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Terminal games with 3 terminals have proper Nash equilibria in pure positional strategies

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Terminal games with 3 terminals have proper Nash equilibria in pure positional strategies

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Abstract. We prove that every connected terminal game with 3 terminals has a proper Nash equilibrium in pure positional strategies.

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1 Introduction

Let us start with the description of a family of games, we call *terminal games with cycles*. These games are n-person finite positional games with perfect information, defined as follows:

Let us consider a directed graph $G = (V, E)$, an *initial vertex* $v_0 \in V$ and a set of *terminal vertices* $T \subseteq V$. Denoting by $N^+(u) = \{v \in V \mid (u, v) \in E\}$ the set of *out-neighbors* for vertices $u \in V$, we assume that $N^+(t) = \emptyset$ for all $t \in T$. We also consider a set of *players* $I = \{1, 2, ..., n\}$ and a fixed ordered partition $\mathcal{U} = \{U_i \mid i \in I\}$, $U_1 \cup U_2 \cup \cdots \cup U_n = U = V \setminus T$ of the set of non-terminal vertices. We say that player $i \in I$ *controls* the vertices $u \in U_i$.

A *strategy* of player $i \in I$ is a selection of a single outgoing edge for each of the vertices player i controls, or in other words, a subset $S_i \subseteq E$ of the edges of the form $S_i = \{(u, \sigma_i(u)) \mid u \in U_i\}$, where $\sigma_i(u) \in N^+(u)$ for all $u \in U_i$. Since the domains of the mappings σ_i are pairwise disjoint, it will not cause any ambiguity if we simply use σ instead, and call it the strategy set of the players. Let us remark that according to standard terminology of game theory, we consider in this paper *pure positional* strategies. In other words, players choose the outgoing arcs $(u, \sigma(u))$ based only on the position u, and this choice is not influenced by additional information, e.g., by other player's earlier choices, etc. In fact we can assume in this paper that all payers make up their mind simultaneously, in advance, and choose σ at once.

The subgraph (V, S) where $S = S_1 \cup \cdots \cup S_n$ is called a *situation*, or in other words a situation is when all the players have chosen a strategy. We shall also use the notation $S = \sigma(E)$ to emphasize that S is a result of the players choosing a particular set σ of strategies.

Since in any given situation $S = \sigma(E)$ there is at most one edge leaving every vertex (exactly one for vertices in U and none for terminal vertices), there is a unique path $P = P(S) \subseteq F$ starting at the initial vertex v_0. Since G is finite, this path either ends at a terminal vertex t (e.g., when $v_0 \in T$, then $t = v_0$), or it ends in a directed cycle $C \subseteq S$. The path P is called the *play* corresponding to the situation S, and its ending (t or C) is called the *outcome* of the game denoted by $\alpha(S) \in T \cup \mathcal{C}$, where $\mathcal{C}$ is the family of directed cycles of the graph G. We denote the set of outcomes of the game by $A = T \cup \mathcal{C}$.

The *payoff function* of a terminal game with cycles is a mapping $\pi : I \times A \to \mathbb{R}$, where $\pi(i, a)$ denotes the amount player $i \in I$ has to pay if the outcome of the game is $a \in A$.

Finally, we call the tuple $\mathcal{G} = (G, T, v_0, \mathcal{U}, \pi)$ satisfying the above definitions a *terminal game with cycles*.

Given such a game $\mathcal{G}$, a situation $S = \sigma(E)$ is called a *Nash equilibrium* (NE) if for all players $i \in I$ and all strategies S'_i of player i, we have

$$\pi(i, \alpha(S)) \leq \pi(i, \alpha(S')),$$

where S' is obtained from S by replacing S_i by S'_i. Such a Nash equilibrium is called *cyclic* if $\alpha(S) \in \mathcal{C}$ and it is called *proper* otherwise.

The main problem considered in this paper is to characterize subfamilies of terminal games with cycles in which NE always exists, regardless the actual payoff function. In fact several such results exist in the literature. Before formulating our main result, we recall some of the related results and discuss some simplifications which do not restrict generality.

