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Why Chess and Back Gammon can be solved in pure positional uniformly optimal strategies

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Abstract. We study existence of (subgame perfect) Nash equilibria (NE) in *pure positional* strategies in finite positional n-person games with perfect information and terminal payoffs. However, these games may have moves of chance and cycles. Yet, we assume that *All Infinite Plays Form One Outcome* a_∞, in addition to the set of *Terminal* outcomes V_T, the so-called AIPFOOT condition. For example, Chess or Back Gammon are AIPFOOT games, since every infinite play is a draw, by definition. All terminals and a_∞ are ranked arbitrarily by the n players.
It is well-known that in each finite *acyclic* positional game, a subgame perfect NE exists and it can be easily found by backward induction, suggested by Kuhn and Gale in early 50s. In contrast, there is a two-person game with only one cycle, one random position, and without NE in pure positional strategies. In 1912, Zermelo proved that each two-person zero-sum AIPFOOT game without random moves (for example, Chess) has a saddle point in pure strategies. Yet, there are cycles in Chess. Zermelo's result can be extended in two directions:

(i) Each two-person (not necessarily zero-sum) AIPFOOT game without random moves has a (not necessarily subgame perfect) NE in pure *positional* strategies; although, the similar statement remains a conjecture for the n-person case.

(ii) Each two-person zero-sum AIPFOOT game (which might have random moves) has saddle point in pure *positional* uniformly optimal strategies.

Surprisingly, to prove (ii), it appears convenient to treat such a game (for example, Back Gammon or even Chess) as a *stochastic* game with perfect information.

Key words: Back Gammon, Chess, pure strategies, positional strategies, Nahh equilibrium, ergodic, subgame perfect, uniformly optimal, stochastic game, perfect information, position, random move, Zermelo, Kuhn, Gale

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1 Introduction

We study existence of (subgame perfect) Nash equilibria in pure *positional* strategies in *finite* positional n-person games with perfect information and terminal payoffs. The games under consideration may have moves of chance and cycles. Yet, we assume that *All Infinite Plays Form One Outcome* a_∞, in addition to the *Terminal* outcomes $V_T = \{a_1, \ldots, a_p\}$; this assumption will be called the AIPFOOT condition. For example, Chess or Back Gammon are AIPFOOT games, since the set of positions is finite and every infinite play is a draw, by definition. In general, a_∞ and all terminal positions form the set of outcomes, $A = V_T \cup \{a_\infty\} = \{a_1, \ldots, a_p, a_\infty\}$, which are ranked arbitrarily by n players $I = \{1, \ldots, n\}$.

In 1950, Nash introduced his concept of equilibrium [33, 34] (so called Nash equilibrium; NE, for short) . In the same year, Kuhn [28, 29] suggested a construction showing that a *subgame perfect* NE exists in every acyclic game, which might have moves of chance but cannot have cycles. Accurately speaking, Kuhn restricted his analysis to trees, yet, the method, so-called *backward induction*, can be easily extended to the acyclic digraphs; see, for example, Section 2.5. Thus, the acyclic case is simple. In this paper we focus on the AIPFOOT games, which naturally may have cycles.

In Section 3.3 we shall extend the backward induction procedure to the *two-person zero-sum* AIPFOOT games without random moves, like Chess, and prove existence of a saddle point in pure positional uniformly optimal strategies.

Furthermore, every n-person AIPFOOT game without random moves, has a NE in pure stationary (but not necessarily positional) strategies. The proof is simple; see, for example, Section 2.11. The graph G of such a game can be "unfolded" as a finite tree T whose vertices are the debuts of the original game. In particular, a position of G might appear in T several times. Then, one can apply backward induction to T, to obtain a subgame perfect NE. However, this NE, "projected" onto the original game, might not be *subgame perfect* and, although the corresponding strategies are pure, they might not be *positional*; indeed, by construction, a move in a position v in G depends not only on v but also on the debut that brought the play to v, that is, on the previous positions and moves.

The lack of these two important properties is not a fault of the above algorithm. Subgame perfect NE in pure positional strategies may just fail to exist, in presence of cycles.

In particular, a unique NE in pure positional strategies is not subgame perfect in a very simple two-person game, with only one cycle and two non-terminal positions, none of which is random; see Example 1. Furthermore, there are games with only one cycle, one position of chance, and no NE in pure positional strategies; see Examples 2 and 3.

Yet, by the so-called Folk Theorem, a (not necessarily subgame perfect) NE in pure (but not necessarily positional) strategies exists in every n-person AIPFOOT game, which might have both cycles and positions of chance; see Sections 2.11.

In 1912, Zermelo proved that each two-person zero-sum AIPFOOT game without moves of chance has a saddle point in pure strategies [37]. He considered Chess as an example (let us notice that there are cycles in Chess). Zermelo's result can be extended in two ways:

(i) Each two-person AIPFOOT game without moves of chance has a (not necessarily subgame perfect) NE in pure *positional* strategies.

(ii) Each two-person zero-sum AIPFOOT game has a *subgame perfect* saddle point in pure *positional* uniformly optimal strategies.

Yet, it remains an open question whether (i) can be extended to the n-person case; see Section 2.7. Let us also note that (ii) is applicable to games with random moves and cycles; for example, to Back Gammon.

Remark 1 *By definition, a saddle point is subgame perfect if and only if its two strategies are uniformly (that is, independently on the initial position) optimal. In general, a strategy that does not depend on the initial position is called* ergodic.

It may sound surprising, but to prove (ii), it is convenient to treat the corresponding game (for example, Back Gammon or even Chess) as a *stochastic* game with perfect information.

We make use of the so-called BWR model, where B, W, and R stand for Black, White, and Random positions, respectively. The BWR games were introduced in [24] and studied recently in [4, 5]. In particular, it was shown that every BWR game can be reduced to a canonical form by a potential transformation of local rewards. This transformation does not change the normal form of the game; yet, it makes the existence of uniformly optimal (ergodic) pure positional strategies obvious, since in the canonical form each locally optimal move is provably optimal. These results were obtained for BW games in [24] and extended to the BWR case in [4], where it was also shown that BWR model is polynomially equivalent with the classical Gillette model [14]. A pseudo-polynomial algorithm that gets the canonical form by a potential transformation was obtained in [5]; see Sections 4.2 – 4.4 and [24, 4, 5] for definitions and more details.

The reduction of a two-person zero-sum AIPFOOT game $\mathcal{G}$ to a BWR game is simple. Let $u(a)$ denote the payoff (of player 1, White) in case of an outcome $a \in A = V_T(\mathcal{G}) \cup \{a_\infty\} = \{a_1, \ldots, a_p, a_\infty\}$. For Chess and Back Gammon, $u(a_\infty) = 0$, since a_∞ is a draw. In general, let us subtract constant $u(a_\infty)$ from $u(a)$ for all $a \in A$. Obviously, the obtained zero-sum game $\mathcal{G}'$ is equivalent with $\mathcal{G}$ and $u'(a_\infty) = 0$. Thus, without any loss of generality, we may assume that each infinite play is a draw, like in Chess or Back Gammon. Then, let us set the local rewards to 0 for all moves of $\mathcal{G}'$; furthermore, to each terminal a in $\mathcal{G}'$ add a loop and assign the local reward $u'(a)$ to it. Then, $\mathcal{G}'$ and the obtained BWR game are equivalent; see Section 4.5 for more details.

Thus, the statement of (ii) follows. For a similarly simple proof of (i) see Section 3.

Remark 2 *However, none of these two proofs is "from scratch": (i) is based on the results of [19], or it can be alternatively reduced to the BW model [31, 11, 24], while (ii) makes use of the more complicated machinery from [26, 14, 2, 30].*

Since Gillette and BWR models are equivalent, it might be possible to derive (ii) directly from Gillette's theorem [14]. However, we have only found a reduction via the BWR model. In fact, Chess and Back Gammon fit ideally BW and BWR models, respectively.

2 Main concepts, results, and conjectures

2.1 Modelling positional games by directed graphs

Given a finite directed graph (digraph) $G = (V, E)$ in which loops and multiple arcs are allowed, a vertex $v \in V$ is a *position* and a directed edge (or arc) $e = (v, v') \in E$ is a *move* from v to v'. A position of out-degree 0 (that is, with no moves) is called a *terminal*. We denote by V_T the set of all terminals. Let us also fix an *initial* position $v_0 \in V$.

Furthermore, let us introduce a set of n players $I = \{1, \ldots, n\}$ and a partition $P : V = V_1 \cup \ldots \cup V_n \cup V_R \cup V_T$. Each player $i \in I$ controls all positions in V_i, and V_R is the set of random positions, in which moves are not controlled by a players but by nature[33, 34]. For each $v \in V_R$ a probability distribution over the set of outgoing edges is fixed.

Let $C = C(G)$ denote the set of all simple directed cycles (dicycles) of digraph G. For instance, a loop $c_v = (v, v)$ is a dicycle of length 1, and a pair of oppositely directed edges $e = (v, v')$ and $e' = (v', v)$ form a dicycle of length 2. A directed path (dipath) p that begins in v_0 is called a *walk*. It is called a *play* if it ends in a terminal vertex $a \in V_T$, or it is infinte. Since the considered digraph G is finite, every infinite play contains infinitely repeated positions. For example, it might consist of an initial part and a dicycle repeated infinitely. Finally, a walk is called a *debut* if it is a simple path, that is no vertex is repeated.

