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METRIC AND ULTRAMETRIC SPACES OF RESISTANCES [a]

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Metric and ultrametric spaces of resistances [1]

Vladimir Gurvich

Abstract. Given an electrical circuit each edge e of which is an isotropic conductor with a monomial conductivity function $y_e^* = y_e^r/\mu_e^s$. In this formula, y_e is the potential difference and y_e^* current in e, while μ_e is the resistance of e; furthermore, r and s are two strictly positive real parameters common for all edges. In particular, $r = s = 1$ correspond to the standard Ohm low.
In 1987, Gvishiani and Gurvich [Russian Math. Surveys, 42:6(258) (1987) 235–236] proved that, for every two nodes a, b of the circuit, the effective resistance $\mu_{a,b}$ is well-defined and for every three nodes a, b, c the inequality $\mu_{a,b}^{s/r} \leq \mu_{a,c}^{s/r} + \mu_{c,b}^{s/r}$ holds. It obviously implies the standard triangle inequality $\mu_{a,b} \leq \mu_{a,c} + \mu_{c,b}$ whenever $s \geq r$ and it turns into the ultrametric inequality $\mu_{a,b} \leq \max(\mu_{a,c}, \mu_{c,b})$ as $r/s \to 0$. For the case $s = r = 1$ these results were rediscovered in 90s. Now, in 22 years, I venture to reproduce the proof of the original result for the following reasons:

- (i) it is more general than the case $r = s = 1$ and one can get several interesting examples of metric and ultrametric spaces playing with parameters r and s;
- (ii) several very unpleasant typos are corrected in two of these examples;
- (iii) Communications of the Moscow Math. Society in Russian Math. Surveys were (and still are) limited to two pages, here more details will be given;
- (iv) although translation in English of the Russian Math. Surveys is available, it is not free in the web and not that easy to find out.

Key words: distance, metric, ultrametric; conductivity function, conductance, resistance, potential, voltage, current, flux; Ohm low, Joule-Lenz heat, Maxwell principle; maximum flow, shortest path, bottleneck path

1 Introduction

We consider an electrical circuit modeled by a (non-directed) *connected* graph $G = (V, E)$ in which each edge $e \in E$ is an isotropic conductor with the monomial conductivity law $y^*_e = y^r_e / \mu^s_e$. Here y_e is the voltage, or potential difference, y^*_e current, and μ_e is the resistance of e; furthermore, r and s are two strictly positive real parameters independent on $e \in E$.

In particular, the case $r = 1$ corresponds to Ohm's low, while $r = 0.5$ is the standard square law of resistance typical for hydraulics or gas dynamics.

Parameter s, in contrast to r, is redundant; yet, it will play an important role too.

It is not difficult to show that the effective resistance $\mu_{a,b}$ is well-defined for any two nodes a, b; also $\mu(a, b) = \mu(b, a)$, due to symmetry (isotropy) of the conductivity functions.

Furthermore, $\mu(a, b) > 0$ whenever nodes a and b are distinct.

Finally, by definition we set $\mu(a, b) = 0$ whenever $a = b$.

In [4], it was shown that for every three nodes a, b, c the following inequality holds.

$$\mu^{s/r}_{a,b} \le \mu^{s/r}_{a,c} + \mu^{s/r}_{c,b}\,. \tag{1}$$

In [6], it was also shown that equality in (1) holds if and only if note c belongs to every path between a and b. Clearly, if $r \le s$ then (1) implies the standard triangle inequality

$$\mu_{a,b} \le \mu_{a,c} + \mu_{c,b}. \tag{2}$$

Furthermore, if $r/s \to 0$ then (1) turns into the ultrametric inequality

$$\mu_{a,b} \le \max(\mu_{a,c}, \mu_{c,b}). \tag{3}$$

Thus, in these two cases, we obtain respectively metric and ultrametric spaces, in which distance between a and b is the effective resistance $\mu_{a,b}$. Playing with parameters r and s, one can obtain several interesting examples. In particular, we obtain:

- (j) the effective electric resistance between poles a and b, when $r = s = 1$, or more generally, $s \to 1$, $r \to 1$;
- (jj) the standard length (or travel time) of a shortest path between terminals a and b, when $r = s \to 0$, or more generally, $s \to 0$, $r/s \to 1$;
- (jjj) the inverse capacity, that is, the inverse value of a maximum flaw per unit time from source a to sink b, or (equivalently) vice versa, when $s = 1, r \to 0$, or more generally, $s \to 1$, $r \to 0$;
- (jv) the width of a bottleneck path between terminals a and b when $s \to 0, r/s \to 0$.

All four example define metric spaces, since $s/r \ge 1$, at least, when both parameters are sufficiently close to their limits. Moreover, the last two examples define ultrametric spaces, since $r/s \to 0$.