1.1 Historical Background

Many 2-player classical zero-sum games belong to the family of terminal games with cycles. Consider for instance the play of chess. We can model it by associating a vertex with every feasible placement of the chess figures on the board together with a designation of a next mover. All placements corresponding to a checkmate can be merged into two terminal vertices "white wins" and "black wins". Though one player could, theoretically, make a different move when presented with the same configuration for a second time, we assume in this paper that this is not the case. In fact most "common" players play in this way, and this is also the hidden assumption behind the compositions of many chess problems. A consequence of this assumption is that any repetition of any configuration leads to infinite repetitions, which by the rules of chess implies that the game ends in a "tie" (or "draw"). Let us further add that a stalmate (when one party has no legal move) is either interpreted as a win for the other party, or it is a "draw", which we can model by adding a loop to the corresponding position, forming a cycle on its own. Thus, chess can be viewed as an example of a game where all cyclic outcomes are equivalent for both players, and we have $|I| = 2$, $|T| = 2$, and $|A| = 3$. Zermelo [5] showed that such (zero sum) 2-person games always have a Nash equilibrium (using an equivalent definition, not this name) in pure strategies[1].

Let us remark here that our definitions introduced above use a somewhat simplified game theory terminology, based on standard terminology of graph theory in order to simplify our rather combinatorial proofs. First of all, in a typical game players may choose moves (edges) in a given position (vertex) randomly. Such a strategy is called *mixed*, while a deterministic selection is called a *pure* strategy. Note that a strategy, as we defined it above, is pure. Secondly, in usual games players move in a certain sequence, and the choice of their next move may depend on the choices they and the other players made earlier. A strategy is called *stationary* if every move depends only on the previous moves (and not some additional information, e.g., the number of times the game was played before, or some random parameters, etc.). A strategy is called *positional* if it depends only on the current position itself (regardless of the previous choices made); in particular, a positional strategy is also stationary. Let us emphasize that, according to our definitions, we consider only pure positional strategies in this paper. Let us also emphasize that a strategy also depends on the game itself, and thus in particular on the initial vertex. Consequently, a NE (in pure positional strategies) may not be a NE if we change the initial position (that is we change the game). Let us finally refer the reader to a recent article [2] for further clarifications on terminology and game models.

In [1] a more general family of games were considered, in which the payoff for each player is defined as the sum of prescribed local payoffs associated to the edges along the play $P = P(S)$. Clearly, if the outcome is a cycle, and all edge-payoffs are positive, then all

[1] Finding such a NE is still the "Holy Grail" of chess players and developers of chess playing programs.

cycles mean an infinitely large payoff for each players, implying that all cycles counts again as a single outcome, dreaded most by all players. There was a recent failed attempt[2] to prove that all such *additive payoff games with cycles* have a NE in pure positional strategies. However, it was demonstrated in [1] that this proof is not correct, and in fact even very special cases of this problem remain still open.

Let us finally remark that in the special case when $\mathcal{C} = \emptyset$ (that is when the graph G is acyclic) all of the above mentioned types of games have proper NE (in pure positional strategies). In fact such NE can be obtained by an efficient dynamic programming type approach, called backward induction introduced by Kuhn in [3, 4].

1.2 Main Results

Inspired by chess and similar 2-person games [1] considered a special subclass of terminal games with cycles, in which all cycles count as the same outcome γ which is the worst for all players that is for which we have

$$\pi(i,t) < \pi(i,\gamma) \tag{1}$$

for all terminals $t \in T$ and players $i \in I$. We call these games *terminal.*

Let us first observe some simplifications. Let us call a terminal game *connected* if there is a directed path from v_0 to the terminal set T, and note that an unconnected terminal game obviously has a NE. Namely, every strategy in an unconnected game must end in a cycle, hence γ is the only achievable outcome, so every strategy constitutes a (cyclic) NE. Let us also remark that even connected terminal games may have a cyclic NE, as the example in figure 1 shows.