The interpretation of this model is standard. The game starts at $v = v_0$ and a walk is constructed as follows. The player who controls the endpoint v of the current walk can add to it a move $(v, w) \in E$. If $v \in V_R$ then a move $(v, w) \in E$ is chosen according to the given probability distribution. The walk can end in a terminal position or it can last infinitely. In both cases it results in a play.

2.2 Outcomes and payoff

We study the AIPFOOT games, in which all infinite plays form a single outcome, which we will denote by a_∞ or c. Thus, $A = V_T \cup \{a_\infty\}$ is the set of outcomes, while $V_T = \{a_1, \ldots, a_p\}$ is the set of terminal positions, or terminals, of G.

A *payoff or utility function* is a mapping $u : I \times A \to \mathbb{R}$ whose value $u(i, a)$ is standardly interpreted as a profit of player $i \in I$ in case of outcome $a \in A$. A payoff is called *zero-sum* whenever $\sum_{i \in I} u(i, a) = 0$ for every $a \in A$.

The quadruple (G, P, v_0, u) will be called a *positional game*, and we call the triple (G, P, v_0) a positional game form.

Remark 3 *It is convenient to represent a game as a game form plus the payoffs. In fact, several structural properties of games, like existence of a NE, may hold for some families of game forms and all possible payoffs.*

Two-person zero-sum games will play an important role. Chess and Back Gammon are two well-known examples. In both, every infinite play is defined as a draw.

Another important special case is provided by the n-person games in which the infinite outcome a_∞ is the *worst* for all players $i \in I$. These games will be called the AIPFOOW games. They were introduced in [6] in a more general setting of additive payoffs, which is a generalization of the terminal case; see Section 2.9.

Suppose, for example, that somebody from a family I should clean the house. Whenever $i \in I$ makes a terminal move, it means that (s)he has agreed to do the work. Although such a move is less attractive for i than for $I \setminus \{i\}$, yet, an infinite play means that the house will not be cleaned at all, which is unacceptable for everybody.

Remark 4 *In absence of random moves, the values $u_i = u(i, *)$ are irrelevant, only the corresponding pseudo-orders $\succ_i$ over A matter. Moreover, in this case, ties can be eliminated, without any loss of generality. In other words, we can assume that $\succ_i$ is a complete order over A and call it the* preference *of the player $i \in I$ over A. The set of n such preferences is called the* preference profile. *However, in presence of random moves, the values $u(i, a)$ matter, since their probabilistic combinations will be compared.*

2.3 Pure, Positional, and Stationary strategies

A *(pure) strategy* x_i of a player $i \in I$ is a mapping that assigns a move $e = (v, v') \in E$ to each walk that starts in v_0 and ends in v provided $v \in V_i$. In other words, it is a "general *plan*" of player i for the whole game. A strategy x_i is called *stationary* if every time the walk ends at vertex $v \in V_i$, player i chooses the same moved. Finally, strategy x_i is called *positional* if for each $v \in V_i$ the chosen move depends only on this position v, not on the previous positions and/or moves of the walk. By definition, all positional strategies are stationary and all stationary strategies are pure. Note also that when all players are restricted to their positional strategies, the resulting play will consist of an initial part (if any) and a simple dicycle repeated infinitely. This dicycle appears when a position is repeated.

In this paper we consider mostly positional strategies. Why to do so? This restriction needs a motivation. The simplest answer is "why not?" or, to state it more politely, why to apply more sophisticated strategies in cases when positional strategies would suffice?

Remark 5 *In 1950, Nash introduced his concept of equilibrium [33, 34] and proved that it exists, in mixed strategies, for every n-person game in normal form. Yet, finite positional games with perfect information can be always solved in pure strategies. For this reason, we restrict all players to their pure strategies and do not even mention the mixed ones.*

However, restriction of all players to their pure positional strategies is by far less obvious. In some cases the existence of a Nash equilibrium (NE) in positional strategies fails; in some other it becomes an open problem; finally, in several important cases it holds, which, in our view justifies the restriction to positional strategies.

To outline such cases is one of the goals of the present paper. There are also other arguments in favor of positional strategies; for example, "poor memory" can be a reason.

- In parlor games, not many individuals are able to remember the whole debut. Solving a Chess problem, you are typically asked to find an optimal move in a given position. No Chess composer will ever specify all preceding moves. Yet, why such an optimal move does not depend on the debut, in the presence of dicycles ? This needs a prove.
- In other, non-parlour, models, the decision can be made by automata without memory.
- The set of strategies is doubly exponential in the size of a digraph, while the set of positional strategies is "only" exponential.

Remark 6 *In [6], we used term "stationary" as a synonym to "positional". Yet, it is better to reserve the first one for the repeated games or positions.*

2.4 Normal form and Nash equilibria

Let X_i denote the set of all pure positional strategies of a player $i \in I$ and let $X = \prod_{i \in I} X_i$ be the set of all *strategy profiles or situations*.

In absence of random moves, given $x \in X$, a unique move is defined in each position $v \in V \setminus V_T = \cup_{i \in I} V_i$. Furthermore, these moves determine a *play* $p = p(x)$ that begins in the initial position v_0 and results in a terminal $a = a(x) \in V_T$ or in a simple dicycle $c = c(x) \in C(G)$, which will be repeated infinitely.

The obtained mapping $g : X \to A = \{c\} \cup V_T$ is called a *positional game form*. Given a payoff $u : I \times A \to \mathbb{R}$, the pair (g, u) defines a *positional game in normal form*.

In general, random moves can exist. In this case, a Markov chain appears for every fixed $x \in X$. (Now, a play is a probabilistic realization of this chain.) One can efficiently compute the probabilities $q(x, a)$ to come to a terminal $a \in V_T$ and $q(x, c)$ of an infinite play; of course, $q(x, c) + \sum_{a \in V_T} q(x, a) = 1$ for every situation $x \in X$. Furthermore, $u(i, x) = u(i, c)q(i, c) + \sum_{a \in V_T} u(i, a)q(x, a)$ is the effective payoff of player $i \in I$ in situation $x \in X$.

Standardly, a situation $x \in X$ is called a *Nash equilibrium* (NE) if $u(i, x) \geq u(i, x')$ for every player $i \in I$ and for each strategy profile x' which might differ from x only in the ith coordinate, that is, $x'_j = x_j$ for all $j \in I \setminus \{i\}$. In other words, x is a NE, whenever no player $i \in I$ can make a profit by replacing x_i by a new strategy x'_i, provided all other players $j \in I \setminus \{i\}$ keep their old strategies x_j. Let us note that this definition is applicable in absence of random moves, as well.

A NE x, in a positional game (G, P, v_0, u) (with any type of payoff u) is called *subgame perfect or ergodic* if x remains a NE in game (G, P, v, u) for every initial position $v \in V \setminus V_T$.

Remark 7 *If G is an acyclic digraph, in which v_0 is a source (that is, each position $v' \in V$ can be reached from v_0 by a directed path) then the name "subgame perfect" is fully justified. Indeed, in this case any game (G, P, v, u) is a subgame of (G, P, v_0, u). Yet, in general, in presence of dicycles, terms* "ergodic" *or* "uniformly optimal" *would be more accurate.*

Let us call a game form (G, P, v_0) *Nash-solvable* if the corresponding game (G, P, v_0, u) has a NE for every possible utility function u.

2.5 Acyclic case: Kuhn and Gale Theorems, backward induction

In the absence of dicycles, every finite n-person positional game (G, P, v_0, u) with perfect information has a subgame perfect NE in pure positional strategies. In 1950, this theorem was proved by Kuhn [28]; see also [29]. Strictly speaking, he considered only trees, yet, the suggested method, so-called *backward induction* can easily be extended to any acyclic digraphs; see, for example, [12].

The moves of a NE are computed recursively, position by position. We start with the terminal positions and proceed eventually to the initial one. To every node and every player we shall associate a value, initialized by setting $u_i(v) = u(i, v)$ for all terminals $v \in V_T$.

We proceed with a position $v \in V$ after all its immediate successors $w \in S(v)$ are done. If $v \in V_i$ then we set $u_i(v) = \max(u_i(w) \mid w \in S(v))$, and chose $w \in S(v)$ realizing this maximum, and set $u_j(v) = u_j(w)$ for all players $j \in I$. If $v \in V_R$ then we set

$$u_i(v) = mean\,(u_i(w) \mid w \in S(v)) \;\; = \sum_{w \in S(v)} p(v, w) u_i(w) \;\text{ for all }\; i \in I.$$

By construction, the obtained situation x is a subgame perfect NE. From now on, we will assume that the considered games have dicycles, otherwise there is nothing to prove.

Let us note that backward induction may fail whenever the digraph G contains a dicycle. Yet, in Section 3.3 we will extend this procedure to the two-person zero-sum AIPFOOT games, which naturally may have dicycles.

Remark 8 *In 1953, Gale proved that backward induction for an acyclic positional game is equivalent with eliminating dominated strategies in its normal form. He proved that the procedure results in a single situation, which is a special NE, [12], so-called domination or sophisticated equilibrium. In his proof, Gale considered acyclic digraphs, not only trees. He did not consider positions of chance but mentioned that they could be included, as well.*

2.6 Zermelo's Theorem

In 1912, Zermelo gave his seminal talk "On applications of set theory to Chess" [37]. However, the results and methods are applicable not only to Chess but to any two-person zero-sum game with perfect information and without moves of chance. The main result was the existence of the value and a saddle point in pure strategies for a fixed initial position.

Naturally, in 1912, Zermelo considered only two-person zero-sum case, since the concept of NE for n-person games was introduced much later, only in 1950. Yet, unlike Kuhn and Gale, Zermelo did not exclude dicycles. Indeed, a position can be repeated in a Chess-play. In this case the play is defined as a draw. In other words, Chess is an AIPFOOT game.

