ-1-edge.*

Proof By (i), let us replace this edge by Q and the corresponding two pathes by Q' and Q'' such that all three pathes have common ends, j_0 and j_k.

Furthermore, by (ii), let Q and Q' (respectively, Q'') form simple cycles c' (respectively, c''). Yet, Q' and Q'' might form several simple cycles, not necessarily one. Nevertheless, we can repeat all arguments of Section 5.5, A and B, modulo the following three remarks:

(i) There is a unique payoff defined on the short dicycles of Q' (respectively, Q'') such that this path will be passed through from j_0 to j_k (respectively, from j_k to j_0), despite Q' and Q'' might share 0-pathes. Indeed, by Lemma 8 (iii), there is a unique payoff such that a given 0-path can be passed through both ways.

(ii) The orientations of Q' and Q'' form a family $\mathcal{C}$ of "short and long dicycles". Let us set $u(c) = +1$ for all $c \in \mathcal{C}$. Clearly, the obtained triplet (G, P, u) is uniformly non-solvable.

(iii) One more special case to consider: Q is a 1-path and $Q' = Q''$ is a 0-path.

Obviously, in this case, the assumptions hold and (G, p) is not solvable, since it contains a 1-cycle formed by Q and $Q' = Q''$. □

By Proposition 8, for the **monochromatic** pairs, the inverse implication holds too, that is, a **monochromatic** pair $(\mathcal{G}, \mathcal{P})$ is **solvable** whenever graph $\mathcal{G}_0$ contains no θ-1-edges.

5.6 Orienting $(\mathcal{G}_0, \mathcal{P})$ and more criteria of solvability

To each pair (G, P) we assigned a pair $(\mathcal{G}, \mathcal{P})$. Then we replaced each 0-edge in $\mathcal{G}$ by a pair of parallel edges getting pair $(\mathcal{G}_0, \mathcal{P})$. Let us proceed further with these transformations: orient the edges of $(\mathcal{G}_0, \mathcal{P})$ arbitrarily and denote the obtained pair $(\overrightarrow{\mathcal{G}_0}, \mathcal{P})$.

Similarly to (G, P), we call $(\overrightarrow{\mathcal{G}_0}, \mathcal{P})$ *solvable* (*uniformly non-solvable*) if the corresponding zero-sum ± 1 game $(\overrightarrow{\mathcal{G}_0}, \mathcal{P}, j, u)$ is solvable (not solvable) for each initial position j of digraph $\overrightarrow{\mathcal{G}}$ and for every (for some) payoff $u : C(\overrightarrow{\mathcal{G}_0}) \to \{-1, +1\}$.

Remark 22 *Without any loss of generality, we will assume that $u(c) = (-1)^i$ for each dicycle $c = (e_0', e_0'')$ corresponding to a 0-edge e_0 in $\mathcal{G}$ controlled by player $i \in I = \{1, 2\}$. Indeed, otherwise player i would immediately win on c.*

Furthermore, digraph $\overrightarrow{\mathcal{G}}$ might have dead-ends. Naturally, we should assume that a player wins whenever the opponent cannot move.

Obviously, a game is solvable when its initial position is a dead-end or it belongs to a winning cycle (e_0', e_0''). Yet, let us also recall that solvability of (G, P) is ergodic (it cannot depend on the initial position) whenever graph G is connected.

The above results easily imply the following conditions, each of which is either necessary or sufficient for (uniform) solvability of a 2-connected pair (G, P).

Proposition 12 *(i) If pair (G, P) is solvable then every oriented pair $(\overrightarrow{\mathcal{G}_0}, \mathcal{P})$ is solvable.*

(ii) A pair (G, P) is solvable whenever it contains no 1-path and pair $(\overrightarrow{\mathcal{G}_0}, \mathcal{P})$ is solvable for every orientation of graph $\overrightarrow{\mathcal{G}_0}$.

(i') If pair (G, P) is not solvable and it contains no 1-path then there is an orientation of $\overrightarrow{\mathcal{G}_0}$ such that the obtained pair $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ is uniformly non-solvable.

(ii') A pair (G, P) is (uniformly) non-solvable whenever there is an orientation of $\overrightarrow{\mathcal{G}_0}$ such that pair $(\overrightarrow{\mathcal{G}_0}, \mathcal{P})$ is not solvable. □

6 "If part" of Theorem 2; proof of Propositions 5 - 8

We will show that all pairs from the list $\mathcal{L}$ of Theorem 2 are solvable, in accordance with Propositions 5 - 8.

First, we assume that pair $(\mathcal{G}, \mathcal{P})$ has no 0- or 1-edges and verify that every its orientation $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ is solvable. By Propositions 12, it is sufficient to check that it cannot be uniformly non-solvable. In particular, we can ignore an orientation whenever it has a dead-end j_T, since solvability obviously holds when j_T is the initial position, $j_0 = j_T$. Solvability also holds when one of two players is a dummy, that is, (s)he has only one move in every position and, hence, only one strategy. The corresponding orientations can be ignored, too.

