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Abstract. The class of equistable graphs is defined by the existence of a weight structure on the vertices where maximal stable sets are characterized by their weights. This class, not contained in any nontrivial hereditary class, has been studied both from a structural and an algorithmic point of view, however no combinatorial characterization is known for these graphs. We present some structural results about equistable graphs and related graph classes.

1 Introduction

In 1977, Chvátal and Hammer introduced threshold graphs in [4] as those graphs $G = (V, E)$ for which there exists a positive integer t and a weight function $\mathbf{w} : V \to \mathbb{N}$, given by $\mathbf{w} = (w(v) : v \in V)$, such that a subset $S \subseteq V$ is a stable set of G if and only if $\sum_{v \in S} w(v) \le t$. In 1980, Payan introduced equistable graphs in [17] as a generalization of threshold graphs: A graph $G = (V, E)$ is called *equistable* if and only if there exists a positive integer t and a weight function $\mathbf{w} : V \to \mathbb{N}$ on the vertices of G such that a subset $S \subseteq V$ is a maximal stable set of G if and only if $\sum_{v \in S} w(v) = t$. In this case $\mathbf{w}$ is called an *equistable weight function*, while the pair $(\mathbf{w}, t)$ is called an *equistable weight structure*.

Threshold graphs are well understood, admitting several characterizations, efficient recognition algorithms, and efficient algorithms for many optimization problems (see, e.g., [16, 7, 8, 14, 9] and [12, 6, 2]). However, this is not the case for equistable graphs: Finding the maximum cardinality of a stable set or the minimum cardinality of a maximal stable set in an equistable graph is APX-hard [15], while the problems of determining the complexity of recognizing an equistable graph, or providing a combinatorial characterization of equistable graphs are both open. It is not even known whether recognizing an equistable graph is in NP.[1] Moreover, verifying whether a given weight function on the vertices of a graph defines an equistable weight structure is a hard problem [15], indicating that any polynomial time recognition algorithm of equistable graphs would most probably have to rely on the structural properties of equistable graphs.

Our current (non-)understanding of the structure of equistable graphs provides ample motivation for further investigation of their structural properties, initiated for general equistable graphs in [13] and continued for particular graph classes in [13, 10, 18, 11]. In this paper, we provide some new structural results for equistable graphs and their generalization *interstable graphs*, introduced in [15] as the graphs $G = (V, E)$ admitting a weight function $\mathbf{w} : V \to \mathbb{N}$ on their vertices and two integers a, b such that a subset $S \subseteq V$ is a maximal stable set of G if and only if its total weight is between a and b. More specifically, we show that the equistable graphs are closed under the disjoint union, the join with the single vertex graph, the substitution with a graph containing an isolated vertex, and under deleting or contracting a module. We also provide a new decomposition result for equistable graphs based on a particular form of a separating clique. Some of the results extend to interstable graphs. While still far from achieving the ambitious goal of providing a complete structural characterization of equistable graphs, these results complement and extend some of the results by Mahadev et al. [13].

We also show that, if instead of considering real-valued weight functions, we allow weight functions with values in $\mathbb{N}^d$, for some $d \in \mathbb{N}$, then the class of so representable *"d-equistable"* graphs coincides with the class of equistable graphs. On the other hand, every graph can

[1] As mentioned in [11], referring to a remark by Igor Zverovich, there is an exponential time algorithm based on linear programming to recognize an equistable graph.

be realized as a d-interstable graph for a sufficiently large value of d, which leads naturally to the notion of the *interstable dimension* of a graph, that is, the minimum value of d such that G is a d-interstable graph.

The rest of the paper is structured as follows: In the remainder of this section, we give the necessary definitions and notation used throughout the paper. In Section 2, we present some preliminary results about weight functions of equistable and interstable graphs. In Section 3 we consider graphs defined analogously as equistable and interstable graphs but whose maximal stable sets are characterized by means of weight functions with values in $\mathbb{N}^d$, for some fixed $d \in \mathbb{N}$. The main results of this paper, certain graph transformations preserving equistability or interstability, are presented in Section 4; some of the proofs in this section rely on results developed in Sections 2 and 3. We conclude the paper with a brief discussion outlining some possible directions for further research.