For the case $s = r = 1$, inequality (1) was rediscovered in [9]. Then, several interesting related results were obtained in [1, 10, 12, 15, 17] and surveyed in [3, 18, 19]. Now, in 22 years, I venture to reproduce the original proof of (1), for the following four reasons:

- (i) The original inequality (1) is slightly more general than (2) and one can get several interesting metric and ultrametric spaces playing with parameters r and s.

- (ii) Two very unpleasant typos appeared in [4] and then were copied in [6, 7]:

  $r = s \to \infty$, in case (jj) and $s \to \infty, r = const$ in case (jv).

- (iii) Communications of the Moscow Math. Society in Russian Math. Surveys were (and still are) limited to two pages; so the proofs in [4] were sketched; here we give more details.

- (iv) Although translation of [4] in English is available, yet, it is not free on the web and not that easy to find out.

Recently, these results were presented as a sequence of problems and exercises for high-school students in the Russian journal Math. Prosveschenie (Education) [7]. Here, these problems and exercises are given with solutions and in English.

2 Conductivity law

Let e be an electrical conductor with the monomial conductivity law

$$y_e^* = f_e(y_e) = \lambda_e^s |y_e|^r \text{sign}(y_e) = \frac{|y_e|^r}{\mu_e^s} \text{sign}(y_e), \tag{4}$$

where y_e is the voltage or potential difference, y_e^* current, and μ_e is the resistance of e; furthermore, r and s are two strictly positive real parameters independent on e.

It is easy to see that the monomial function f_e is

- continiuos, strictly monotone increasing, and taking all real values;

- symmetric (isotropic), or odd, that is, $f_e(-y_e) = -f_e(y_e)$;

- the inverse function f_e^{-1} is also monomial with parameters $r' = r^{-1}$ and $s' = s^{-1}$.

3 Main variables

An electrical circuit is modeled by a *connected* weighted non-directed graph $G = (V, E, \mu)$ in which weights of the edges are *positive* resistances $\mu_e, e \in E$. Let us introduce the following four groups of real variables; two for each vertex $v \in V$ and edge $e \in E$:

potentials x_v; difference of potentials, or voltage y_e; current y_e^*; sum of currents, or flux x_v^*.

We say that the first Kirchhoff law holds for a vertex v whenever $x_v^* = 0$.

The above variables are not independent. By (4) , the current y_e^* depends on voltage y_e.

Furthermore, the voltage (flux) is a liner function of potentials (currents).

To define these linear functions, we shall have to fix an arbitrary orientation of edges.

Then, let us introduce the vertex-edge incidence function as follows:

$$\mathrm{inc}(v,e) = \begin{cases} +1, & \text{if vertex } v \text{ is the beginning of } e; \\ -1, & \text{if vertex } v \text{ is the end of } e; \\ 0, & \text{in every other case.} \end{cases} \tag{5}$$

We assume that the next two systems of linear equations always hold:

$$y_e = \sum_{v \in V} \mathrm{inc}(v,e) x_v; \tag{6}$$

$$x_v^* = \sum_{e \in E} \mathrm{inc}(v,e) y_e^*. \tag{7}$$

Let us notice that equation (6) for edge $e = (v', v'')$ can be simplified and reduced to $y_e = \mathrm{inc}(e, v')x_{v'} + \mathrm{inc}(e, v'')x_{v''}$ and even further to $y_e = x_{v'} - x_{v''}$, yet, in the last case it should be assumed that e is directed from a to b.

Let us introduce four vectors, one for each group of variables:

$$x = (x_v \mid v \in V),\ x^* = (x_v^* \mid v \in V),\ \ y = (y_e \mid e \in E),\ y^* = (y_e^* \mid e \in E).$$

Obviously, $x, x^* \in \mathbb{R}^n; y, y^* \in \mathbb{R}^m$, where $n = |V|$ and $n = |E|$ are numbers of vertices and edges of graph $G = (V, E)$. Let $A = A_G$ be the edge-vertex $m \times n$ incidence matrix of

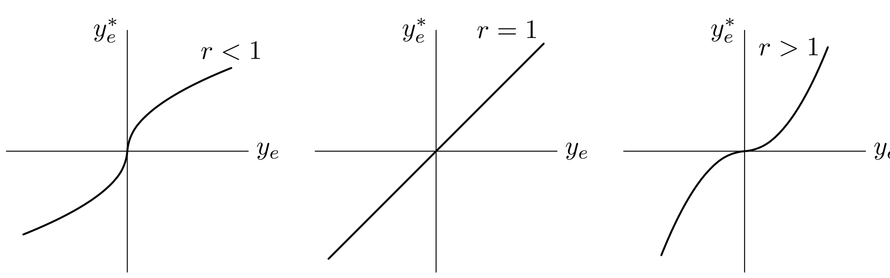

Figure 1: Monomial conductivity law.

graph G, that is, $A(v, e) = inc(v, e)$ for all $v \in V$ and $e \in E$. Equations (6) and (7) can be rewritten in this matrix notation as $y = Ax$ and $x^* = A^T y^*$, respectively.