Remark 9 *Accurately speaking, a play in Chess is claimed a draw only after a position is repeated* three times. *However, if both players are restricted to their positional strategies, a position will be repeated infinitely whenever it appears twice.*

2.7 On Nash-solvability in absence of random moves

What can we say about existence of a NE in pure *positional* strategies in finite n-person positional AIPFOOT games that are deterministic (have no moves of chance), yet, might have dicycles? Somewhat surprisingly, this question was not asked until very recent times. The answer is positive in the two-person case.

Theorem 1 *Every two-person AIPFOOT game without random moves has a NE in pure positional strategies. Moreover, these strategies can be chosen uniformly optimal in the zero-sum case, or in other words, a subgame perfect saddle point exists in this case.*

The first claim was shown in [6], for the AIPFOOW games. For the AIPFOOT games it was proven by Gimbert and Sørensen; private communications. With their permission, a simplified version of the proof is given in Sections 5 of [1]. This version is based on an algebraic criterion of Nash-solvability for the normal two-person game forms [19]; see also [20, 3], and Section 3.1 for more details. The second statement of Theorem 1 is shown in Section 3.3. The proof is based on an extension of the backward induction procedure to the two-person zero-sum AIPFOOT games, which naturally may have dicycles. Let us notice that this statement results also from Theorem 2, which will follow; see Sections 2.10 and 3.1. Yet, the proof of Theorem 2 is much more difficult than one in Section 3.3.

Furthermore, even a unique NE might be not subgame perfect in a *non-zero-sum* two-person game without moves of chance. The following example is borrowed from [1].

Example 1 *Consider the game shown in Figure 1. None of its four situations is a NE for both initial positions. Yet, for each* fixed *initial position v_0 there is a unique NE.*

In the n-person case, even the existence of a NE in pure positional strategies (Nash-solvability) becomes an open problem.

Conjecture 1 *An n-person AIPFOOT game without moves of chance has a (not necessarily subgame perfect) NE in pure positional strategies.*

This conjecture is confirmed only in a few special cases. In [1], it was shown for the so-called flower games; see Sections 1.3, 2.4, and 5 of [1] for definitions and more deatails. In [6], it was shown for the AIPFOOW deterministic n-person games such that:

(iii) each player controls only one position, so-called *play-once* games; *or*

(iv) there are at most two players, $n \leq 2$; *or*

(v) there are at most two terminals, $|V_T| \leq 2$.

The last result was recently strengthened in [8] to (v′) $|V_T| \leq 3$.

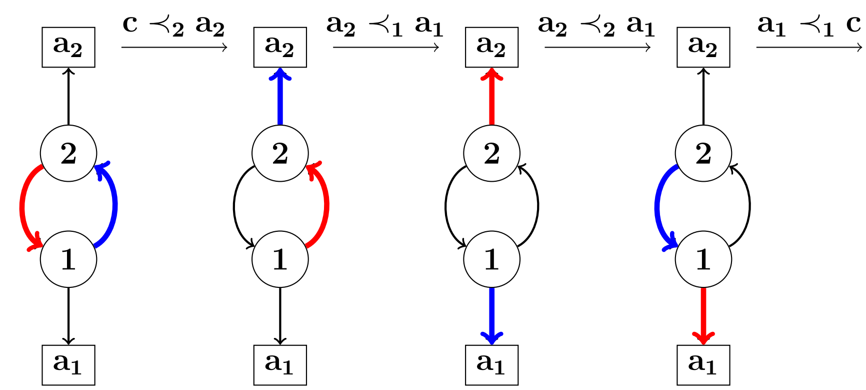


Figure 1: **A two-person game with one dicycle, two terminals, and no subgame perfect NE; the preferences are as follows:** $u_1: \ a_2 \prec a_1 \prec c$ **and** $u_2: \ c \prec a_2 \prec a_1$**.** In other words, player 1 prefers the cycle c the most, while player 2 dislikes it the most, and both players prefer a_1 to a_2. Thick lines represent the chosen strategies; the red one indicates the choice which the player could switch and get a profit. Between the situations we indicated the preference which make such a switch possible. In the first and third (the second and fourth) situations player 2 (respectively, 1) can improve. Since there are only four situations, none of them is a NE for both initial positions simultaneously. Yet, for each *fixed* initial position v_0 there is a NE. If v_0 is the position controlled by player 1 then the third situation is a NE, while if v_0 is controlled by player 2 then the second one is a NE.

2.8 Additive payoffs

In fact, claims (iii) and (iv) were proven in [6] in a more general setting. Given a digraph $G = (V, E)$, let us define a *local reward* as a mapping $r : I \times E \to \mathbb{R}$. Standardly, the value $r(i, e)$ is interpreted as the profit obtained by player $i \in I$ whenever the play passes $e \in E$.

Let us recall that, in absence of random moves, each situation $x \in X$ defines a unique play $p = p(x)$ that begins in the initial position v_0 and either terminates at $a(x) \in V_T$ or results in a simple dicycle $c = c(x)$. The *additive effective payoff* $u : I \times X \to \mathbb{R}$ is defined in the former case as the sum of all local rewards of the obtained play, $u(i, x) = \sum_{e \in p(x)} r(i, e)$, and the latter case it is $u(i, x) \equiv -\infty$ for all $i \in I$. In other words, all infinite plays are equivalent and ranked as the worst by all players, that is, we obtain a natural extension of AIPFOOW games. Let us note however that in the first case payoffs depend not only on the terminal position $a(x)$ but on the entire play $p(x)$.

The following two assumptions were considered in [6]:

(vi) all local rewards are non-positive, $r(i, e) \leq 0$ for all $i \in I$ and $e \in E$ and

(vii) all dicycles are non-positive, $\sum_{e \in c} r(i, e) \leq 0$ for all dicycles $c \in C = C(G)$.

Obviously, (vi) implies (vii). Moreover, it was shown in 1958 by Gallai [13] that in fact these two assumptions are equivalent, since (vi) can be enforced by a potential transformation whenever (vii) holds; see [13] and also [6] for definitions and more details.

Remark 10 *In [6], all players $i \in I$ minimize cost function $-u(i, x)$ instead of maximizing payoff $u(i, x)$. Hence, conditions (vi) and (vii) turn into non-negativity conditions in [6].*

In [6], statements (iii) and (iv) are proven for the additive payoffs under assumptions (vi) and (vii); also several examples are given showing that these assumptions are essential.

It was also observed in [6] that the terminal AIPFOOW payoffs is a special case of the additive AIPFOOW payoffs. To see this, let us just set $r(i, e) \equiv 0$ unless e is a terminal move and notice also that no terminal move can belong to a dicycle. Hence, conditions (vi) and (vii) hold automatically in the special case of terminal AIPFOOW games. Thus, statements (iii) and (iv) follow from the corresponding claims proven in [6].

Remark 11 *This reduction is similar to one that was sketched in the end of Introduction; it will be instrumental in the proof of Theorem 2.*

In [6], Conjecture 1 is suggested for the additive AIPFOOW payoffs.

Conjecture 2 *An AIPFOOW game with additive payoffs and without moves of chance has a (not necessarily subgame perfect) NE in pure positional strategies whenever conditions (vi) and (vii) hold.*

It was demonstrated in [6] that restrictions (vi), (vii), and AIPFOOW are essential.

Remark 12 *Additive payoffs are considered only in this section; furthermore, other modifications, such as cyclic and mean payoffs, will be considered in Sections 2.11 and 4.2 – 4.3, respectively; while the rest of the paper deals only with terminal AIPFOOT payoffs.*

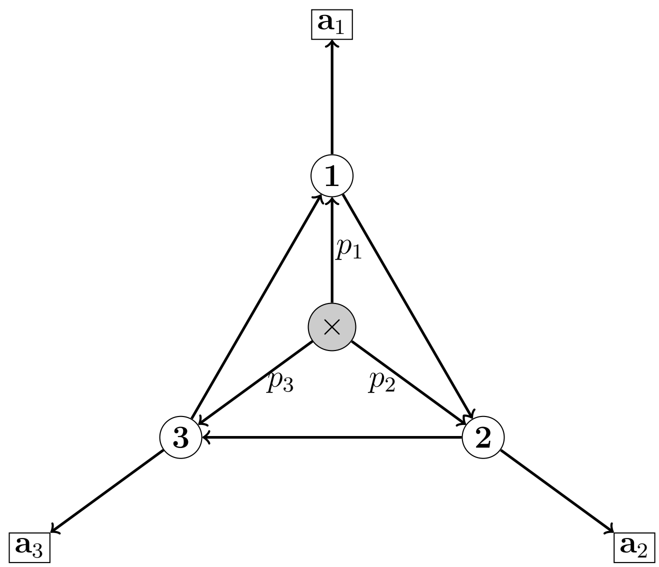


Figure 2: **A three-person AIPFOOW game with one dicycle, one random position, and without NE in pure positional strategies.** Each player $i \in I = \{1, 2, 3\}$ has only two options : either to move along the dicycle c or to terminate in a_i. The last option is the second in the preference list of i; it is better (worse) if the next (previous) player terminates, while the dicycle itself is the worst option for all. In other words, the preference profile is $u_1 : a_2 \succ a_1 \succ a_3 \succ c$, $u_2 : a_3 \succ a_2 \succ a_1 \succ c$, $u_3 : a_1 \succ a_3 \succ a_2 \succ c$.
Finally, there is a position of chance (in the middle) in which there are three moves to v_1, v_2, v_3 with *strictly positive* probabilities p_1, p_2, p_3, respectively.
This game has no NE in pure positional strategies; see Example 2.
However, there is a NE in pure stationary, but not positional, strategies.