Then we assume that there is a 0- or 1-edge in $(\mathcal{G}, \mathcal{P})$ and produce an orientation of $\overrightarrow{\mathcal{G}_0}$ such that the obtained pair $(\overrightarrow{\mathcal{G}_0}, \mathcal{P})$ is (uniformly) non-solvable. By Proposition 2, it is sufficient to find out a non-solvable subpair $(\overrightarrow{\mathcal{G}}', \mathcal{P}') < (\overrightarrow{\mathcal{G}_0}, \mathcal{P})$. In all cases, it will be one of the subpairs given by Figures 3 (1), 5, and 6 (A, B), or more precisely, by Proposition 11.

6.1 Pairs $(\theta_k, \mathcal{P}_k)$; proof of Proposition 5

Let us consider a bipartite pair $(\theta_k, \mathcal{P}_k)$ for an integer $k \geq 1$ in Figure 3 and redraw it with more detail and notation, as in Figure 8. The bipartite graph $\theta_k = (V_k, E_k)$ contains $2k$ vertices (positions) and $3k$ edges (moves). The vertex-set $V_k = \{j^1_\ell, j^2_\ell \mid \ell \in [k] = \{1, \dots k\}\}$ is partitioned into two subsets: $\mathcal{P}_k : V_k = V^k_1 \cup V^k_2$, where $V^k_i = \{j^i_\ell \mid \ell \in [k]\}$ are k positions of player $i \in I = \{1, 2\}$: White and Black, respectively. The edge-set $E_k = \{e_\ell, e'_\ell, e''_\ell \mid \ell \in [k]\}$ consists of $3k$ edges: k of type 1, $e_\ell = (j^1_\ell, j^2_\ell)$, and $2k$ of type 2, $e'_\ell = (j^2_\ell, j^1_{\ell+1})'$, $e''_\ell = (j^2_\ell, j^1_{\ell+1})''$, where $\ell \in [k]$ and $k + 1 = 1$, by convention.

Let us notice that case $k = 1$ is "slightly degenerated": θ_1 consists of three parallel edges.

Let us also note that θ_k is a proper subgraph of θ_{k+1}; moreover, it is easy to see that $(\theta_{k+1}, \mathcal{P}_{k+1}) > (\theta_k, \mathcal{P}_k)$ for each integral positive k and, hence, pairs $(\theta_k, \mathcal{P}_k)$ form an infinite chain of solvable 2-connected pairs that has no maximal element. Finally, since pairs $(\theta_k, \mathcal{P}_k)$ are bipartite, they cannot contain 0-edges (hence, $\theta_k = (\theta_k)_0$), yet, they can contain 1-edges.

Clearly, for each $k \geq 1$, a pair $(\theta_k, \mathcal{P}_k)$ is uniformly non-solvable whenever it contains a 1-edge of type 2. To show this, one can just refer to Section 5.5 (B) and Proposition 11.

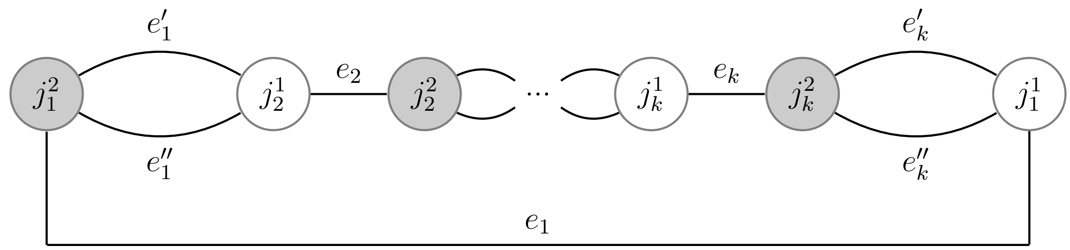


Figure 8: Pair $(\theta_k, \mathcal{P}_k)$

By induction on k, let us show that $(\overrightarrow{\theta_k}, \mathcal{P}_k)$ is solvable for every orientation of θ_k.

Clearly, $(\overrightarrow{\theta_1}, \mathcal{P}_1)$ is solvable since for each orientation of θ_1 one of two players is a dummy.

Let us assume indirectly that $(\theta_k, \mathcal{P}_k)$ is (uniformly) non-solvable for some $k > 1$.

Case A: there is an $\ell \in [k]$ such that the edges e_ℓ and $e_{\ell+1}$ are oriented oppositely, one towards the other, that is, $\overrightarrow{e_\ell} = (j_\ell^1, j_\ell^2)$, while $\overrightarrow{e}_{\ell+1} = (j_{\ell+1}^2, j_{\ell+1}^1)$. Clearly, in this case game form $(\overrightarrow{\theta_k}, \mathcal{P}_k, j)$ is solvable whenever $j = j_\ell^2$ or $j = j_{\ell+1}^1$ and we get a contradiction.