It is both obvious and well known that these two equations imply the identity

$$(x, x^*) = \sum_{v \in V} x_v x_v^* = \sum_{e \in E} y_e y_e^* = (y, y^*).$$

Let us also recall that vectors y and y^* uniquely define each other, by (4). Thus, given x, the remaining three vectors y, y^*, and x^* are uniquely defined by (6), (7), and (4).

Lemma 1 *For a positive constant c, two quadruples (x, y, y^*, x^*) and $(cx, cy, c^r y^*, c^r x^*)$ can satisfy all equations of (6), (7), and (4) only simultaneously.*

Proof is straightforward. □

4 Two-pole circuits

Let us fix two distinct vertices $a, b \in V$ and call them *poles*. Then, let us add to equations (6), (7), and (4) the first Kirchhoff law

$$x_v^* = 0, \quad \text{for all } \; v \in V \setminus \{a, b\} \tag{8}$$

and fix the potentials in both poles

$$x_a = x_a^0, \qquad x_b = x_b^0. \tag{9}$$

Lemma 2 *The obtained system of equations (4)-(9) has a unique solution.*

Respectively, we will say that the corresponding unique potential vector $x = x(G, a, b)$ solves the circuit (G, a, b) for $x_a = x_a^0$ and $x_b = x_b^0$.

Proof of existence. Given x_a^0 and x_b^0, let us assume without any loss of generality that $x_a^0 \geq x_b^0$ and apply the method of successive approximations to compute x_v for all remaining vertices $v \in V \setminus \{a, b\}$. To do so, let us order these vertices and initialize $x_v = x_a^0$ for all $v \in V \setminus \{b\}$. Then, obviously,

$$x_v^* \geq 0 \;\text{ for all }\; v \in V \setminus \{b\}. \tag{10}$$

Moreover, the inequality is strict whenever v is adjacent to b and $x_a^0 > x_b^0$, In this case, there is a unique potential x'_v such that the corresponding flux x'^*_v becomes equal to 0 after we replace x_v with x'_v leaving all other potentials unchanged. Finally, it is clear that (10) still holds and, moreover,

$$x_a^0 \geq x_v \geq x'_v \geq x_b^0 \;\text{ for all }\; v \in V. \tag{11}$$

We shall consider the vertices of $V \setminus \{a, b\}$ one by one in the defined (cyclical) order and apply in turn the above transformation to each vertex. Obviously, equations (10) and (11) hold all time. In particular, $x_a \equiv x_a^0$, $x_b \equiv x_b^0$, and x_v, for each $v \in V \setminus \{a, b\}$, is a monotone non-increasing sequence bounded by x_b^0 from below. Hence, it has a limit $x_v^0 \in [x_a^0, x_b^0]$.

The limit values of potentials uniquely define the values of all other variables.

Let us show that these limit values satisfy all equations (4)-(9).

To do so, we shall watch x_v^* for all $v \in V$. First, let us notice that x_a^* is non-negative and monotone non-decreasing, while x_b^* is non-positive, and monotone non-increasing.

[Moreover, the voltage y_e and current y_e^* are non-negative and monotone non-decreasing for each $e = (a, v)$ and non-positive and monotone non-increasing for each $e = (v, b)$.]

Then, $x_v^* \geq 0$ all time for all $v \in V \setminus \{b\}$. Yet, the value of x_v^* is not monotone in time: it becomes zero when we treat v and then it monotone increases, while we treat other vertices of $V \setminus \{a, b\}$. Finally, $\sum_{v \in V} x_v^* = 0$ all time, by the conservation of electric charge.

If the first Kirchhoff law holds, that is, $x_v^* = 0$ for all $V \setminus \{a, b\}$, then $x_a^* + x_b^* = 0$, all equations are satisfied, and we stop. Yet otherwise, we obviously can proceed with the potential reduction. Thus, the limit values of x solve (G, a, b) for $x_a = x_a^0$ and $x_b = x_b^0$. □

Remark 1 *A very similar monotone potential reduction, or pumping, algorithm for stochastic games with perfect information was recently suggested in [2].*

Remark 2 *Let us note that the connectivity of G is an essential assumption. Suppose for a moment that G is not connected. If a and b are in one connected component then, obviously, all potentials of any other component must be equal. Yet, the corresponding constants might be arbitrary. If a and b are in two distinct connected components then, obviously, all potentials in these two components must be equal to x_a^0 and x_b^0, respectively, and to an arbitrary constant for another component, if any. Clearly, in this case $x_v^* = 0$ for all $v \in V$.*

Let us also note that the above successive approximation method does not prove the uniqueness of solution. For example, it needs to be shown that the limit potential values do not depend on the order of vertices in the above successive approximations. Moreover, even then it is not clear whether one can get another solution by a different method.