To strengthen the general conjecture we claim:

Conjecture 1 *All connected terminal games have a proper NE (in pure positional strategies).*

This conjecture was shown in the affirmative in [1] for some special cases:

Theorem 1 ([1]) *Connected terminal games have a proper NE in any of the following cases:*

- $|I| = 2$*;*
- $|T| = 2$*;*
- $|U_i| = 1$ *for all* $i \in I$ *(*play-once*).*

In all of these cases a proper NE can be found efficiently in the size of the graph G*.*

[2] R.Boliac, D.Lozovanu, and D.Solomon Optimal paths in network games with p players, Discrete Applied Mathematics 99 (2000) 339-348.

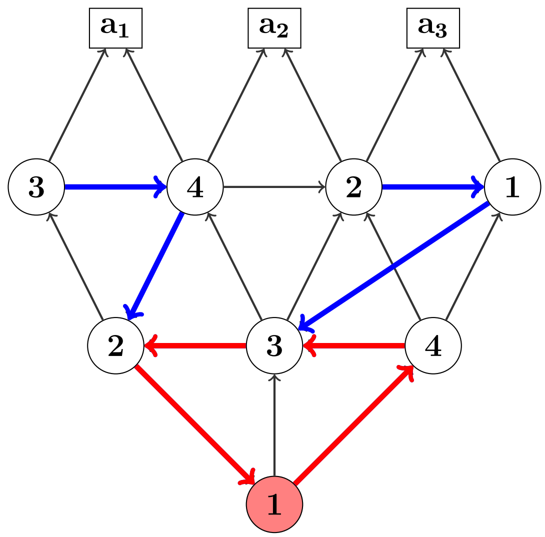

Figure 1: An example of a connected terminal game where $T = \{a_1, a_2, a_3\}$, $I = \{1, 2, 3, 4\}$, players ID-s shown inside nodes they control, and with the shaded node at bottom as the initial vertex v_0. Here, thick lines form a strategy S ending in a cycle. It is easy to verify that S is a NE.

Our main result is to prove the main conjecture for another special case:

Theorem 2 *Connected terminal games with* $|T| = 3$ *have a proper NE (in pure positional strategies).*

The rest of this short note provides the proof for Theorem 2. The proof is an induction on the number of players. We can use $|I| = 1$ as a base case, since then the claim is trivial, or we could start with $|I| = 2$, since the claim was shown for this case in [1]. In what follows, first we show that it is enough to prove the theorem for a special case, and then show that we can fix the strategy of some player $n \in I$ such that the resulting $|I|-1$-player terminal game has a proper NE which together with the fixed strategy of player n constitutes a proper NE.

2 Proof of Theorem 2

Let us first note that it is enough to prove Theorem 2 for terminal games $\mathcal{G} = (G, T, v_0, \mathcal{U}, \pi)$ in which no player can reach directly its best outcome, and no position is forced to move to T. More precisely, we can assume that the following conditions hold:

$$\min_{r \in T} \pi(i, r) \;<\; \min_{r \in T \cap N^+(u)} \pi(i, r) \qquad \text{players } i \in I \text{ and for all vertices } u \in U_i. \tag{2}$$

$$N^+(u) \not\subseteq T \qquad \text{for all vertices } u \in U. \tag{3}$$

Let us remark that however natural the above conditions look, it needs a careful proof that these assumptions do not restrict generality. For instance, in the case of condition (2) we may have an edge (u,t) in the graph where u is controlled by player i and where t is the best terminal for player i. Still, it is possible to have a NE in which player i has to chose a different outgoing edge at vertex u in order to prevent another player on the play from improving. See Figure 2 for such an example. Still, it can be shown quite simply that conditions (2)-(3) can be assumed without any restriction, due to the following lemma:

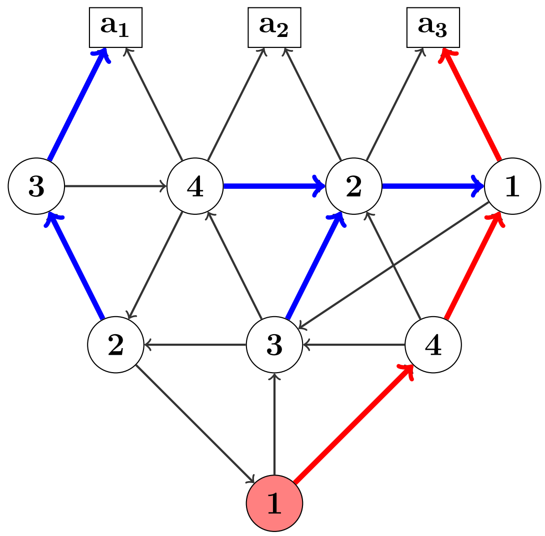

Figure 2: An example of a connected terminal game with $T = \{a_1, a_2, a_3\}$, $I = \{1, 2, 3, 4\}$, players ID-s shown inside nodes they control, and with the shaded node at bottom as the initial vertex v_0. Here, thick lines form a strategy S ending at terminal a_3. It is easy to verify that if $\pi(1, a_2) < \pi(1, a_3) < \pi(1, a_1)$ then this is a NE, regardless of the preferences of the other players. Let us note that even if a_2 is player 2's best outcome, he cannot switch to directly reaching a_2 without destroying the current equilibrium (player 1 could then improve).

Lemma 1 *Assume that all connected terminal games* $\mathcal{G} = (G, T, v_0, \mathcal{U}, \pi)$ *with* $|T| \leq k$ *satisfying conditions* (2)-(3) *have a proper NE. Then, all connected terminal games with* $|T| \leq k$ *have proper NE.*

Proof. Assume indirectly that there are counter examples, and consider a minimal counter example $\mathcal{G} = (G, T, v_0, \mathcal{U}, \pi)$, where minimality is with respect to the number of nodes $|U|$. Since $\mathcal{G}$ is a connected terminal game which does not have a proper NE, by our assumption it must violate one of the conditions (2)-(3), that is we have a player $i \in I$ and node $u \in U_i$ such that either $(u,t) \in E$ for the terminal $t \in T$ which is player i's best, or we have $N^+(u) \subseteq T$, in which case we denote by $t \in N^+(u)$ the terminal which is player i's best within the set $N^+(u)$. Let us now *merge* vertex u into terminal t, and denote the resulting terminal game by $\mathcal{G}'$. Formally, we define $\mathcal{G}' = (G', T, v_0', \mathcal{U}', \pi)$, where $G' = (V \setminus \{u\}, E')$, $E' = (E \setminus \{(x,y) \in E \mid \text{ where either } y = u \text{ or } x = u\}) \cup \{(x,t) \mid \text{ for all edges } (x,u) \in E\}$, and where $v_0' = v_0$ if $v_0 \neq u$, and $v_0' = t$ if $v_0 = u$, and $\mathcal{U}' = \{U_1, ..., U_{i-1}, U_i', U_{i+1}, ..., U_n\}$ with $U_i' = U_i \setminus \{u\}$.

It is immediately apparent that $\mathcal{G}'$ is a connected terminal game with one less vertex than $\mathcal{G}$. Since G was a minimal counter example, $\mathcal{G}'$ must have a NE $S' = \sigma'(E')$. Let us then define $S = \sigma(E)$ by

$$\sigma(x) = \begin{cases} \sigma'(x) & \text{if } x \neq u \text{ and } \sigma'(x) \neq t, \\ u & \text{if } x \neq u \text{ and } \sigma'(x) = t, \\ t & \text{if } x = u \end{cases}$$

It is then straightforward to check that $S = \sigma(E)$ is a NE of $\mathcal{G}$, contradicting the choice of $\mathcal{G}$. This contradiction, together with the trivial fact that the number of terminals did not change proves the claim. □

In the sequel we shall assume conditions (1)-(3). The following claim, interesting on its own, will be instrumental in proving Theorem 2.