Case B: there is an $\ell \in [k]\}$ such that edges e_ℓ and $e_{\ell+1}$ are oriented oppositely, one from the other, that is, $\overrightarrow{e}_\ell = (j_\ell^2, j_\ell^1)$, while $\overrightarrow{e}_{\ell+1} = (j_{\ell+1}^1, j_{\ell+1}^2)$. It is easy to see that this case can be reduced to the previous one. Indeed, if Case B holds for some $\ell \in [k]$ then Case A holds for some other $\ell' \in [k]$ and again we get a contradiction.

Thus, all edges of type 1 must be oriented in the same way, say, clockwise, that is, $\overrightarrow{e}_\ell = (j_\ell^1, j_\ell^2)$ for all $\ell \in [k]$. Now let us consider the edges of type 2.

Case C: there is a $\ell \in [k]$ such that both edges e'_ℓ and e''_ℓ are oriented counter-clockwise. Then, obviously, game form $(\theta_k, \mathcal{P}_k, j_\ell^2)$ is solvable, since j_ℓ^2 is a dead-end.

Case D: there is an $\ell \in [k]$ such that e'_ℓ and e''_ℓ are oppositely oriented, say, $\overrightarrow{e}'_\ell = (j_\ell^2, j_{\ell+1}^1)$, while $\overrightarrow{e}''_\ell = (j_{\ell+1}^1, j_\ell^2)$. Then, these two arcs form a short dicycle c. Let us also note that $\overrightarrow{e}'_\ell$ is a forced move in j_ℓ^2 and consider two subcases.

Subcase D1: $u(c) = +1$. Then, obviously, player 1 wins in game $(\theta_k, \mathcal{P}_k, j, u)$ when $j = j_\ell^2$ or $j = j_{\ell+1}^1$.

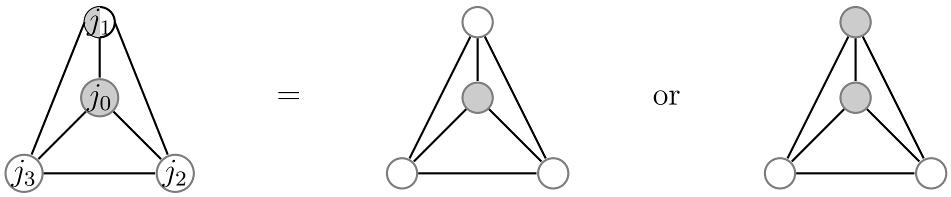
Subcase D2: $u(c) = -1$. In this case $\overrightarrow{e}''_\ell = (j_{\ell+1}^1, j_\ell^2)$ is definitely a losing move for player 1. Hence, we can delete this arc from digraph $\overrightarrow{\theta_k}$. Obviously, the reduced pair is equivalent to $(\overrightarrow{\theta}_{k-1}, \mathcal{P}_{k-1})$ and, hence, it is solvable, by the induction hypothesis.

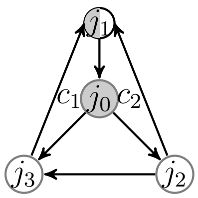
Thus, the problem is reduced to the last Case E: all edges are oriented clockwise.

In this case, game forms $(\theta_k, \mathcal{P}_k, j)$ are solvable for all j, since player 1 is a dummy. $\square$

6.2 Pairs $(\mathcal{K}_4, \mathcal{P}')$ and $(\mathcal{K}_4, \mathcal{P}'')$; proof of Proposition 6

We will prove that each of these two pairs is solvable, unless it contains a 0- or 1-edge.





	↙	↘	
↖	c_1	c_2	(j_2, j_1)
←	c_1	c_3	(j_2, j_3)
	(j_0, j_3)	(j_0, j_2)	

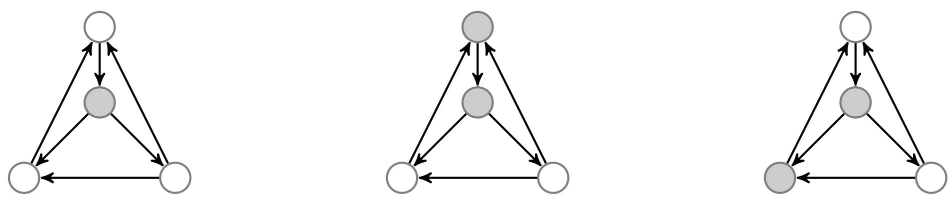
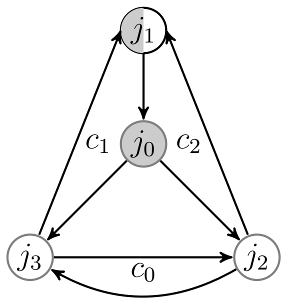


Figure 9: $F_1 = c_1c_2 \vee c_1c_3$, $F_2 = c_1 \vee c_2c_3$, $F_1 = F_2^d$; $c_1 = ((j_0, j_3), (j_3, j_1), (j_1, j_0))$, $c_2 = ((j_0, j_2), (j_2, j_1), (j_1, j_0))$, $c_3 = ((j_0, j_2), (j_2, j_3), (j_3, j_1), (j_1, j_0))$.