Unfortunately, I have no elementary proof for uniqueness. Of course, existence and uniqueness are both well known; see, e.g., [14, 16, 5, 6]. For example, uniqueness results from the Maxwell principle of minimum dissipation of energy: potential vector x that solves the circuit (G, a, b) for $x_a = x_a^0$ and $x_b = x_b^0$ must minimize the "Joule-Lenz heat"

$$F(y) = \sum_{e \in E} F_e(y_e) = \sum_{e \in E} \int f_e(y_e)\, dy_e, \tag{12}$$

where $y = A_G x$, by (6), and f_e is the conductivity function of edge e.

Obviously, F_e is (strictly) convex if and only if f_e is (strictly) monotone increasing. In particular, strict monotonicity and convexity hold when f_e is defined by (4). In this case

$$F_e(y_e) = \int f_e(y_e)\,dy_e = \frac{|y_e|^{r+1}}{(r+1)\mu_e^s}. \tag{13}$$

Let us notice that (13) turns into the Joule-Lenz formula when $r = s = 1$.

Clearly, $F(A_G x)$ is a strictly convex function of x, since $r > 0$. In remains to recall that if a strictly convex function has a minimum then it is reached in a unique vector. □

The difference $y_{a,b} = x_a - x_b$ is called the voltage, or potential difference, and the value $y^*_{a,b} = x^*_a = -x^*_b$ is called the current in the two-pole circuit (G, a, b).

Proposition 1 *The current $y^*_{a,b}$ and voltage $y_{a,b}$ are still related by a monomial conductivity law with the same parameters r and s:*

$$y^*_{a,b} = f_{a,b}(y_{a,b}) = \lambda^s_{a,b}|y_{a,b}|^r sign(y_{a,b}) = \frac{|y_{a,b}|^r}{|\mu_{a,b}|^s} sign(y_{a,b}). \tag{14}$$

Proof follows immediately from Lemmas 1 and 2. □

The values $\lambda_{a,b}$ and $\mu_{a,b} = \lambda^{-1}_{a,b}$ are called the conductance and, respectively, resistance of the two-pole circuit (G, a, b).

Remark 3 *In fact, we restricted ourselves by the monomial conductivity law (4), because Proposition 1 cannot be extended to any other family of continuous monotone non-decreasing functions, as it was shown in [6].*

Remark 4 *Again, the connectivity of graph G is an essential assumption. Suppose for a moment that G is not connected and poles a and b belong to distinct connected components. Then, obviously, $y^*_{a,b} \equiv 0$.*

Remark 5 *Given a two-pole circuit (G, a, b), where $G = (V, E, \mu)$, and an edge $e_0 \in E$, let us replace a resistance μ_{e_0} by a larger resistance μ'_{e_0} and denote by $G' = (V, E, \mu')$ the obtained circuit. Of course, the total resistance will not decrease either, that is, $\mu'_{a,b} \geq \mu_{a,b}$.*

Yet, how to prove this "intuitively obvious" statement? Somewhat surprisingly, according to [11], the simplest way is to apply the Maxwell principle of the minimum energy dissipation.

Let x and x' be unique potential vectors that solve (G, a, b) and (G', a, b), respectively, while y and y' be the corresponding voltage vectors defined by (6). Let us consider G' and vector x, instead of x'. Since $\mu_{e_0} \leq \mu'_{e_0}$, inequality $F'_e(y_{e_0}) \leq F_e(y_{e_0})$ is implied by (13). Furthermore, $F'_e(y_e) = F_e(y_e)$ for all other $e \in E$ and, hence, $F'(y) \leq F(y)$. In addition, $F'(y') \leq F'(y)$, by the Maxwell principle. Thus, $F'(y') \leq F(y)$ and, by (13), $\mu'_{a,b} \geq \mu_{a,b}$.

Let us say that a vertex v is *between* a and b if $v \neq a, v \neq b$, and v belongs to a path (without self-intersections) between a and b. Then, Lemma 2 can be extended as follows.