2.9 On Nash-solvability in presence of dicycles and random moves

Let us start with an example from [6], which we reproduce here in Figure 2.

Example 2 *Let us consider the game described in Figure 2. Its normal form is of size* $2 \times 2 \times 2$*, since each of three players has two positional strategies. Let us show that none of the eight situations is a NE. First, let us consider two situations: all three players terminate or all three move along the dicycle* c*. Obviously, each of these two situations can be improved by any one of the three player.*

Now, let us show that none of the remaining six situations is a NE either. For example, consider the situation in which player 1 *terminates, while* 2 *and* 3 *proceed. Then, player* 2 *is unhappy and can improve the situation by choosing termination. Yet, after this, player* 1 *can switch to move along* c *and improve again. Thus, we arrive to a situation in which player* 2 *terminates, while* 3 *and* 1 *proceed. Clearly, this situation is just the clockwise shift by* $120°$ *of the one we started with. Hence, after repeating the same procedure two more times, we get the so-called improvement cycle including all six considered situations.*

Yet, this game has a NE $x = (x_1, x_2, x_3)$ *in pure stationary, but not positional, strategies. This strategy* x_i*,* $i \in I = \{1, 2, 3\}$*, requires to terminate in* a_i *whenever the play comes to* v_i *from* v_0*, and to proceed along* c *to* v_{i+1} *whenever the play comes to* v_i *from* v_{i-1} *(where standardly the indices are taken modulo* 3*).*

By definition, all these strategies x_i*,* $i \in I$*, are pure and stationary but not positional.*

Let us show that the obtained situation x *is a NE. Indeed, each player* i *could try to improve and get his best outcome* a_{i+1} *instead of* a_i*, which is his second best. Yet, to do so, this player* i *needs to proceed along* c *rather than terminate at* a_i*. Then, by definition of* $x = (x_1, x_2, x_3)$*, the other two players would also proceed along* c*. Thus, the play would result in* c*, which is the worst outcome for all. (Compare this example to the house-cleaning example from Section 2.2.)*

Let us note that the above game has only one random position and one dicycle, which is the worst outcome for all players. Furthermore, the game is play-once, that is, each of the three players controls only one position. Thus, this example leaves no hopes for Nash-solvability of n-person AIPFOOW games with dicycles and random moves, when $n \geq 3$.

Therefore, our main result (and hopes) are related to the two-person case; yet, even then one should not be too optimistic, as the following example shows.

Example 3 *Let us consider the game form given in Figure 3, which is similar to the Example 2, except* n *is reduced from* 3 *to* 2*. The corresponding game has no NE in pure positional strategies only in cases indicated in Figure 3. Yet, even in these cases, there is a NE in pure (but not stationary and, hence, not positional) strategies. Indeed, the profiles* $u = (u_1, u_2)$ *shown in Figure 3 and their reverse are symmetric; hence, without any loss of generality, we can restrict ourselves to the first one.*

Then, let us consider the following situation $x = (x_1, x_2)$*: strategy* x_1 *instructs player* 1 *always to terminate in* a_1*, while* x_2 *requires player* 2 *to move along the dicycle* c *whenever*

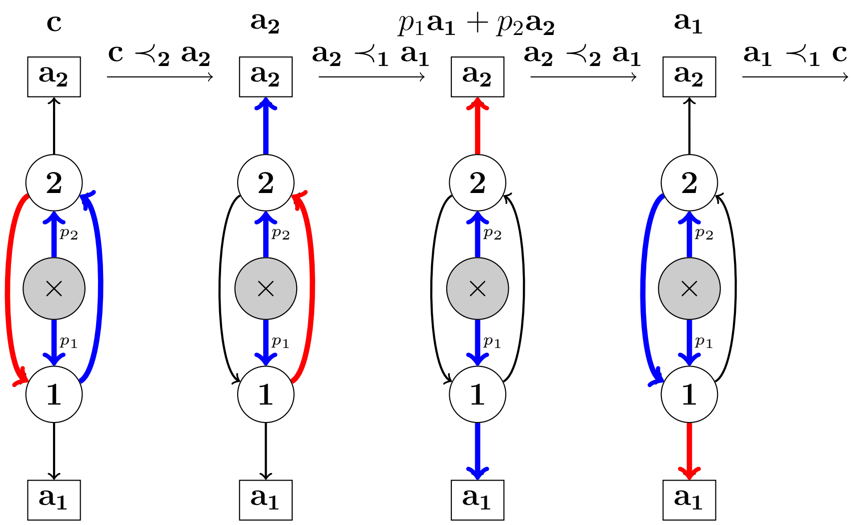


Figure 3: **A two-person game form with one dicycle and one position of chance.** The corresponding game has a NE in pure positional strategies unless all its four situations form an improvement cycle. This happens, indeed, if and only if the preference profile is $u_1 : c \succ a_1 \succ a_2$ and $u_2 : a_1 \succ a_2 \succ c$ or, their reverse permutations. Yet, in both cases the obtained game is not zero-sum and not AIPFOOW. The initial position v_0 is in the center (shaded) and it is a random node moving down with probability $p_1 > 0$ and up with probability $p_2 > 0$, where $p_1 + p_2 = 1$. Thick lines represent the chosen strategies; the red line indicates the choice that the corresponding player can change to get a profit. Above each situation, we indicate the corresponding outcome and the preference relation allowing the improvement. In the first and third (the second and fourth) situations player 2 (respectively, 1) can improve. Thus, for the initial position v_0, none of the four situations is a NE.

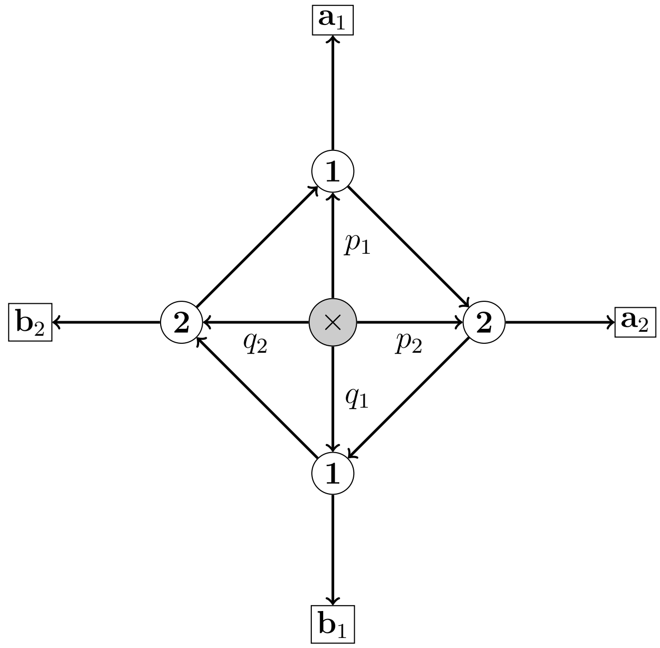


Figure 4: **A two-person terminal game form with one position of chance and one dicycle** c that contains four positions v_1, v_2, w_1, w_2 controlled by players 1 and 2, respectively. In each of these positions there are two options: either to proceed along c or terminate in a_1, a_2, b_1, b_2, respectively. Finally, the initial position v_0 is a random one. The four moves from it lead to v_1, v_2, w_1, w_2 with probabilities p_1, p_2, q_1, q_2, respectively. Each of the two players has four strategies; the 4×4 normal form is given in Table 1.

the play came to v_2 from the random position v_0 and to terminate in a_2 if the play came to v_2 from v_1. By definition, x_1 is a pure positional strategy, while x_2 is pure but not even stationary. Let us show that x is a NE. Indeed, x always result in a_1. Player 2 cannot improve, because a_1 is his best outcome. Although player 1 could try to improve, and get c rather than a_1 by switching to the move (v_1, v_2) from (v_1, a_1), as required by x_1, but then player 2 would terminate in a_2 as required by x_2, resulting in a worse outcome for both.

Remark 13 *In both Examples, 2 and 3, the move in a current position v depends only on v and the previous position v'. In other words, the considered equilibrium strategies are not positional but they require to remember only the last move.*

In Example 3 a NE exists when the payoff is zero-sum or when it is AIPFOOW, that is, the dicycle c is the worst outcome for both players. Is this true in general?

Open question: Whether each two-person AIPFOOW game (possibly with dicycles and random moves) has a NE in pure positional strategies?

The next example indicates that the answer might be positive.

Example 4 *Let us consider the game given in positional and normal forms in Figure 4 and Table 1, respectively. By analysis of the Table, it is not difficult to verify that a NE exists whenever c is the worst outcome for both players (the AIPFOOW case), as well as in the zero-sum case. In the latter case, simple criteria obtained in [23] can be applied.*

2.10 Main Theorem

In the zero-sum case Nash-solvability holds, indeed:

Theorem 2 *Every two-person zero-sum AIPFOOT game (which might have moves of chance and dicycles) has a saddle point in pure positional uniformly optimal strategies.*

The proof will be given in Section 4.5. It is based on recent results on solvability of two-person zero-sum stochastic games with perfect information represented by the so-called BWR model [4, 5]. Let us note that Theorem 2 implies the second part of Theorem 1.

Let us also remark that Chess and Back Gammon satisfy all conditions of Theorem 2.