For brevity we can represent both pairs by one $(\mathcal{K}_4, \mathcal{P})$ in which j_1 is an uncertain position, as in Figure 9. First, let us suppose that there are no 0- and 1-edges and show that all orientations $(\overrightarrow{\mathcal{K}}_4, \mathcal{P})$ are solvable. Let us assume indirectly that there is a (uniformly) not solvable one. In particular, it has no-dead-ends and no player is a dummy. The following simple case analysis shows that there are only three such orientations; see Figure 9.

First let us show that out-degree of each vertex j_ℓ is 1 or 2. Indeed, if it is 0 then j_k is a dead-end and if it is 3 then there are two options: (a) one of the remaining three vertices is a dead-end, or (b) the orientation forms a simple cycle on these three vertices. Yet, in case (b) the player that controls j_ℓ has three strategies and his opponent is a dummy.

Since in $\mathcal{K}_4$ each vertex is of degree 3, simple counting shows that there are two vertices, say, j_0 and j_2, of out-degree 2 and in-degree 1 and the remaining two, j_1 and j_3, of out-degree 1 and in-degree 2. Clearly, j_0 and j_2 cannot belong to the same player, since in this case



	c_0	c_3	c_1	c_1	(j_0, j_3)
	c_0	c_2	c_4	c_2	(j_0, j_2)
	(j_2, j_3) (j_3, j_2)	(j_2, j_1) (j_3, j_2)	(j_2, j_3) (j_3, j_1)	(j_2, j_1) (j_3, j_1)	

Figure 10: $F_1 = c_0c_1c_3 \vee c_0c_2c_4$, $F_2 = c_0 \vee c_1c_2 \vee c_1c_4 \vee c_2c_3$, $F_1 \neq F_2^d$;
$c_0 = ((j_2, j_3), (j_3, j_2)), c_1 = ((j_0, j_3), (j_3, j_1), (j_1, j_0)), c_2 = ((j_0, j_2), (j_2, j_1), (j_1, j_0))$,
$c_3 = ((j_0, j_3), (j_3, j_2), (j_2, j_1), (j_1, j_0)), c_4 = ((j_0, j_2), (j_2, j_3), (j_3, j_1), (j_1, j_0))$.

the opponent would be a dummy. Thus, in $(\overrightarrow{K}_4, \mathcal{P})$ each player has two strategies and the corresponding normal game form is tight; see Figure 9.

Let us remark that we can "recolor" vertices j_1 and j_3 and get $j_1 \in V_1$, while j_3 becomes uncertain. However, this transformation does not change the normal form, since both vertices j_1 and j_3 are of out-degree 1, i.e., there is only one (forced) move in each of these two positions. Assigning a player to each uncertain position we obtain three slightly different pairs; yet, all three have the same normal game form; see Figure 9.

Now, let us assume that pair $(\mathcal{K}_4, \mathcal{P})$ contains a 1-edge e. It may be of type A or B. Without any loss of generality, let us assume that

(a) (j_0, j_2) is a 1-edge of type A or (b) (j_2, j_3) is a 1-edge of type B.

In each case it is easy to show that the considered pair is uniformly non-solvable. Indeed, in case (a) (respectively (b)) it is enough to delete edge (j_0, j_1) (respectively, (j_1, j_3)) and recall that the reduced subpair was considered in Section 5.5 (A) (respectively, (B)).

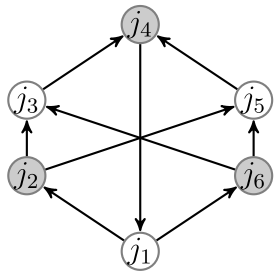
Finally, let us assume that pair $(\mathcal{K}_4, \mathcal{P})$ contains a 0-edge e, say, $e = (j_2, j_3)$. Let us replace e by two oppositely oriented arcs and orient all other edges as in Figure 10.

It is easy to see that the corresponding normal game form does not depend on the initial position and it is not tight. This completes the proof of Proposition 6. □

6.3 Pair $(\mathcal{K}_{3,3}, \mathcal{P})$; proof of Proposition 7

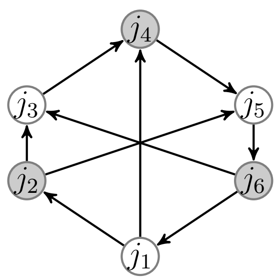
We will prove that this pair is solvable unless it contains a 1-edge. (Clearly, it cannot contain 0-edges, since it is bipartite.) First, let us suppose that there is no 1-edge and show that all orientations $(\overrightarrow{K}_{3,3}, \mathcal{P})$ of $(\mathcal{K}_{3,3}, \mathcal{P})$ are solvable.