Lemma 3 *(o) If $x^0_a = x^0_b$ then $x^0_v = x^0_a = x^0_b$ for all $v \in V$;*

Otherwise, let us assume without any loss of generality that $x^0_a > x^0_b$. Then

- *(i) Inequalities $x_a^0 \geq x_v^0 \geq x_b^0$ holds for all $v \in V$;*
- *(i') If v is between a and b then $x_a^0 > x_v^0 > x_b^0$.*
- *(ii) The voltage y_e and current y_e^* are non-negative whenever $e = (a, v)$ or $e = (v, b)$.*
- *(ii') Moreover, they are strictly positive if also v is between a and b.*

Proof Claim (i),(ii), and (o) result immediately from Lemma 2, yet, connectivity is essential.

In fact, the same is true for (i') and (ii'). Indeed, let us recall the successive approximations, which were instrumental in the proof of Lemma 2; consider a path between a and b and any vertex v in it, distinct from a and b. . Obviously, potential x_v will be strictly reduced from x_a but it cannot reach x_b. □

Remark 6 *If v is not between a and b then inequalities in the above Lemma might be still strict, yet, they might be not strict, too.*

5 Proof of the main inequality and related claims

Theorem 1 *Given an electrical circuit, that is, a* connected *graph $G = (V, E, \mu)$ with strictly positive weights-resistances $(\mu_e | e \in E)$, three arbitrary vertices $a, b, c \in V$, and strictly positive real parameters r and s, then inequality (1) holds: $\mu_{a,b}^{s/r} \leq \mu_{a,c}^{s/r} + \mu_{c,b}^{s/r}$.*
It holds with equality if and only if vertex c belongs to every *path between a and b in G.*

Remark 7 *The proof of the first statement was sketched in [4]; see also [7]. Both claims were proven in [6]. Here we shall follow the plan suggested in [4], yet, give more details.*

Proof Let us fix arbitrary potentials x_a^0 and x_b^0 in vertices a and b. Then, by Proposition 1, all variables, and in particular all remaining potentials, are uniquely defined by equations (4)-(9). Let x_c^0 denote the potential in c.

Without any loss of generality, let us assume that $x_a^0 \geq x_b^0$. Then, $x_a^0 \geq x_c^0 \geq x_b^0$, by Lemma 2. Let us consider the two-pole circuit (G, a, c) and fix in it $x_a = x_a^0$ and $x_c = x_c^0$.

Lemma 4 *The currents in the circuits (G, a, b) and (G, a, c) satisfy inequality $y_{a,b}^* \geq y_{a,c}^*$.*
Moreover, the equality holds if and only if c belongs to every path between a and b.

Proof As in the proof of Lemma 2, we will apply successive approximations to compute a (unique) potential vector $\bar{x} = x(G, a, c)$ that solves the circuit (G, a, c) for $\bar{x}_a = x_a^0$ and $\bar{x}_c = x_c^0$. Yet, as an initial approximation, we shall now take the unique potential vector $x = x(G, a, b)$ that solves the circuit (G, a, b) for $x_a = x_a^0$ and $x_b = x_b^0$. As we know, x uniquely defines all other variables, in particular, $x^* = x^*(G, a, b)$. Obviously, for x^* the first Kirchhoff law holds for all vertices of $V \setminus \{a, b\}$. Yet, for b, it does not hold: $x_b^* < 0$. Let us replace the current potential x_b by x'_b to get $x_b^{'*} = 0$. Obviously, there is a unique such x'_b and $x'_b > x_b$. Yet, after this, the value x_v^* will become negative for some $v \in V \setminus \{a, c\}$.

Let us order the vertices of $V \setminus \{a, c\}$ and repeat the same iterations as in the proof of Lemma 2. By the same arguments, we conclude that in each $v \in V \setminus \{a, c\}$, the potentials x_v form a monotone non-decreasing sequence that converges to a unique solution $\bar{x}_v = x_v(G, a, c)$. By construction, potentials $\bar{x}_a = x_a^0$ and $\bar{x}_c = x_c^0$ remain constant.

Thus, the value x_a^* is monotone non-increasing and the inequality $y_{a,b}^* \geq y_{a,c}^*$ follows.

Let us show that it is strict whenever there is a path P between a and b that does not contain c. Without loss of generality, we can assume that path P is simple, that is, it has no self-intersections. Also without loss of generality, we can order $V \setminus \{a, c\}$, so that vertices of $V(P) \setminus \{a\}$ go first in order from b towards a. Obviously, after the first $|P|$ successive approximations, potentials will strictly increase in all vertices of P, except a. Thus, the value x_a^* will be strictly reduced. Let us remark, however, that the above arguments do not work when c belongs to P, since potential $x_c = x_c^0$ cannot be changed.