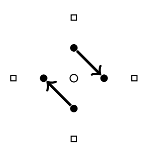
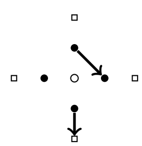
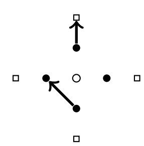
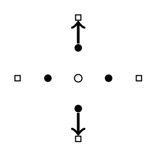
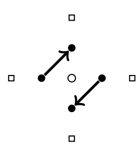
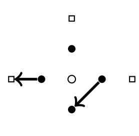
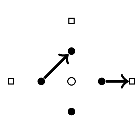
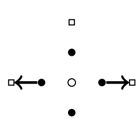
				
	$\mathbf{c}$	$\mathbf{b}_1$	$\mathbf{a}_1$	$(q_2+p_1)\mathbf{a}_1$ $+(p_2+q_1)\mathbf{b}_1$
	$\mathbf{b}_2$	$(p_1+p_2+q_1)\mathbf{b}_1$ $+q_2\mathbf{b}_2$	$(p_2+q_1+q_2)\mathbf{b}_2$ $+p_1\mathbf{a}_1$	$(p_2+q_1)\mathbf{b}_1$ $+p_1\mathbf{a}_1+q_2\mathbf{b}_2$
	$\mathbf{a}_2$	$(q_2+p_1+p_2)\mathbf{a}_2$ $+q_1\mathbf{b}_1$	$(q_2+p_1+q_1)\mathbf{a}_1$ $+p_2\mathbf{a}_2$	$(p_1+q_2)\mathbf{a}_1$ $+p_2\mathbf{a}_2+q_1\mathbf{b}_1$
	$(p_1+p_2)\mathbf{a}_2$ $+(q_1+q_2)\mathbf{b}_2$	$(p_1+p_2)\mathbf{a}_2$ $+q_1\mathbf{b}_1+q_2\mathbf{b}_2$	$(q_1+q_2)\mathbf{b}_2$ $+p_1\mathbf{a}_1+p_2\mathbf{a}_2$	$p_1\mathbf{a}_1+p_2\mathbf{a}_2$ $+q_1\mathbf{b}_1+q_2\mathbf{b}_2$

Table 1: Normal form of the 2-person game of Figure 4.

2.11 Nash-solvability of n-person AIPFOOT games in pure but not necessarily positional strategies; Folk Theorem

It is easy to get a (not necessarily subgame perfect) NE in pure and stationary (but not necessarily positional) strategies in any AIPFOOT n-person game $\mathcal{G} = (G, P, v_0, u)$ without random moves, for example, in Chess. The digraph G of such a game can be "unfolded" and, thus, represented as a tree: Let us assume, without any loss of generality, that each position v of G can be reached by a simple directed path (dipath) d from v_0; recall that d is called a *debut*. Let us add to d one more move (v, v'); the obtained dipath d' will be called a *lasso* if v' belongs to d. Let us assign a position $v(d)$ to every debut d (including all finite plays) and a terminal position $v(d')$ to each lasso d'. Furthermore, let us draw a directed edge from v to v'' whenever $d(v'')$ is an extension (a lasso or not) of $d(v)$. It is easily seen that the obtained digraph G' is a directed *tree* in which $d(v_0)$ is a root. Let us also note that several positions of G' may correspond to debuts of G ending at the same position.

A game $\mathcal{G}' = (G', P', v_0', u')$ is defined in an obvious way. The terminal vertices of G' correspond to the lassos or finite plays, terminating at $a \in V_T$. Let us assign payoffs by associating $u(i, a_\infty)$ for to a lasso and $u(i, a)$ to a finite play, for all $i \in I$.

A NE in the obtained game $\mathcal{G}'$ can be determined by the classical backward induction. Obviously, the corresponding strategies in the original game $\mathcal{G}$ are pure (and even stationary) but they may not be positional. Indeed, by construction, a move in a position $v = v(d)$ is chosen as a function of the debut d, not only of v.

Moreover, a subgame perfect NE in pure (but not necessarily positional or even stationary) strategies exists in each AIPFOOT n-person game, which might have both dicycles and random moves; see Examples 2 and 3. This result is usually referred to as the "Folk Theorem"; so we shall not provide any other references. Let us derive it from Theorem 2.

For each player $i \in I$ denote by u_i^0 the maximal payoff that player i can guarantee in pure positional strategies and let x_i be such a strategy. The corresponding situation $x = (x_i \mid i \in I)$ defines a Markov chain. Let u_i denote the expected payoffs of the player $i \in I$ in this Markov chain. By definition, $u_i \geq u_i^0$ for all $i \in I$. Furthermore, by the same definition and Theorem 2, each other player $j \in I \setminus \{i\}$ has a *positional and ergodic* "punishing" strategy x_j^i such that i gets at most u_i in the situation $(x_i) \cup (x_j^i \mid j \in I \setminus \{i\})$. This follows, because "punishing of player i" can be viewed as a two-person zero-sum game of i against the complementary coalition $I \setminus \{i\}$, where all coalitionists have the same preference, which is opposite to the preference of player i.

Then, let us modify the strategies x_i "slightly" to get strategies y_i defined for all $i \in I$ as follows: $y_i = x_i$ until each player $i \in I$ applies x_i; yet, if a player $i \in I$ deviates from x_i then all other players $j \in I \setminus \{i\}$ immediately switch to their joint punishing strategies x_j^i. Let us underline that the player who brakes the rule *first* is to get punished.

By construction, the situation $y = (y_i \mid i \in I)$ is a subgame perfect NE; moreover, all strategies y_i, $i \in I$, are pure and ergodic but not necessarily positional or even stationary.

Remark 14 *By Theorem 2, all strategies x_i and x_j^j were chosen positional. However, the resulting strategies y_i are not even stationary, anyway. Hence, to prove Folk Theorem, it* *would suffice to solve the considered two-person zero-sum "punishing games" in pure but not necessarily positional strategies. Of course, the existence of such strategies can be shown much simpler, without Theorem 2.*

2.12 On Nash-solvability of bidirected cyclic two-person games

For completeness, let us survey one more result on Nash-solvability obtained in [7].

Let payoff $u : I \times (C \cup V_T) \rightarrow \mathbb{R}$ be an arbitrary function, where $C = C(G)$ denotes the set of dicycles of G. In this model, every dicycle $c \in C$, as well as each terminal $a \in V_T$, is a *separate* outcome, in contrast to the AIPFOOT case.

A digraph G is called *bidirected* if (v, v') is its edge whenever (v', v) is.

Necessary and sufficient conditions of Nash-solvability were announced in [15, 16, 17] for the bidirected *bipartite* cyclic two-person game forms. Recently, it was shown that the bipartitedness is in fact irrelevant. Necessary and sufficient conditions for Nash-solvability of bidirected cyclic two-person game forms were shown in [7].

3 Proof of Theorem 1

3.1 Equivalence of Nash- zero-sum- and ±1-solvabilities

Let us recall some of the basic definitions. Given a set of players $I = \{1, \dots, n\}$ and outcomes $A = \{a_1, \dots, a_p\}$, an n-person game form g is a mapping $g : X \rightarrow A$, where $X = \prod_{i \in I} X_i$ and X_i is a finite set of strategies of player $i \in I$. Furthermore, a utility or payoff function is a mapping $u : I \times A \rightarrow \mathbb{R}$. Standardly $u(i, a)$ is interpreted is a profit of player $i \in I$ in case of outcome $a \in A$. A payoff u is called *zero-sum* if $\sum_{i \in I} u(i, a) = 0$ for all $a \in A$.

The pair (g, u) is called a *game in normal form.*

Given a game (g, u) a strategy profile $x \in X$ is a NE if $u(i, g(x)) \geq u(i, g(x'))$ for every $i \in I$ and every x' that differs from x only in coordinate i. A game form g is called *Nash-solvable* if for every utility function u the obtained game (g, u) has a NE.

Furthermore, a two-person game form g is called:

- *zero-sum-solvable* if for each zero-sum utility function the obtained zero-sum game (g, u) has a NE (which is called a saddle point for the zero-sum games);

- ± 1-*solvable* if zero-sum solvability holds for each u that takes only values $+1$ and -1.

Necessary and sufficient conditions for zero-sum solvability were obtained by Edmonds and Fulkerson [9] in 1970; see also [18]. Somewhat surprisingly, these conditions remain necessary and sufficient for Nash-solvability as well [19], see also [20] and [3]. Moreover, all three types of solvability are equivalent for the two-person game forms but, unfortunately, not for the three-person ones [19, 20, 3].

3.2 Proof of the first statement of Theorem 1

We want to prove that every two-person AIPFOOT game without random moves has a NE in pure positional strategies.

Let $\mathcal{G} = (G, P, v_0, u)$ be such a game, in which $u : I \times A \to \{-1, +1\}$ is a zero-sum ± 1 utility function. As we just mentioned, it would suffice to prove solvability in this case [19].

Let $A_i \subseteq A$ denote the outcomes winning for player $i \in I = \{1, 2\}$. Let us also recall that $V_i \subseteq V$ denotes the subset of positions controlled by player $i \in I = \{1, 2\}$.

Without any loss of generality, we can assume that $c \in A_1$, that is, $u(1, c) = 1$, while $u(2, c) = -1$, or in other words, player 1 likes dicycles. Let $W^2 \subseteq V$ denote the set of positions in which player 2 can enforce (not necessarily in one move) a terminal from A_2, and let $W^1 = V \setminus W^2$. By definition, player 2 wins whenever $v_0 \in W^2$. Let x_2 denote such a winning strategy; note that x_2 can be defined arbitrarily in $V_2 \cap W^1$.