Let us assume indirectly that there is a not solvable orientation $(\overrightarrow{\mathcal{K}}_{3,3}, \mathcal{P})$. Then it is uniformly non-solvable. In particular, there is no-dead-end and no player is a dummy. The following simple case analysis shows that there are only two such orientations; see Figure 11.



(a)

	(j_2, j_3) (j_6, j_3)	(j_2, j_3) (j_6, j_5)	(j_2, j_5) (j_6, j_3)	(j_2, j_5) (j_6, j_5)
(j_1, j_2)	c_1	c_1	c_2	c_2
(j_1, j_6)	c_3	c_4	c_3	c_4



(b)

	(j_2, j_3) (j_6, j_3)	(j_2, j_5) (j_6, j_3)	(j_2, j_3) (j_6, j_1)	(j_2, j_5) (j_6, j_1)
(j_1, j_2)	c_1	c_1	c_2	c_3
(j_1, j_4)	c_1	c_1	c_4	c_4

Figure 11: (a) $F_1 = c_1c_2 \vee c_3c_4$, $F_2 = c_1c_3 \vee c_1c_4 \vee c_2c_3 \vee c_2c_4$, $F_1 = F_2^d$;
$c_1 = ((j_1, j_2), (j_2, j_3), (j_3, j_4), (j_4, j_1)), c_2 = ((j_1, j_2), (j_2, j_5), (j_5, j_4), (j_4, j_1)),$
$c_3 = ((j_1, j_6), (j_6, j_3), (j_3, j_4), (j_4, j_1)), c_4 = ((j_1, j_6), (j_6, j_5), (j_5, j_4), (j_4, j_1)).$
(b) $F_1 = c_1(c_2c_3 \vee c_4)$, $F_2 = c_1 \vee c_2c_4 \vee c_3c_4$, $F_1 = F_2^d$;
$c_1 = ((j_3, j_4), (j_4, j_5), (j_5, j_6), (j_6, j_1)), c_2 = ((j_1, j_2), (j_2, j_3), (j_3, j_4), (j_4, j_5), (j_5, j_6), (j_6, j_1)),$
$c_3 = ((j_1, j_2), (j_2, j_5), (j_5, j_6), (j_6, j_1)), c_4 = ((j_1, j_4), (j_4, j_5), (j_5, j_6), (j_6, j_1)).$

First let us show that out-degree of each vertex j_k is 1 or 2. Indeed, if it is 0 then j_k is a dead-end and we get a contradiction. If it is 3 then j_k is transient position. In this case we can reduce $\mathcal{K}_{3,3}$ to $\mathcal{K}_{2,3}$ by deleting j_k. Furthermore, it is easy to see that the bipartite pair $(\mathcal{K}_{2,3}, \mathcal{P})$ is, in fact, equivalent to the monochromatic pair $(\theta_1, \mathcal{P}')$. By Proposition 8, this pair is solvable, unless it contains a 1-edge.

By simple counting, we prove that there are three positions of out-degree 2 and three of out-degree 1. If the first three belong to one player and the last three to the other one then the latter is a dummy. Hence, without loss of generality we can assume that $j_1, j_3, j_5 \in V_1$ and $j_2, j_4, j_6 \in V_2$; furthermore, j_1, j_2, j_6 have out-degree 2, while j_3, j_4, j_5 have out-degree 1. It is easy to verify that, up to an isomorphism, there are only two such orientations of $\mathcal{K}_{3,3}$.

They are given in Figure 11. The corresponding two normal game forms are tight.

Now let us show that pair $(\mathcal{K}_{3,3}, \mathcal{P})$ is not solvable whenever it contains a 1-edge. Due to symmetry, we can assume without loss of generality that (j_1, j_2) is such an edge. In this case we can delete edges (j_4, j_5) and (j_3, j_6) and obtain a pair from Section 5.5 (B), which, as we already know, is not solvable. This completes the proof of Proposition 7. $\square$

6.4 Monochromatic pairs; proof of Proposition 8

By Proposition 11, a pair $(\mathcal{G}, \mathcal{P})$ is not solvable whenever graph $\mathcal{G}_0$ contains a 1-θ-edge. Proposition 8 claims that, in monochromatic case, the inverse holds too.

Moreover, if $(\mathcal{G}, \mathcal{P})$ is an **edge-minimal** non-solvable monochromatic (say, White) pair then **every** 1-edge of $\mathcal{G}_0$ is a θ-edge. (This claim is slightly stronger than the previous one, since solvability is a hereditary property of pairs.)

Since $(\mathcal{G}, \mathcal{P})$ is not solvable, there is a uniformly non-solvable orientation $(\overrightarrow{G_0}, \mathcal{P})$. Let $\mathcal{C}^+$ and $\mathcal{C}^-$ denote the sets of dicycles in $\overrightarrow{G_0}$ respectively winning and losing for player 1 (White). Every $c \in \mathcal{C}^+$ contains a 1-edge, since otherwise player 1 would just win on c.