Moreover, if c belongs to every path between a and b then clearly $y_{a,b}^* = y_{a,c}^* = y_{c,b}^*$. □

By the obvious symmetry, we conclude that $y_{a,b}^* \geq y_{c,b}^*$, too, and obtain two inequalities

$$y_{a,b}^* = \frac{(x_a^0 - x_b^0)^r}{\mu_{a,b}^s} \geq \frac{(x_a^0 - x_c^0)^r}{\mu_{a,c}^s} = y_{a,c}^*; \quad y_{a,b}^* = \frac{(x_a^0 - x_b^0)^r}{\mu_{a,b}^s} \geq \frac{(x_c^0 - x_b^0)^r}{\mu_{c,b}^s} = y_{c,b}^*, \tag{15}$$

which can be obviously rewritten as follows

$$\left(\frac{\mu_{a,c}}{\mu_{a,b}}\right)^{s/r} \geq \frac{x_a^0 - x_c^0}{x_a^0 - x_b^0}; \quad \left(\frac{\mu_{c,b}}{\mu_{a,b}}\right)^{s/r} \geq \frac{x_c^0 - x_b^0}{x_a^0 - x_b^0} \tag{16}$$

Summing up these two inequalities we obtain (1).

Obviously, (1) holds with equality if and only if $y_{a,b}^* = y_{a,c}^* = y_{c,b}^*$, which, by Lemma 4, happens if and only if c belongs to every path between a and b. □

Remark 8 *As a corollary, we obtain that* $y_{a,b}^* = y_{a,c}^*$ *if and only if* $y_{a,b}^* = y_{c,b}^*$.

Let us also note that $\mu(a, b) = \mu(b, a)$ for all $a, b \in V$. This easily follows from the fact that conductivity functions f_e are odd for all $e \in E$. Furthermore, obviously, $\mu(a, b) > 0$ whenever vertices a and b are distinct. By definition, let us set $\mu(a, b) = 0$ whenever $a = b$.

As we already mentioned, the main inequality (1) obviously implies the triangle inequality (2) whenever $s \geq r$ and it turns into the ultrametric inequality (3), as $s/r \to +\infty$.
Thus, in these two cases the resistances form the metric and ultrametric space, respectively.

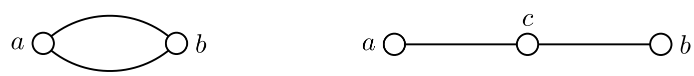

Figure 2: Parallel and series connection

6 Parallel and series connection of edges

Let us consider two simplest two-pole circuits given in Figure 2.

Lemma 5 *The resistances of these two circuits can be determined from formulas*

$$\mu_{a,b}^{-s} = (\mu_{e'}^{-s} + \mu_{e''}^{-s}) \text{ and } \mu_{a,b}^{s/r} = (\mu_{e'}^{s/r} + \mu_{e''}^{s/r}), \tag{17}$$

respectively, where e' and e'' denote two edges of each circuit.

Proof If $r = s = 1$ then (17) turns into familiar high-school formulas. The general case is not much more difficult. Without loss of generality let us assume that $y_{a,b} = x_a - x_b \geq 0$.

In case of parallel connection we obtain the following chain of equalities.

$y_{a,b}^* = f_{a,b}(y_{a,b}) = \frac{y_{a,b}^r}{\mu_{a,b}^s} = f_{e'}(y_{a,b}) + f_{e''}(y_{a,b}) = \frac{y_{e'}^r}{\mu_{e'}^s} + \frac{y_{e''}^r}{\mu_{e''}^s} = \frac{y_{a,b}^r}{\mu_{e'}^s} + \frac{y_{a,b}^r}{\mu_{a,b}^s}.$

To arrive at (17) it is sufficient to compare the third and the last terms.

In case of series connection we stat with determining x_c from the first Kirchhoff law:

$y_{a,b}^* = f_{a,b}(y_{a,b}) = \frac{y_{a,b}^r}{\mu_{a,b}^s} = \frac{(x_a - x_b)^r}{\mu_{a,b}^s} =$
$y_{e'}^* = f_{e'}(y_{e'}) = f_{e'}(x_a - x_c) = \frac{(x_a - x_c)^r}{\mu_{e'}^s} = y_{e''}^* = f_{e''}(y_{e''}) = f_{e''}(x_c - x_b) = \frac{(x_c - x_b)^r}{\mu_{e''}^s}.$

It is sufficient to compare the last and eighth terms to get

$$x_c = \frac{x_b \mu_{e'}^{s/r} + x_a \mu_{e''}^{s/r}}{\mu_{e'}^{s/r} + \mu_{e''}^{s/r}}.$$

Then, we compare the last and forth terms, substitute the obtained x_c, and get (17). □

Let us consider convolution $\mu(t) = (\mu_{e'}^t + \mu_{e''}^t)^{1/t}$. It is well known and easy to see that

$$\mu(t) \to \max(\mu_{e'}, \mu_{e''}) \text{ as } t \to +\infty \quad \text{and} \quad \mu(t) \to \min(\mu_{e'}, \mu_{e''}) \text{ as } t \to -\infty. \tag{18}$$

7 Four examples of resistance distances

A two-pole circuit (G, a, b), where $G = (V, E, \mu)$ is a weighted graph (*non-directed* and *connected*, can model the following four situations.