We have to prove that player 1 wins whenever $v_0 \in W^1$. Indeed, for an arbitrary vertex v, if $v \in W^1 \cap V_2$ then player 2 cannot leave W^1, that is, $v' \in W^1$ for every move $(v, v') \in E$. Furthermore, if $v \in W^1 \cap V_1$ then player 1 can stay in W^1, that is, (s)he has a move $(v, v') \in E$ such that $v' \in W^1$. Let player 1 choose such a move for every position $v \in W^1 \cap V_1$ and arbitrary moves in all remaining positions, from $W^2 \cap V_1$. This rule defines a strategy x_1 of player 1. Let us show that x_1 wins whenever $v_0 \in W^1$. Indeed, in this case the play cannot enter W^2. Hence, it either will terminate in A_1 or result in a di cycle; in both cases player 1 wins. Thus, player 1 wins when $v_0 \in W^1$, while player 2 wins when $v_0 \in W^2$. □

Remark 15 *We proved a little more than we planed to, namely, in case of ± 1 zero-sum payoffs the obtained strategies x_1 and x_2 are positional and uniformly optimal, or in other words, that situation $x = (x_1, x_2)$ is a subgame perfect saddle point. Moreover, in the next section we will extend this result to all (not only ± 1 zero-sum games. However, it cannot be extended further, since a non-zero-sum two-person AIPFOOT game might have, in pure positional strategies, a unique NE, which is not subgame perfect; see Example 1.*

3.3 Backward induction in presence of dicycles and proof of the second statement of Theorem 1

As we already mentioned, the second part of Theorem 1 is a special case of Theorem 2. Yet, it can be proved simpler, by a generalization of the arguments from the previous section.

We want to show that every two-person zero-sum AIPFOOT game without random moves has a saddle point in pure positional uniformly optimal strategies. Let $\mathcal{G} = (G, P, v_0, u)$ be such a game. The results of the previous two sections imply that the value $\mu = \mu(v)$ and optimal positional pure strategies $x_1(v)$ and $x_2(v)$ exist for every initial position v. Yet, we still have to prove that there are *positional uniformly optimal* strategies, as well.

In the previous section, this was already shown for the zero-sum ± 1 payoffs. Now, we extend this proof to the case of arbitrary zero-sum payoffs. In fact, we extend the backward induction procedure to work in the presence of dicycles. However, this extension will work only for the two-person zero-sum AIPFOOT games without moves of chance.

Let us first determine the value $\mu(v)$ for each possible initial position $v \in V$.

Since the game is zero-sum, we simplify notation by introducing $u(1, a) = u(a)$ and $u(2, a) = -u(a)$ for every outcome $a \in A = V_T \cup \{a_\infty\}$, and view player 1, White, as a maximizer, and player 2, Black, as a minimizer.

We shall apply two reduction phases, recursively:

Phase 1. Let us start with the standard backward induction and proceed until possible: Given a position $v \in V_1$ (respectively, $v \in V_2$) such that every move (v, a) from v leads to a terminal position $a \in V_T$, let us chose in v a move $(v, a) = x_1(v)$ maximizing (respectively, $(v, a) = x_2(v)$ minimizing) the value $u(a)$. Obviously, such a move should be required by any uniformly optimal strategy. Then let us delete all moves from v, thus, making it a terminal position, and define payoff $u(v) = u(a)$ in it. This (backward induction) procedure can be repeated until either:

(viii) we got rid of all arcs, or

(ix) a non-terminal move exists in every non-terminal position $v \in V_1 \cup V_2 = V \setminus V_T$.

Case (viii) means that G is an acyclic digraph and pure positional uniformly optimal strategies x_1 and x_2 in game $\mathcal{G}$ are found by the standard backward induction. Otherwise, we obtain a reduced game $\mathcal{G}'$ satisfying condition (ix).

Furthermore, each pair (x'_1, x'_2) of pure positional uniformly optimal strategies in game $\mathcal{G}'$ can obviously be extended to such strategies in the original game $\mathcal{G}$, by selecting the above chosen moves. Hence, condition (ix) can be assumed without any loss of generality.

Phase 2. Let us set $\mu_1 = \max\{\mu(v)\}$ and $\mu_2 = \min\{\mu(v)\}$ where max and min are taken over all positions $v \in V_1 \cup V_2 = V \setminus V_T$, and define $W_1 = \{v \mid \mu(v) = \mu_1\}$ and $W_2 = \{v \mid \mu(v) = \mu_2\}$. Let us recall that $u(a_\infty)$ is the payoff in case of an infinite play. By (ix), we obtain:

$$\begin{array}{ll} W_1 \cap V_1 \neq \emptyset & \text{if and only if } \mu_1 \geq u(a_\infty) \text{ and} \\ W_2 \cap V_2 \neq \emptyset & \text{if and only if } \mu_2 \leq u(a_\infty). \end{array}$$

If $W_i \cap V_i \neq \emptyset$ for some $i \in I = \{1, 2\}$ then we must have a position $v \in W_i \cap V_i$ in which there is a terminal move (v, a), $a \in V_T$, such that $u(a) = \mu(v) = \mu_i$. In other words, move (v, a) gives player i the best possible result; in particular, it is not worse than an infinite play. Obviously, such a move can be chosen by any uniformly optimal strategy.

Then, let us set $x_i(v) = (v, a)$, delete all moves from v, thus, making it a terminal position, and define payoff $u(v) = u(a)$ in it.

Let us note that, after this, we may be able to perform some reductions by Phase 1 again.

If we arrive again to condition (ix), we just repeat the above Phase 2 reduction again.

Let us also remark that, by repeating these steps, we shrink the $[\mu_1, \mu_2]$ interval.

It is easy to see that, after a finite number of steps, either

(x) we get rid of all arcs, or

(xi) each terminal move is strictly worse for the controlling player than an infinite play.

In both cases, we can obtain pure positional uniformly optimal strategies.

In case (x), it is obvious that the strategies x_1 and x_2 have all required properties. Otherwise, we obtain a reduced game $\mathcal{G}''$ satisfying condition (xi). In this case pure positional uniformly optimal strategies x_1'' and x_2'' exist, since it is enough for both players to choose a non-terminal move in each position, thus, yielding an infinite play which realizes the value $\mu = u(a_\infty)$. The constructed strategies x_1'' and x_2'') in $\mathcal{G}''$ can obviously be extended to pure positional uniformly optimal strategies in the original game $\mathcal{G}$. □

Remark 16 *In fact, our arguments combine and extend the original Zermelo, Kuhn, and Gale approaches [37, 28, 12]. We just treat dicycles properly, to obtain strategies that are both* positional *and* uniformly optimal.

4 Proof of Theorem 2

4.1 Mean payoff n-person games

Let us recall the model of Section 2.1 and introduce the following modifications. First, let us get rid of the terminals; to do so, we just add a loop (v, v) to each terminal $v \in V_T$.

Then, as in Section 2.9, let us introduce a local reward $r : I \times E \to \mathbb{R}$, which value $r(i, e)$ is standardly interpreted as a profit obtained by the player $i \in I$ whenever the play passes the arc e. A reward function r is called

- *zero-sum* if $\sum_{i \in I} r(i, e) = 0$ for all $e \in E$.
- *terminal* if $r(e) = 0$ unless e is a loop.

Assume that all players are restricted to their pure *positional* strategies. Then, each situation $x = (x_i \mid i \in I) \in X = \prod_{i \in I} X_i$ defines a Markov chain on G. This chain has a limit distribution, which defines the expected reward $u(x) = (u(i, x) \mid i \in I)$. Function $u(i, x)$ is the *effective payoff* in situation x, and called the *mean* payoff.

As we know, in the deterministic case, when $V_R = \emptyset$, a situation x defines a play $p(x)$ that results in a lasso, since there are no terminals. In this case, the effective payoff is the average (mean) payoff over the dicycle $c = c(x)$ of this lasso:

$$u(i, x) = |c(x)|^{-1} \sum_{e \in c(x)} r(i, e) \text{ for all players } i \in I \text{ and situations } x \in X,$$

where $|c|$ is the length of a dicycle c, that is, the number of its vertices or edges.

In the literature on mean-payoff games, usually the two-person zero-sum case is considered, since not much is known in other cases. We also will restrict ourselves to this case too, except for Section 4.3. Thus, we arrive to the BWR model, as it was defined earlier. Recall that B and W stand for "Black" (player 2) who is minimizing, and "White" (player 1) who is maximizing, respectively, while R means "random". Since $r(1, e) + r(2, e) = 0$ for all $e \in E$, it will be convenient to set $r(e) = r(1, e)$ just remembering that $r(2, e) = -r(e)$.

4.2 Mean payoff games, BW model

In the deterministic case, when $V_R = \emptyset$, we obtain the so called mean payoff BW games.

In 1976, Moulin introduced these games in his PhD Thesis [31]; see also [32]. In fact, he studied strategic extensions of matrix games, rather than stochastic games, and asked the question: which extensions have always a saddle point? The classical mixed extension gives an example. Yet, this extension is infinite. Can it be finite?

Given a $k \times \ell$ matrix, Moulin introduced a large, but finite, $k^\ell \times \ell^k$ extension, in which a strategy of a player is a function of (called as reply to) the opponent's strategy. Given such two reply functions and an initial strategy v_0 in the original $k \times \ell$ matrix game, both players take turns choosing their replies and this process results in an infinite play. Since the matrix is finite, this play consists of an initial part, if any, and an infinitely repeated cycle. The effective payoff is defined as the mean payoff over this cycle. Then, from the Brouwer Fixed Point Theorem, Moulin derived the existence of a saddle point in pure strategies for the obtained $k^\ell \times \ell^k$ matrix game.