Let us assume indirectly that e is a 1- but not θ-edge of $\mathcal{G}_0$. Then, after elimination of e, solvability still fails, in contradiction with edge-minimality. Indeed, let us reverse the orientation of e. By Lemma 9, in the obtained digraph $\mathcal{G}'_0$ no dicycle contains e. Hence, player 2 can safely block passing e in $\overrightarrow{G_0}$. Doing so, (s)he might reduce $\mathcal{C}^+$ but cannot add anything to it. $\square$

Remark 23 *Partition $\mathcal{P} : \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is monochromatic: $\mathcal{V}_1 = \mathcal{V}, \mathcal{V}_2 = \emptyset$. Hence, given an orientation $(\overrightarrow{G_0}, \mathcal{P})$ player 1 has an almost full control; she can choose any dicycle or terminal; yet, player 2 controls 1-edges. If there is a 1-θ-edge then solvability fails. Otherwise, for each 1- but not θ-edge e, by Lemma 9, all dicycle through e go in one direction. Obviously, player 2 is to block this direction and, thius, reduce $\mathcal{C}^+$. Clearly, he wins if he can eliminate the whole set $\mathcal{C}+$, otherwise the opponent wins. Yet, the game is solvable in both cases.*

7 Proof of Theorem 2 by Lovasz' ear decomposition

Thus, a pair $(\mathcal{G}, \mathcal{P})$ is solvable whenever it belongs to the list $\mathcal{L}$ defined by Propositions 4-8. Still, we have to prove that there are no other 2-connected solvable pairs.

By Lovasz' "Ear-Decomposition" Lemma [27], each 2-connected graph $\mathcal{G}$ whose every vertex is of degree at least 3 can be obtained from $\mathcal{G}_1 = \theta_1$ by successive addition of new vertices and edges, as follows. By each step $k = 1, 2, \ldots$, we add to a current graph $\mathcal{G}_k$ at most two new vertices and exactly one new edge $e_k = (j'_k, j''_k)$. There are the three options:

(a) j'_k and j''_k are two "old" vertices of graph $\mathcal{G}_k$;
(b) j'_k is an old vertex, while j''_k is a new vertex subdividing an edge of $\mathcal{G}_k$;
(c) both j'_k and j''_k are new vertices subdividing an edge or two distinct edges of $\mathcal{G}_k$.

In all cases e_k is not a loop, that is, $j' \neq j''$. This inequality automatically holds for (b) and we assume it for (a) and (c). Then, after each step, we obtain a 2-connected graph.

This procedure needs only a slight modification to generate all 2-connected **pairs** $(\mathcal{G}, \mathcal{P})$: let us assign a player, 1 or 2, to each new vertex, in cases (b) and (c). Applying such assignments and ear extensions in all possible ways, we will obtain all 2-connected pairs, by the Lovasz Lemma. Let us proceed with this procedure verifying solvability of every obtained new pair. Since, by Proposition 2, solvability is anti-monotone, we should extend further only solvable pairs, while non-solvable pairs should be eliminated.

For example, we get the bipartite solvable pair $(\theta_k, \mathcal{P}_k)$ from $(\theta_1, \mathcal{P}_1)$ by $k-1$ successive extensions of type (c). Furthermore, the remaining solvable pairs $(\mathcal{K}_4, \mathcal{P}')$, $(\mathcal{K}_4, \mathcal{P}'')$, and $(\mathcal{K}_{3,3}, \mathcal{P})$ are also obtained from $(\theta_1, \mathcal{P}_1)$ by, respectively, one and two extensions of type (c).

The complete case analysis is routine, elementary but long and it requires very many diagrams. To save space, we refer the reader to Section 7 of [5].

Let us only remark that in all cases but one, the non-solvability of a pair $(\mathcal{G}, \mathcal{P})$ under consideration is guaranteed, because this pair contains a construction from Example 2.

Only one ear extension of $(\mathcal{K}_{3,3}, \mathcal{P})$ requires the example (2) of Figure 2.

8 Another three problems related to Nash-solvability

In this section we will consider n-person games and will not assume that $V_T = \emptyset$.
Thus, the set of positions is partitioned into $n+1$ subsets, $P : V = V_1 \cup \ldots \cup V_n \cup V_T$.

8.1 All dicycles form a unique outcome

It seems that Nash-solvability is implied by the following assumption:

(0) all dicycles are equivalent, that is, they form a unique outcome c; in other words, the set of all outcomes A is $V_T \cup \{c\}$.

In contrast, the main subject of the present paper was the case when each dicycle is a separate outcome. In both cases, a utility function is an arbitrary mapping $u : I \times A \to \mathbb{R}$.

Conjecture 1 *Assumption (0) implies Nash-solvability.*

It is known that the conjecture holds in the following two cases:

(i) digraph $\overrightarrow{G}$ is acyclic, that is, it has no dicycles at all, and

(ii) for the two-person case, $|I| = 2$.