Main definitions and notation

$\mathbb{N}$ is the set of positive integer numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ is the set of non-negative integers, and $\mathbb{R}_+$ is the set of non-negative real numbers. For $n \in \mathbb{N}$, we write $[n]$ for the set $\{1, \ldots, n\}$. All the graphs considered are finite, undirected and simple (without loops and multiple edges). A *stable set* in a graph is a subset of pairwise non-adjacent vertices, and a *clique* is a subset of pairwise adjacent vertices. For a vertex v of a graph $G = (V, E)$, we denote by $N_G(v)$ (or simply $N(v)$, if no confusion can arise) the *neighborhood of* v, that is, the set of vertices in G that are adjacent to v, and by $N[v]$ the set $N(v) \cup \{v\}$. Also, for a set $X \subseteq V$, we define by $N_G(X)$ (or simply $N(X)$) the set of all vertices in $V \backslash X$ that are adjacent to at least one vertex of X, by $G[X]$ the subgraph of G induced by X, that is, $G[X] = (X, \{uv : uv \in E, u \in X, v \in X\})$, and by $G - X$ the graph $G[V - X]$. We denote by $\overline{G}$ the complement of a graph $G = (V, E)$, that is, the graph $(V, \{uv : u \in V, v \in V, u \neq v, uv \notin E\})$.

As usual, we denote by K_n, C_n and P_n the complete graph, the chordless cycle and the induced path on n vertices, respectively. For $n = 4$ we also write $P_4(a, b, c, d)$ for a P_4 to indicate that its vertex set is $\{a, b, c, d\}$ and its edges are ab, bc, cd. For a graph H, an H-free graph is a graph without an induced copy of H. If $G_i = (V_i, E_i), i \in \{1, 2\}$, are graphs with $V_1 \cap V_2 = \emptyset$, their *disjoint union* $G_1 \cup G_2$ is the graph $(V_1 \cup V_2, E_1 \cup E_2)$, and their *join* $G_1 + G_2$ is the graph $(V_1 \cup V_2, E_1 \cup E_2 \cup E_{12})$, where $E_{12} = \{v_1v_2 : v_1 \in V_1, v_2 \in V_2\}$. Moreover, if x is a vertex of G_1, the *substitution* $H = G_1(x \to G_2)$ of G_2 for x in G_1 is defined as the graph obtained by deleting x from G_1 and joining each vertex of G_2 to each neighbor of x in G_1.

A graph $G = (V, E)$ is said to be *equistable* if there exists a mapping $\mathbf{w} : V \to \mathbb{R}_+$ and a number $t \in \mathbb{R}_+$ such that for all $S \subseteq V$,

$$S \text{ is a maximal stable set of } G \iff w(S) := \sum_{v \in S} w(v) = t\,.$$

This definition is equivalent to the one given above (with integer weights). We will refer to equistable weight functions (and structures) irrespectively of their integrality. A graph $G = (V, E)$ is said to be *interstable* if there exists a mapping $\mathbf{w} : V \to \mathbb{R}_+$ and two positive numbers a, b such that for all $S \subseteq V$,

$$S \text{ is a maximal stable set of } G \iff a \leq w(S) \leq b\,.$$

If this is the case, then $\mathbf{w}$ is called an *interstable weight function*, while the triple $(\mathbf{w}, a, b)$ is called an *interstable weight structure*. In most of our proofs, we will make use of real-valued equistable and interstable weight functions and structures. However, for some of the proofs, it will be more convenient to assume that they are integer-valued.

Given a graph G, we will denote by $\mathcal{S}(G)$ the set of all (inclusion-wise) maximal stable sets of G and by $\mathcal{T}(G)$ the set of all other nonempty subsets of $V(G)$.

2 Weight functions of equistable and interstable graphs

In this section, we give some results about weight functions of equistable and interstable graphs. We start with a characterization of equistable and interstable graphs by means of the corresponding weight functions only (that is, without target weight sets such as $\{t\}$ or $[a, b]$). This characterization will be used later, in some of the proofs in Section 4. The interstable graphs are precisely those graphs that admit a weight structure on the vertices such that no subset of vertices that is not maximal stable has weight between the weights of two maximal stable sets. A similar observation holds for equistable graphs. More precisely:

Lemma 1. *Let G be a graph, and let $\mathbf{w} : V(G) \to \mathbb{R}_+$. Then:*

1. $\mathbf{w}$ *is not an interstable weight function of G if and only if $w(S_1) \leq w(T) \leq w(S_2)$ for some $S_1, S_2 \in \mathcal{S}(G)$ and $T \in \mathcal{T}(G)$.*