Lemma 2 *Assume that $\mathcal{G} = (G, T, v_0, \mathcal{U}, \pi)$ is a connected terminal game with $|T| \leq 3$, and that it has a proper NE. Then, it has a proper NE, $S = \sigma(E)$, which does not enter a terminal node except the outcome $\alpha(S)$ corresponding to situation S. In other words, it satisfies*

$$\sigma(u) \notin T \setminus \{\alpha(S)\} \text{ for all } u \in U. \tag{4}$$

Proof. If $|T| < 3$, then this follows from the cited results of [1], thus we can assume that $|T| = 3$. Let us denote by $S^* = \sigma^*(E)$ a proper NE of the connected terminal game $\mathcal{G}$, which exists by our assumptions. Let us define a new strategy σ by

$$\sigma(u) = \begin{cases} \sigma^*(u) & \text{if } \sigma^*(u) \notin T \setminus \{\alpha(S^*)\} \\ v \in N^+(u) \setminus T & \text{whenever } \sigma^*(u) \in T \setminus \{\alpha(S^*)\} \end{cases}$$

According to property (2) the above definition is feasible and yields a situation $S = \sigma(E)$ for $\mathcal{G}$. Let us also note that S has the same outcome as S^*, and moreover, both have the same play $P = P^* \subseteq S \cap S^*$.

We claim that S is a proper NE of $\mathcal{G}$ satisfying property (4). Since the latter is obvious, we only need to argue that S is a NE. To simplify our notations in proving this claim, let us call edges of $S \setminus S^*$ *blue*, and assume indirectly that there is a player $i \in I$ such that $P \cap U_i \neq \emptyset$ and that player i can change its strategy at nodes in U_i so that to achieve a better outcome $t' \succ_i \alpha(S) = \alpha(S^*)$. Since S^* is a NE, the new play P' must contain some blue edges. Let (u, v) be the first blue edge along the path P' when traversing from the initial point v_0. Since (u, v) is a blue edge, we must have $\sigma^*(u) \in T \setminus \alpha(S)$. Let us also note that player i could divert play $P = P^*$ from strategy S^* to reach $\sigma^*(u)$ by changing this strategy along the vertices in P up to vertex u, since this initial segment of P does not involve any blue edges. Since S^* is a NE, this implies that we have $\alpha(S^*) \succ_i t'' = \sigma^*(u)$. Consequently, we have $t' \succ_i \alpha(S) = \alpha(S^*) \succ_i t''$, implying that t' is i's best outcome, since we have $|T| = 3$. Due to the construction of S, the last move along play P' must be controlled by player i, meaning that player i can directly reach its best outcome t', contradicting condition (2). This contradiction proves that no player can improve on situation S, and hence completes the proofs of our claim. □

For a subset $X \subseteq T$ of the terminal set and a player $i \in I$ let us denote by $W(i, X) \subseteq U$ the set of vertices from which player i can (and will) *force* an outcome in X. In other words, in any situation S the pay P either avoids the set $W(i, X)$, or it must terminate in X. It is straightforward to verify that the recursive application of the following two rules defines $W(i, X)$ uniquely, if we start with $W(i, X) = X$ initially. More precisely, let us associate a *rank* $\rho(u)$ to all vertices of V. Initially we define $\rho(t) = 0$ for $t \in X$, and $\rho(u) = \infty$ for $u \in V \setminus X$. At any moment in the sequel we define $W(i, X)$ as the set of nodes of finite rank:

(i) if there is a vertex $u \in U_i$ such that $(u, v) \in E$ for some vertex $v \in W(i, X)$ then set $\rho(u) = |W(i, X)| + 1$ and redefine $W(i, X)$;

(ii) if there is a vertex $u \in U \setminus U_i$ such that $N^+(u) \subseteq W(i, X)$, then set $\rho(u) = |W(i, X)|+1$ and redefine $W(i, X)$.