In fact, Moulin's games are the BW games on complete bipartite digraphs. In addition, the equality $r(e) = r(e')$ must hold for every pair of "opposite" moves $e = (v_1, v_2)$ and $e' = (v_2, v_1)$, where positions V_1 and V_2 are the strategies of the players 1 and 2, respectively.

Moulin also proved that such a game is ergodic, that is, its value does not depend on the original strategy v_0. For this reason, he called the obtained games *ergodic extensions of matrix games*. Yet, this name appears not very lucky, because ergodicity holds for the complete bipartite digraphs but for some other digraphs, the value might depend on the initial position. In fact, ergodic game forms are fully characterized in [25]. However, it is important to note that in any BW game there are *ergodic* (or *uniformly optimal*) strategies, which form a saddle point for every initial position v_0.

Moreover, such pure positional uniformly optimal strategies exist not only in Moulin's model but in general, for any BWR game. This fact will imply Theorem 2.

In 1979, Ehrenfeucht and Mycielski introduced BW games for all (not only complete) *bipartite* graphs and gave a combinatorial proof of the existence of a saddle point [11]. Their, very short abstract appeared already in 1973 [10].

In 1988, the model was extended to arbitrary (not only bipartite) digraphs in [24].

Given a real potential $\pi : V \to \mathbb{R}$, the local reward function $r : E \to \mathbb{R}$ can be transformed by the formula $r_\pi(v, v') = r(v, v') + \pi(v) - \pi(v')$. Such transformation changes the normal form of the game in a trivial way: the constant $\pi(v_0)$ is added to all efficient payoffs.

In [24] an algorithm of potential reduction was suggested that brings the game to a canonical form in which the solution is obvious, since every locally optimal move is just optimal. Thus, we obtain *uniformly* optimal strategies of both players. In contrast, the value of a BW game might depend on the initial position.

BW games are of serious interest for complexity theory, because the decision problem "whether the value of a BW game is positive" belongs to NP $\cap$ coNP [27], yet, no polynomial algorithm is known yet; see, for example, the survey [36].

4.3 A NE-free non-zero-sum two person mean payoff game

Let us continue by observing that Nash-solvability fails for the non-zero-sum mean payoff BW games. Already the ergodic extension of a 3×3 bimatrix game might not have NE. The following example was given in [21]; see also [24]:

$$\begin{matrix} 0 & 0 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \end{matrix} \qquad\qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-\varepsilon & 0 & 1 \end{matrix}$$

Here ε is a small positive number, say, 0.1. Standardly, the first and second 3×3 matrices define the payoffs for players 1 and 2, respectively. The corresponding normal form is of size $3^3 \times 3^3 = 27 \times 27$. Its entries are mean payoffs defined, in accordance with the above two matrices, on the dicycles of the complete bipartite 3×3 digraph. Let us choose the best dicycle for player 1 (respectively, 2) in each column (row) of the normal form. The obtained two sets of dicycles are disjoint [21]. Hence, there is no NE. In contrast, it was shown in [22] that the ergodic extensions of $2 \times k$ bimatrix games are always Nash-solvable.

In other words, a mean payoff BW game has a NE in pure positional ergodic strategies whenever its digraph is complete bipartite and of size $2 \times k$, yet, this statement cannot be extended to the case of the 3×3 digraphs. This negative result shows that our proof of Theorem 2 cannot be extended to the non-zero-sum case, although, by Theorem 1, in this case, Nash-solvability holds. However, in Example 1, a unique NE exists for each of the two initial positions but the corresponding two NE are distinct.

4.4 Stochastic games with perfect information and the BWR model

The mean payoff BWR games were introduced in [24], yet, main results were obtained in [24] only for the BW games; they were extended to the BWR case recently, in [4, 5].

In particular, it was shown that for the BWR games a canonical form exists and it can be obtained by a potential transformation method. The main two properties of canonical forms for the mean payoff BW and BWR games are similar:

- The normal form of the game is modified in a predictable way, namely, the value $\pi(v_0)$ is added to the effective payoff in each situation $x \in X$;
- In each position, the locally optimal moves (that maximizes for White player 1, or minimizes for Black player 2) are optimal.

The latter property implies that the obtained optimal strategy is *uniformly optimal (ergodic)*, that is, it does not depend on the initial position. In contrast, the value of the game might depend on it.

Yet, the proof for the BWR case is substantially more complicated than for the BW one. Along with mean payoff games δ-discounted payoff must also be considered. For this case, the potential transformation should be "slightly" modified to $r_\pi(v, v') = r(v, v') + \pi(v) - \delta\pi(v')$

[24]. The technical derivation is based on non-trivial results of the Tauberian theory and, in particular, on the Hardy-Littlewood Theorem [26], which provide sufficient conditions for the Abel and Cesaro averages (which correspond to the discounted and mean payoffs, respectively) to converge to the same limit. The verification of these conditions in case of Markov chains is based on deep results by Blackwell [2]. It is also shown in [4] that the mean payoff BWR model is polynomially equivalent with the classical Gillette model [14].

Remark 17 *Of course, the careful reader noticed that we applied similar techniques, as Gillette in his paper [14]. In 1953 Shapley introduced stochastic games with non-zero stoping probability [35]. Then, in 1957 Gillete extended the analysis to a difficult case of zero-stop probability. In particular, he introduced the mean payoff stochastic games with perfect information and, for this case, proved solvability in pure ergodic positional strategies.*

The Hardy-Littlewood Theorem is instrumental in Gillette's arguments, too. By the way, his paper contained a repairable flaw: conditions of Hardy-Littlewood's theorem were not accurately verified. (Let us recall that the Blackwell paper [2] did not appear yet.) This flaw was corrected in 1969 by Liggett and Lippman [30]. As we already mentioned in the Introduction, it may be possible to derive Theorem 2 from Gillete's result rather than the BWR model. Yet, the latter is still of independent interest and also it is "closer to the subject". Indeed, the reduction of parlour games, like Chess and Back Gammon, to mean payoff BWR games is immediate; see the Introduction or the next Section. However, it is not that simple, although it might be possible, to represent these games in the Gilette model.

4.5 Reduction of AIPFOOT games to terminal BWR games and proof of Theorem 2

In fact, we have not much to add to the proof sketched in Introduction, yet, all necessary definitions are available by now. To prove Theorem 2, it would be sufficient to consider only two-person zero-sum AIPFOOT games. Yet, the following reduction works in general.

Let $\mathcal{G} = (G, P, v_0, u)$ be an n-person AIPFOOT game and $u(i, c)$ be the payoff of player $i \in I$ in case of an infinite play. Let us reduce the payoff function u by vector $u(c) = (u(i, c) \mid i \in I)$, that is, for all $i \in I$ let us set $u'(i, v) = u(i, v) - u(i, c)$ for each $v \in V_T$ and $u'(i, c) = 0$. Clearly, the obtained AIPFOOT game $\mathcal{G}' = (G, P, v_0, u')$ is trivially related to $\mathcal{G}$ and every infinite play "becomes a draw" in $\mathcal{G}'$, like in Chess or Back Gammon.

Let us add a loop $e_v = (v, v)$ to each terminal $v \in V_T$, then set $r(i, e_v) = u'(i, v)$ and $r(i, e) = 0$ for all arcs of G and players $i \in I$. The following five statements are obvious:

- the obtained *terminal* BWR game $\mathcal{G}''$ is equivalent to $\mathcal{G}'$ and, hence, to $\mathcal{G}$;
- the mapping $\mathcal{G}'' \leftrightarrow \mathcal{G}$ is a bijection;
- the game $\mathcal{G}$ has no moves of chance if and only if $\mathcal{G}''$ is a BW game;
- the obtained terminal BWR payoff in $\mathcal{G}''$ is *strictly positive*, that is $r(i, e_v) > 0$ for all players $i \in I$ and loops $e_v = (v, v)$, $v \in V$, if and only if $\mathcal{G}$ is an AIPFOOW game;

- games $\mathcal{G}$ and $\mathcal{G}''$ can be zero-sum games only simultaneously.

Thus, if $\mathcal{G}$ is a two-person zero-sum game then so is $\mathcal{G}''$. It was shown in [4, 5] that $\mathcal{G}''$ has a saddle point in pure positional uniformly optimal strategies. Obviously, the same strategies form a saddle point in game $\mathcal{G}$. □

Remark 18 *Note that games without positions of chance, like Chess, are reduced to BW games, which are much easier to solve and analyze than BWR games.*

Moreover, solvability of Chess in pure positional uniformly optimal strategies results also from Theorem 1, which, in its own turn, follows from the equivalence of Nash-, zero-sum, and ± 1*-solvabilities [9, 18, 19]).*

5 State of the art

Finite positional n-person games with perfect information are considered. In absence of dicycles everything is simple. Each acyclic game has a subgame perfect NE in pure positional strategies. One of these NE (so-called sophisticated or dominance equilibrium) can be obtained by the backward induction procedure suggested by Kuhn [28, 29] and Gale [12].

Next, let us assume that dicycles might exist but *All Infinite Plays Form One Outcome*, a_∞ or c, in addition to the *Terminal* outcomes $V_T = \{a_1, \ldots, a_p\}$. This assumption is called the AIPFOOT condition. For example, Chess and Back Gammon are AIPFOOT games, because an infinite play is a draw, by definition.