2. $\mathbf{w}$ *is not an equistable weight function of G if and only if either $w(S_1) \neq w(S_2)$ for some $S_1, S_2 \in \mathcal{S}(G)$, or $w(S) = w(T)$ for some $S \in \mathcal{S}(G)$ and some $T \in \mathcal{T}(G)$.*

Proof. To prove the first part of the lemma, suppose that $\mathbf{w}$ is an interstable weight function of G, and let $(\mathbf{w}, a, b)$ be an interstable weight structure of G. Suppose that there exist sets $S_1\, S_2 \in \mathcal{S}(G)$ and $T \in \mathcal{T}(G)$ such that $w(S_1) \leq w(T) \leq w(S_2)$. It follows from the definition of interstability that $a \leq w(S_1)$ and $w(S_2) \leq b$, which in turn implies $w(T) \in [a, b]$, a contradiction.

Conversely, suppose that whenever three sets $S_1\, S_2, T \subseteq V(G)$ satisfy $S_1, S_2 \in \mathcal{S}(G)$ and $w(S_1) \leq w(T) \leq w(S_2)$, it follows that $T \in \mathcal{S}(G)$. Define positive real numbers a and b by $a = \min\{w(S) : S \in \mathcal{S}(G)\}$ and $b = \max\{w(S) : S \in \mathcal{S}(G)\}$. Then, $(\mathbf{w}, a, b)$ is an interstable weight structure of G. This completes the proof of the first statement.

The second part can be proved similarly. □

Corollary 1. *Let G be a graph and let $\mathbf{w} : V(G) \to \mathbb{R}_+$. If there exist sets $S \in \mathcal{S}(G)$ and $T \in \mathcal{T}(G)$ such that $w(S) = w(T)$, then $\mathbf{w}$ is not an interstable weight function of G.*

Proof. Apply part 1 of Lemma 1 with $S_2 = S_1 = S$. □

The next result collects some observations that can be helpful for verifying whether a graph G is equistable or interstable, in particular whether a given function $\mathbf{w} : V(G) \to \mathbb{R}_+$ is an equistable or an interstable weight function of G.

Proposition 1. *Let $G = (V, E)$ be an interstable graph.*

1. *Let $\mathbf{w}$ be an interstable weight function of G, and let $u, v \in V(G)$, $u \neq v$. If $N(u)\backslash\{v\} \neq N(v)\backslash\{u\}$, then $w(u) \neq w(v)$.*
2. *Let $K \subseteq V(G)$ be a clique such that all vertices in K have the same neighborhood outside K. If G is equistable, then every equistable weight function $\mathbf{w}$ of G satisfies $w(u) = w(v)$ for all $u, v \in K$.*
3. *There exists an interstable weight function $\mathbf{w}$ of G such that no two vertices of G are assigned the same weight.*

Proof. 1. Let $N(u)\backslash\{v\} \neq N(v)\backslash\{u\}$. W.l.o.g., we may assume that $z \in V\backslash\{u, v\}$ is a vertex adjacent to u but not to v. Suppose that $w(u) = w(v)$. Since z is not adjacent to v, there is a maximal stable set $S \in \mathcal{S}(G)$ containing z and v. However, replacing v by u in S results in a set from $\mathcal{T}(G)$ with the same weight as S, contradicting the statement of Corollary 1.

2. Let $\mathbf{w}$ be an equistable weight function of G. Let $u, v \in K$, $u \neq v$, and consider a maximal stable set $S \in \mathcal{S}(G)$ such that $u \in S$. Then $v \notin S$, and the set S' obtained by replacing u by v in S is again maximal stable. Since all maximal stable sets have the same weight, we conclude that $w(u) = w(v)$.

3. Let $\mathbf{w}'$ be an interstable weight function of G. We will show how to modify $\mathbf{w}'$ to obtain an interstable weight function with all weights distinct. Let $\delta > 0$ be defined as

$$\delta = \min\{|w(S) - w(T)| : S \in \mathcal{S}(G), T \in \mathcal{T}(G)\}\,,$$

and let $\varepsilon = \delta/(1 + \delta) \in (0, 1)$. Let $V = \{v_1, \ldots, v_n\}$ and let $\alpha_1, \ldots, \alpha_n$ be n real numbers algebraically independent over the rational numbers, which are also algebraically independent of all the numbers $\{w'(v) : v \in V\}$ and satisfy $\varepsilon^{i+1} < \alpha_i < \varepsilon^i$, for all $i \in [n]$. Define a weight function $\mathbf{w}$ by:

$$w(v_i) = w'(v_i) + \alpha_i\,, \quad \text{for all} \;\; i \in [n]\,.$$

By the algebraic independence of the α_i's, no two vertices are assigned the same weight. To complete the proof, we need to show that $\mathbf{w}$ is an interstable weight function. For each set $U \subseteq V$, we have:

$$0 \le w(U) - w'(U) \le \sum_{i=1}^{n} \alpha_i < \sum_{i=1}^{\infty} \varepsilon^i = \frac{1}{1-\varepsilon} - 1 = \delta\,.$$