Remark 24 *In case (i), the result is frequently referred to as the Zermelo-Kuhn theorem.*

The concept of equilibrium was introduced in 1950 by Nash [32]; see also [33].

Soon after this, in 1953, Kuhn [26] suggested a constructive proof for case (i). Strictly speaking, he considered only trees, yet, the algorithm, so-called **backward induction**, *can be easily extended to acyclic digraphs.*

Let us note that in case (i) assumption (0) is irrelevant, since there are no dicycles.

In fact, studying case (ii) started much earlier. In 1912, Zermelo gave his seminal talk "Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels" [38].

In fact, his analysis is applicable not only to Chess but to any two-person **zero-sum** *game modeled by a finite digraph. Moreover, this digraph* might *contain dicycles. Indeed, a position can be repeated in a Chess play and in this case the play results in a draw. Thus, in Chess all dicycles are equivalent, since they form a unique outcome, namely, a draw.*

Accurately speaking, a play in Chess results in a draw only when the same position appears **three times**. *Yet, we assume that both players are restricted to their positional strategies. Hence, a position will be repeated infinitely whenever it appears just twice.*

In case (ii), Conjecture 1 easily results from Theorem 1. The following nice proof was suggested by Gimbert and Sørensen, private communications; see also [1] for more details.

Proof By Theorem 1, it is sufficient to prove ± 1-solvability rather than Nash-solvability.

Hence, we can assume that there are two players, $I = \{1, 2\}$, and each outcome $a \in A = V_T \cup \{c\}$ is either winning for player 1 and losing for 2, or vice versa. Without any loss of generality, let us assume that outcome c is winning for 1. Furthermore, let $V_T = V_T^1 \cup V_T^2$ be the partition of all terminals into outcomes winning for players 1 and 2, respectively.

Then, let $V^2 \subseteq V$ denote the set of positions from which player 2 can enforce V_T^2; in particular, $V_T^2 \subseteq V^2$. Finally, let us set $V^1 = V \setminus V^2$; in particular, $V_T^1 \subseteq V^1$.

By these definitions, in every position $j \in V_1 \cap V^1$ player 1 can stay out of V^2, that is, (s)he has a move (j, j') such that $j' \in V^1$. Let us fix a strategy x_1^0 that chooses such a move in each position $j \in V_1 \cap V^1$ and any move in $j \in V_1 \cap V^2$. Then, for any $x_2 \in X_2$, the outcome $g(x_1^0, x_2)$ is winning for player 1 whenever the initial position j_0 is in V^1. Indeed, either $g(x_1^0, x_2) \in V_T^1$, or $g(x_1^0, x_2) = c$. In both cases player 1 wins. Thus, player 1 (respectively, 2) wins whenever $j_0 \in V^1$ (respectively, $j_0 \in V^2$). Yet, in any case a saddle point exists. □

Unfortunately, Theorem 1 is not applicable in the n-person case. In addition to (0), the following extra assumption introduced in [4] might be important too.

(iii) Outcome c is worse than any terminal outcome for each player.

In [4], several interpretations of assumption (iii) are given (see Remark 2 and Section 2.1.1 of [4]) and the following relaxation of Conjecture 1 is suggested.

Conjecture 2 *Assumptions (0) and (iii) imply Nash-solvability.*

Under assumptions (0) and (iii), Nash-solvability was proved in [4] for two more cases:

(iv) play-once games, in which each player controls a unique position;

(v) three outcomes, $|A| \leq 3$, that is, there are at most two terminals, $|V_T| \leq 2$, and c.

Recently in [6], the latter statement was extended to the case (vi) $|A| \leq 4$.

Thus, Nash-solvability results from: (0, i), or (0, ii), or (0, iii, iv), or ((0, iii, vi).

8.2 Additive payoffs

As we already mentioned, Conjecture 2 is a relaxation of Conjecture 1. Yet in [4], Conjecture 2 was strengthen in an alternative way, based on a generalization of the terminal payoffs.

Given a digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, let us define a *local reward* as a mapping $r : I \times \overrightarrow{E} \to \mathbb{R}$.

Standardly, the value $r(i, \overrightarrow{e})$ is interpreted as the profit obtained by the player $i \in I$ whenever the play passes the directed edge $e \in \overrightarrow{E}$.

Let us recall that each situation $x \in X$ defines a unique play $p = p(x)$ that begins in the initial position j_0 and either (a) terminates in V_T or (b) results in a dicycle $c = c(x)$.

The *additive effective payoff* $u : I \times X \to \mathbb{R}$ is defined as follows. In case (a), it equals the sum of all local rewards of the obtained play, $u(i, x) = \sum_{\overrightarrow{e} \in p(x)} r(i, \overrightarrow{e})$ and in case (b) $u(i, x) \equiv -\infty$ for all $i \in I$.

In particular, all dicycles are equivalent and the corresponding outcome is the most unwanted for all players, in accordance with assumption (iii) of the previous subsection.