First, let us assume that there are no positions of chance, like in Chess, for example.

For this case, Zermelo [37] proved his famous theorem that can be reformulated as follows: a two-person zero-sum AIPFOOT game without random moves has a saddle point $x = (x_1, x_2)$ in pure strategies. Theorem 1 extends this result in two directions:

> Strategies x_1 and x_2 can be chosen positional and uniformly optimal.
>
> A (not necessarily subgame perfect) NE in pure positional strategies exists in any two-person AIPFOOT game without random moves, even in the non-zero-sum case.

To prove the first statement, we modify the classical backward induction procedure for the *two-person zero-sum* AIPFOOT games, which might have dicycles; see Section 3.3.

Furthermore, we conjecture that the last statement can be extended to the n-person case, Conjecture 1. It was proved for the play-once AIPFOOW games in [6] and by Gimbert and Sørensen for the two-person AIPFOOT games. Their proof can be simplified by recalling that Nash-solvability and ± 1-solvability are equivalent for the two-person game forms [19]. However, this observation cannot be extended to the n-person case with $n > 2$ [19, 20, 3].

Already for $n = 2$, a unique NE might be not subgame perfect [1]; see Example 1.

Recently in [8], Conjecture 1 was proven for the case of at most three terminals.

In [6], Conjecture 1 was suggested in a more general setting of additive payoffs, yet, only for the AIPFOOW games and non-negative local cost functions, Conjecture 2. It was shown that both above restrictions are essential and that Conjecture 2 is stronger than Conjecture 1. Finally, the first one was proved in two special cases: play-once and two-person AIPFOOW games.

Now, let us assume that both dicycles and moves of chance may exist. Even in this, most general, case the existence of a subgame perfect NE in pure (but not necessarily positional or even stationary) strategies can be derived from the so-called Folk Theorem; Section 2.11.

In absence of random moves, these strategies can be chosen ergodic pure and *stationary*, but still, not necessarily positional. The proof is based on the observation that each finite digraph can be "unfolded" as a tree; see Section 2.11 again.

In this paper, we concentrate on (subgame perfect) NE in pure *positional* strategies. The following negative observations should be taken into account: Example 2 gives a three-person play-once AIPFOOW game with only one dicycle, one position of chance, and no NE in pure positional strategies and, thus, leaves no hope for Nash-solvability in the case of $n \geq 3$. Moreover, Example 3 provides a game that has all above properties but n is reduced from 3 to 2. Yet, this game is not zero-sum and not AIPFOOW. This leaves two open ends.

The first one, whether a NE in pure positional strategies always exists in the *two-person AIPFOOW* games, is still open.

The second one, whether each two-person *zero-some* AIPFOOT game has a saddle point in pure positional uniformly optimal strategies is answered by Theorem 2 in the affirmative. This result is proved by the reduction of the n-person AIPFOOT games to stochastic games with perfect information. In particular, two-person zero-sum AIPFOOT games are reduced to the BWR games. For the latter, the solvability in pure positional uniformly optimal strategies was recently shown in [4, 5].

Finally, let us remark that two-person zero-sum AIPFOOT games without random moves, like Chess, are reduced to the BW games, for which solvability in pure positional uniformly optimal strategies is much easier to show [32, 11, 24] than for the BWR games.

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References

[1] D. Anderson, V. Gurvich, and T. Hansen, On acyclicity of games with cycles, RUTCOR Research Report RRR-18-2008 and DIMACS Technical Report 2009-09, Rutgers University; Algorithmic Aspects in Information and Management (AAIM); Lecture Notes in Computer Science 5564 (2009) 15-28.

[2] D. Blackwell, Discrete dynamic programming, Annals of Mathematical Statistics 33 (1962) 719726.

[3] E. Boros, K. Elbassioni, Gurvich, and K. Makino, On effectity functions of game forms, RUTCOR Research Report, RRR-03-2009; Games and Economic Behaviour, to appear; http://dx.doi.org/10.1016/j.geb.2009.09.002

[4] E. Boros, K. Elbassioni, Gurvich, and K. Makino, Every stochastic game with perfect information admits a canonical form, RUTCOR Research Report, RRR-09-2009.

[5] E. Boros, K. Elbassioni, Gurvich, and K. Makino, A pumping algorithm for ergodic stochastic mean payoff games, RUTCOR Research Report, RRR-19-2009.

[6] E. Boros and V. Gurvich, On Nash-solvability in pure strategies of finite games with perfect information which may have cycles. Math. Social Sciences 46 (2003), 207-241.

[7] E. Boros, V. Gurvich, K. Makino, and Wei Shao, Nash-solvable bidirected cyclic two-person game forms, DIMACS Technical Report, DTR-2008-13 and RUTCOR Research Report, RRR-30-2007, revised in RRR-20-2009, Rutgers University.

[8] E. Boros and R. Rand, Terminal games with three terminals have proper Nash equilibria, RUTCOR Research Report, RRR-22-2009, Rutgers University.

[9] J. Edmonds and D.R. Fulkerson, Bottleneck extrema, J. of Combinatorial Theory, 8 (1970), 299-306.

[10] A. Ehrenfeucht and J. Mycielski, Positional games over a graph, Notices of the American Mathematical Society **20** (1973) A-334, Abstract.

[11] A. Ehrenfeucht and J. Mycielski, Positional strategies for mean payoff games, International Journal of Game Theory **8** (1979), 109-113.

[12] D. Gale, A theory of N-person games with perfect information, Proc. Natl. Acad. Sci. 39 (1953) 496–501.

[13] T. Gallai, Maximum-minimum Sätze über Graphen, Acta Mathematica Academiae Scientiarum Hungaricae 9 (1958) 395–434.

[14] D. Gillette, Stochastic games with zero stop probabilities; in Contribution to the Theory of Games III, in Annals of Mathematics Studies 39 (1957) 179–187; M. Dresher, A.W. Tucker, and P. Wolfe eds, Princeton University Press.

[15] A.I. Gol'berg and V.A. Gurvich, Tight cyclic game forms, Russian Mathematical Surveys, 46:2 (1991) 241-243.

[16] A.I. Gol'berg and V.A. Gurvich, Some properties of tight cyclic game forms, Russian Acad. Sci. Dokl. Math., 43:3 (1991) 898-903.

[17] A.I. Gol'berg and V.A. Gurvich, A tightness criterion for reciprocal bipartite cyclic game forms, Russian Acad. Sci. Dokl. Math., 45 (2) (1992) 348-354.

[18] V. Gurvich, To theory of multi-step games, USSR Comput. Math. and Math. Phys. 13:6 (1973) 143-161.

[19] V. Gurvich, Solution of positional games in pure strategies, USSR Comput. Math. and Math. Phys. 15:2 (1975) 74-87.

[20] V. Gurvich, Equilibrium in pure strategies, Doklady Akad. Nauk SSSR 303:4 (1988) 538-542 (in Russian); English translation in Soviet Math. Dokl. 38:3 (1989) 597-602.

[21] V. Gurvich, A stochastic game with complete information and without equilibrium situations in pure stationary strategies, Russian Math. Surveys 43:2 (1988) 171–172.

[22] V. Gurvich, A theorem on the existence of equilibrium situations in pure stationary strategies for ergodic extensions of $(2 \times k)$ bimatrix games, Russian Math. Surveys 45:4 (1990) 170–172.

[23] V. Gurvich, Saddle point in pure strategies, Russian Acad. of Sci. Dokl. Math. 42:2 (1990) 497–501.

[24] V. Gurvich, A. Karzanov, and L. Khachiyan, Cyclic games and an algorithm to find minimax cycle means in directed graphs, USSR Computational Mathematics and Mathematical Physics 28:5 (1988) 85-91.

[25] V.A. Gurvich and V.N. Lebedev, A criterion and verification of ergodicity of cyclic game forms, Russian Mathematical Surveys, 44:1 (1989) 243-244.

[26] G. H. Hardy and J. E. Littlewood, Notes on the theory of series (XVI): two Tauberian theorems, J. of London Mathematical Society 6 (1931) 281–286.

[27] A.V. Karzanov and V.N. Lebedev, Cyclical games with prohibition, Mathematical Programming **60** (1993), 277-293.

[28] H. Kuhn, Extensive games, Proc. Natl. Acad. Sci. 36 (1950) 286–295.

[29] H. Kuhn, Extensive games and the problem of information, in Contributions to the theory of games, Volume 2, Princeton (1953) 193–216.

[30] T.M. Liggett and S.A. Lippman, Stochastic Games with Perfect Information and Time Average Payoff, SIAM Review 11 (1969) 604607.

[31] H. Moulin, Prolongement des jeux à deux joueurs de somme nulle, Bull. Soc. Math. France, Memoire **45**, 1976.

[32] H. Moulin, Extension of two person zero-sum games, Journal of Mathematical Analysis and Application **55 (2)** (1976) 490-508.

[33] J. Nash, Equilibrium points in n-person games, Proceedings of the National Academy of Sciences 36 (1) (1950) 48-49.

[34] J. Nash, Non-Cooperative Games, Annals of Mathematics 54:2 (1951) 286–295.

[35] L. S. Shapley, Stochastic games, in Proc. Nat. Acad. Science, USA 39 (1953) 1095–1100.

[36] S. Vorobyov, Cyclic games and linear programming, Discrete Applied Mathematics **156** (11) (2008) 2195–2231.

[37] E. Zermelo, Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels; in: Proc. 5th Int. Cong. Math. Cambridge 1912, vol. II; Cambridge University Press, Cambridge (1913) 501-504.