Example 1: effective resistance of an electrical circuit. In this case μ_e is the resistance of edge e and $r = s = 1$. Respectively, $\mu_{a,b}$ is the standard effective resistance between poles a and b.

Example 2: the length of a shortest route. Let $G = (V, E, \mu)$ model a road network in which μ_e is the length (milage or traveling time) of road e.

In this case, $\mu_{a,b}$ is the distance between terminals a and b, that is, the length of a shortest path between a and b. For parallel and series connection of two edges e' and e'', as in Figure 2, from (18) we obtain, respectively

$\mu^P_{a,b} = \min(\mu_{e'}, \mu_{e''})$ and $\mu^S_{a,b} = \mu_{e'} + \mu_{e''}$.

Hence, $-1/s \to -\infty$ and $s/r = 1$; or in other words, $s \to 0$ and $r/s \to 1$.

Example 3: the inverse value of a maximal flaw. Let $G = (V, E, \mu)$ model a pipeline or transport network in which $\lambda_e = \mu_e^{-1}$ is the capacity of the pipe or road e.

Then, $\lambda_{a,b} = \mu^{-1}_{a,b}$ is the capacity of the whole network with respect to terminals a and b.

In this case, for parallel and series connection we obtain, respectively:

$\lambda^P_{a,b} = \lambda_{e'} + \lambda_{e''}$ and $\lambda^S_{a,b} = \min(\lambda_{e'}, \lambda_{e''})$. Hence,

$-1/s = -1$, $s/r \to \infty$; or in other words, $s = 1$, or more generally, $s \to 1$, while $r \to 0$.

Example 4: the width of a bottleneck route. Let $G = (V, E, \mu)$ model a system of passages (rivers, canals, bridges, roads, etc.) in which μ_e is the *"width"* of passage e, that is, the maximum size or weight of a ship or car that can still pass e.

Then, $\lambda_{a,b} = \mu^{-1}_{a,b}$ is the maximum width of a (bottleneck) path between a and b.

In this case, for parallel and series connection we obtain, respectively

$\lambda^P_{a,b} = \max(\lambda_{e'}, \lambda_{e''})$ and $\lambda^S_{a,b} = \min(\lambda_{e'}, \lambda_{e''})$. Hence,

$-1/s \to -\infty$ and $-s/r \to -\infty$; or in other words, $s \to 0$ and $r/s \to 0$.

Theorem 2 *In each of the above four examples, every distances $d(a, b)$ indeed equals the effective resistances $\mu_{a,b}$ of the two-pole circuit (G, a, b), where $G = (V, E\mu)$ is the weighted graph corresponding to the example, and parameters r and s behave as it was explained above.*

Proof (sketch). The claim is obvious for Example 1 and also for the series-parallel circuits. Yet, it holds in general too. Indeed, it is not difficult to demonstrate for Examples 2 and 4 that all currents tend to concentrate on, respectively, the shortest and bottleneck paths between a and b. In particular, if an edge e does nor belong to such a path then $y^*_e \to 0$ as r and s tend to the corresponding limit values. For Example 3, the limit currents form a maximal flaw between a and b. □

As we already mentioned, in all four examples the distances form a metric and in the last too ultrametric space.

8 k-pole circuits with $r = s = 1$

By Proposition 1, in a two-pole circuit (G, a, b), the total current $y^*_{a,b}$ and voltage $y_{a,b} = x_a - x_b$ are related by a (uniquely defined) conductivity function $f_{a,b}$ with the same parameters r and s as in the functions f_e for each $e \in E$. In other words, every two-pole circuit (G, a, b)

with parameters r and s can be effectively replaced by a single edge (a, b) with the same parameters.

Remark 9 *In [14], Minty proved that the last claim holds not only for monomial but for arbitrary monotone conductivity laws, as well. More precisely, if f_e is non-decreasing for each edge $e \in E$ then there is a (unique) non-decreasing conductivity function $f_{a,b}$ such that the whole two-pole circuit (G, a, b) can be effectively replaced by the single edge (a, b).*

In the case of standard electric resistances, $r = s = 1$, the above "effective replacement statement" can be extended from the two-pole circuits to the k-pole ones.