Now, we are ready to start proving our main result. Let us assume (without any restriction of generality) that $\pi(n, a_1) < \pi(n, a_2) < \pi(n, a_3)$, and consider the set $W = W(n, \{a_1, a_2\})$. Let us now fix a strategy for player n. For vertices $u \in U_n \cap W$ rule (i) above provides a vertex with $\rho(v) < \rho(u)$ and $(u, v) \in E$, and let us fix such an outgoing edge from all $u \in U_n \cap W$. For $u \in U_n \setminus W$ let us choose an arc (u, v), $v \in N^+(u)$ such that the initial vertex v_0 is not cut from the terminal set T (clearly, such a choice is possible since $\mathcal{G}$ is connected), and denote by F the chosen set of outgoing edges from vertices controlled by player n. Denoting by $E_n = \{(u, v) \in E \mid u \in U_n\}$ the set of edges controlled by player n, let us delete all edges in $E_n \setminus F$ and let us contract the edges in F. We obtain in this way a new $n-1$-player terminal game $G' = (V', E')$. Due to our careful selection of edges in F, the game G' is again a connected terminal game, with $n - 1$ players. Due to the inductive hypothesis, G' has a proper NE, and thus by Lemma 2 it has one, $S' = \sigma'(E')$, satisfying conditions (4). Consider the set $S = S' \cup F$. By our construction, S is a situation in game G with $\alpha(S) = \alpha(S') \in T$.

Case 1: $\alpha(S) = a_1$. In this case $S^* = S$ is a NE of $\mathcal{G}$, since only player n could improve in S, but a_1 is his best outcome.

Case 2: $\alpha(S) = a_2$. In this case we claim that $S^* = S$ is again a NE of $\mathcal{G}$. Clearly, only player n could improve on situation S, and only if it can reach terminal a_1. For this he would need a vertex $u \in U_n$ such that $(u, a_1) \in E$. However, according to condition (2) such an edge does not exist.

Case 3: $\alpha(S) = a_3$. Let us note that by the definition of subset W, no edge in S can leave W, and since $a_3 \notin W$, the entire play $P \subseteq S$ must be disjoint from W. Let us also note that the set $F \subseteq S$ may include some edges (u, a_2) for some $u \in W \cap U_n$. Let us remark that in this case S may not be a NE of $\mathcal{G}$, since there may be edges in S' entering W, and player n maybe able to change this strategy to reaching such an edge, from which the play will reach W, and by extension terminal a_2 (his more preferred outcome).

Let us now change situation S as follows: Let us replace every edge $(u, a_2) \in F$ (where $u \in W \cap U_n$) by (u, v) for some $v \in N^+(u) \setminus T$. By property (3) this is always possible. Furthermore, let us replace every edge $(u, v) \in S$ where $u \notin W$, $v \in W$ by an edge (u, v'), where $v' \in N^+(u) \setminus W$. According to our definition of W, this is also possible. Let us call the new edges (u, v') introduced in this way *blue edges*, and denote by S^* the resulting situation of G. We claim that S^* is a proper NE of G completing our proof of the main theorem.

To see our claim, note that our changes did not change the play P, and hence we have $\alpha(S^*) = \alpha(S) = a_3$. Let us also note that since $N^+(u) \subseteq V \setminus W$ for all $u \in U_n \setminus W$ by the definition of W, and since no edge of S^* enters the set W, player n cannot improve on S^*. Thus our claim will follow if we can show that no other player can improve on S^*, either. To this end, let us assume indirectly that player $i \in \{1, ..., n-1\}$ can change his strategy and change the play P to P' ending in a terminal $a' \in \{a_1, a_2\}$ such that

$$\pi(i, a') < \pi(i, a_3). \tag{5}$$

Since player i could not improve on situation S, the new play P' must involve some blue edges. Let $(u, v') \in S^*$ be the first blue edge when we traverse P' from the initial vertex v_0. By the definition of blue edges, we must have $(u, v) \in S$ for some $v \in W$. Since the induced subgraph (W, S) is acyclic by our construction, there is a unique path P'' in S connecting v to a terminal node $a'' \in \{a_1, a_2\}$. Concatenating the path segment of P' form v_0 to u, then the edge (u, v), and then the path P'' from v to a'' we get a path P''' from v_0 to a'' such that all edges of $P''' \setminus S$ are controlled by player i. Since player i could not improve on situation S, we must have

$$\pi(i, a_3) < \pi(i, a''). \tag{6}$$

Therefore inequalities (5) and (6) imply that a' is player ith best outcome (since we have $|T| = 3$). Because S^* has no edges entering terminals a_1 or a_2, the last edge of P' must be controlled by player i, contradicting property (2). This contradiction proves our claim and hence concludes the proof of our main theorem. □

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