However, the set of the remaining outcomes is formed by the acyclic plays, that is, by all simple dipaths from j_0 to V_T, rather than by the set V_T itself.

The following two assumptions were suggested in [4]:

(j) all local rewards are non-positive, $r(i, \overrightarrow{e}) \leq 0$ for all $i \in I$ and $\overrightarrow{e} \in \overrightarrow{E}$ and

(jj) all dicycles are non-positive, $\sum_{\overrightarrow{e} \in c} r(i, \overrightarrow{e}) \leq 0$ for all dicycles $c \in C = C(\overrightarrow{G})$.

Obviously, (j) implies (jj). It was shown in 1958 by Gallai [11] that in fact these two assumptions are equivalent, since (j) can be enforced by a potential transformation whenever (jj) holds; see [11] and also [4] for definitions and more details.

Conjecture 3 *Assumptions (j, jj) implies Nash-solvability.*

This conjecture was suggested in [4], where it was also shown that assumptions (j, jj) are essential; see two examples of Section 2.1 on Figures 5 and 6 in [4].

Remark 25 *At least, assumptions (j) and (jj) look logical. Indeed, since a dicycle c is repeated infinitely, it becomes very attractive (unattractive) for a player $i \in I$ whenever the corresponding sum $\sum_{\overrightarrow{e} \in c} r(i, \overrightarrow{e})$ is positive (negative). Assumptions (j, jj) reflect the second case. However, it would not be logical to assume that a positive cycle is the most unwanted outcome for the corresponding player.*

Remark 26 *In [4], all players $i \in I$ minimize cost function $-u(i, x)$ instead of maximizing payoff $u(i, x)$. Hence, conditions (j, jj) turn into non-negativity conditions in [4].*

In [4], Conjecture 3 was proved for the following three special cases:

(jjj) two players, $|I| = 2$;

(jv) play-once games, in which each player controls a unique position;

(v) three outcomes, $|A| \leq 3$, that is, there are at most two terminal moves.

Thus, Nash-solvability is implied by (jj, jjj), or (jj, jv), or (jj, v). Let us show that the last two implications strengthen the last two implications of the previous subsection: (0, iii, iv), (0, iii, vi), respectively; furthermore Conjecture 3, if true, would imply Conjecture 2.

To prove both above claims, it is sufficient to demonstrate that the additive payoffs considered in this subsection generalize the terminal payoffs of the previous one.

A local reward r is called *terminal* if $r(i,e) = 0$ unless e is a terminal move.

Obviously, in the latter case, the additive effective payoff is exactly the payoff of the previous subsection satisfying assumption (iii).

Remark 27 *Obviously, in case of a local terminal payoff one can assume, without loss of generality, that there is only one terminal, that is,* $V_T = \{j_T\}$.

Furthermore, let us assume indirectly that Conjecture 3 fails even under assumption (jj). Then, it is not difficult to show that a minimal example must have the following property:

(vj) in digraph $\overrightarrow{G}$*, every dicycle and every dipath from* j_0 *to* j_T *intersect.*

The class of digraphs defined by (vj) has many interesting properties; in particular, they result in Nash-solvability; see [4].

8.3 Mean payoffs

Let us reassume that $V_T = \emptyset$ or equivalently, that $A = C = C(\overrightarrow{G})$, by Remark 4.

Then, for every situation $x \in X$ the corresponding play $p(x)$ results in a dicycle $c(x) \in C$. Furthermore, let us "slightly" modify the additive payoff setting

$$u(i,x) = |c(x)|^{-1} \sum_{\overrightarrow{e} \in c(x)} r(i, \overrightarrow{e}) \text{ for all } x \in X,$$

where $|c|$ is the length of a dicycle c, that is, the number of its verices or edges.

This formula defines the so-called *mean* effective payoff. It was introduced by Moulin [28, 29], Ehrenfeucht and Mycielski [9, 10], and it is interpreted as the Cesaro average of the local rewards, or in other words, as the local reward per one move of the play $p(x)$.

It appears that the **two-person zero-sum mean-payoff games are solvable**.

This was shown in [28, 29] for the case of complete bipartite digraphs, then in [10], for any bipartite digraphs, and finally in [23], for any digraphs.

Yet, Nash-solvability fails already for two-person (but not zero-sum) mean-payoff games. An example on the complete bipartite 3×3 digraph was constructed in [19]; see also [23]. This example is minimal, since Nash-solvability holds in case of $2 \times k$ bipartite digraphs [20]. However, no other sufficient conditions are known.

In the last decade, two-person zero-sum mean-payoff games and their applications got a lot of attention, mostly, due to their open complexity status. No polynomial algorithm solving such games is known yet, although the problem is in the intersection of NP and co-NP [25]; see the recent survey of Vorobyov [37] for more details.

Acknowledgements. Finally, we would like to recall the fundamental contribution of Andrey I. Gol'berg (1954 - 1985) to characterizing Nash-solvability of bidirected **bipartite** cyclic game forms.

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