Given a weighted graph $G = (V, E, \mu)$, let us fix $k \geq 2$ distinct poles $A = \{a_1, \ldots, a_k\} \subseteq V$ and add to equations (4), (6), (7) the first Kirchhoff law for all non-poles:

$$x^*_v = 0 \text{ for } v \in V \setminus A, \tag{19}$$

while in the k poles let us fix the potentials:

$$x_a = x^0_a \text{ for } a \in A. \tag{20}$$

The above two equations in the two-pole case turn into (8) and (9), respectively.

Lemma 6 *The obtained system of equations (4), (6), (7), (19), (20) has a unique solution.*

As in the two-pole case, we shall say that the corresponding (unique) potential vector $x = x(G, A)$ solves the k-pole circuit (G, A) for $x_a = x^0_a, a \in A$.

Proof of the lemma is fully similar to the proof of Lemma 2. □

Two k-poles circuits $(G; a_1, \ldots, a_k)$ and $(G'; a'_1, \ldots, a'_k)$ are called *equivalent* if in them the corresponding fluxes are equal whenever the corresponding potentials are equal, or more accurately, if $x^*_{a_i} = x^*_{a'_i}$ for all $i \in [k] = \{1, \ldots, k\}$ whenever $x_{a_i} = x_{a'_i}$ for all $i \in [k]$.

Proposition 2 *For every k-pole circuit with n vertices (where $n \geq k$) there is an equivalent k-pole circuit with k vertices.*

Proof To show this, we shall explicitly reduce every k-pole circuit with $n + 1$ vertices to a k-pole circuit with n vertices, whenever $n \geq k$. To do so, let us label the vertices of the former circuit G by $0, 1, \ldots, n$ and denote by $\lambda_{i,j}$ the conductance of edge (i, j). (If there is no such edge then $\lambda_{i,j} = 0$.) Let us construct a circuit G' whose n vertices are labeled by $1, \ldots, n$ and conductances are given by formula

$$\lambda'_{i,j} = \lambda_{i,j} + \frac{\lambda_{0,i}\lambda_{0,j}}{\sum_{m=1}^{n} \lambda_{0,m}}. \tag{21}$$

Lemma 7 *The obtained two k-pole circuits (G, A) and (G', A) are equivalent.*

Proof (sketch). Since $r = 1$ the conductance of a pair of parallel edgers is the sum of their conductances, we can assume, without any loss of generality, that G' is a star the with center at 0, that is, G' consists of n edges: $(0,1),\dots,(0,n)$.

Due to linearity, it is sufficient to consider the n basic potential vectors $x^i = (x^i_1,\dots,x^i_n)$ such that $x^i_m = \delta^i_m$, that is, $x^i_i = 1$ and $x^i_m = 0$ whenever $m \neq i$. For each such vector x_i, by the first Kirchhoff law at vertex 0, we obtain that

$$x^i_0 = \frac{\lambda_{0,i}}{\sum_{m=1}^{n} \lambda_{0,m}}, \tag{22}$$

In its turn, this formula easily implies (21). □

Finally, we derive Proposition 2 applying Lemma 7 successively $n - k$ times. □

Remark 10 *Regarding the above proof, we should notice that:*

- $\lambda'_{i,j}$ *gets the same value for vectors* x^i *and* x^j*;*
- *Let* G' *be an* n*-star, that is,* $\lambda_{i,j} = 0$ *for all distinct* i *and* j*. Then, we obtain a mapping that assigns a weighted* n*-clique* K_n *to each weighted* n*-star* S_n*. Obviously, this mapping is a bijection. In particular, for* $n = 3$*, the obtained one-to-one correspondence between the weighted claws and triangles is known as the* Y*-*Δ *transformation.*

As a corollary, we obtain an alternative proof of the triangle inequality (2) in the linear case. Indeed, every three-pole network can be reduced to an equivalent triangle. In its turn, the triangle is equivalent to a claw and for the latter, the triangle inequality is obvious.

For the two-pole case, we can also obtain an important corollary, namely, an explicit formula for the effective conductance $\lambda_{a,b}$. To get it, let us consider the Kirchhoff $n \times n$ conductivity matrix K defined as follows: $K_{i,j} = \lambda_{i,j}$ when $i \neq j$ and $K(i,i) = -\sum_{j \mid j \neq i} \lambda_{i,j}$.

Applying the reduction of Proposition 2 successively $n-2$ times we represent the effective conductance $\lambda_{a,b}$ as the ratio of two determinants:

$$\lambda_{a,b} = \left| \frac{\det(K'_{a,b})}{\det(K''_{a,b})} \right|, \tag{23}$$

where K' and K'' are two submatrices of K obtained by eliminating (i) row a and column b and, respectively, (ii) two rows a, b and two columns a, b.

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