Suppose that $\mathbf{w}$ is not an interstable weight function. Then, by Lemma 1, there are sets $S_1, S_2 \in \mathcal{S}(G)$ and $T \in \mathcal{T}(G)$ such that $w(S_1) \leq w(T) \leq w(S_2)$. Since $\mathbf{w}'$ is an interstable weight function, we must have either $w'(S_1) < w'(T), w'(S_2) < w'(T)$, or $w'(T) < w'(S_1), w'(T) < w'(S_2)$. In each of these two cases, using the fact that each set's weight change is in the interval $[0, \delta)$, we derive a contradiction with the chain of inequalities $w(S_1) \leq w(T) \leq w(S_2)$. The proof is complete. □

3 Vector weights

The concept of equistable weight functions and structures can easily be generalized to more dimensions, in the sense that each vertex gets assigned a vector from $\mathbb{N}^d$ (or, equivalently, from $\mathbb{R}^d_+$). A graph G is called *vector equistable* if there is a $d \in \mathbb{N}$ such that there is an assignment of vectors from $\mathbb{N}^d$ to the vertices of G such that all maximal stable sets of G have the same weight (which is now an element of $\mathbb{N}^d$), and this weight is not achieved by any other set. Clearly, every equistable graph is vector equistable. As we show next, the converse also holds. This fact will be used in the next section, in our discussion of graph transformations preserving equistability.

Proposition 2. *Let d be a positive integer, let V be a finite set and let $\{\mathbf{w}(v) : v \in V\} \cup \{\mathbf{b}\}$ be a collection of vectors in $\mathbb{Z}^d_+$. Then, there exists a collection of nonnegative integers $\{w(v) : v \in V\} \cup \{\beta\}$ such that, for all $S \subseteq V$, $\mathbf{w}(S) = \mathbf{b}$ if and only if $w(S) = \beta$.*

Proof. The proof is by induction on d, the case $d = 1$ being trivial. Suppose now that $d \geq 2$. For each $v \in V$, write $\mathbf{w}(v)$ as

$$\mathbf{w}(v) = (w_1(v), \mathbf{w}'(v)) \tag{1}$$

where $w_1(v)$ is a nonnegative integer, and $\mathbf{w}'(v) \in \mathbb{Z}^{d-1}_+$. In a similar way, write $\mathbf{b}$ as

$$\mathbf{b} = (\beta_1, \mathbf{b}') . \tag{2}$$

By induction hypothesis, there exists a collection of nonnegative integers $\{w_2(v) : v \in V\} \cup \{\beta_2\}$ such that, for all $S \subseteq V$,

$$\mathbf{w}'(S) = \mathbf{b}' \quad \text{if and only if} \quad w_2(S) = \beta_2 . \tag{3}$$

Now, let N be an integer satisfying

$$N > \max\{w_2(V), \beta_2\} , \tag{4}$$

and define the integers $\{w(v) : v \in V\} \cup \{\beta\}$ by

$$w(v) = N w_1(v) + w_2(v), \quad \beta = N\beta_1 + \beta_2 . \tag{5}$$

We claim that for all $S \subseteq V$,

$$\mathbf{w}(S) = \mathbf{b} \quad \text{if and only if} \quad w(S) = \beta\,.$$

Consider a set $S \subseteq V$. Assume first that the equality $\mathbf{w}(S) = \mathbf{b}$ holds. By reading it according to (1) and (2) we obtain $w_1(S) = \beta_1$ and $\mathbf{w}'(S) = \mathbf{b}'$. By (3), we have $w_2(S) = \beta_2$. The equality $w(S) = \beta$ now follows from (5).

Conversely, assume that $S \subseteq V$ satisfies $w(S) = \beta$. First, observe that suffices to show that $w_1(S) = \beta_1$. Indeed, if this is the case, then the equality $w(S) = \beta$ and the relations (5) imply that $w_2(S) = \beta_2$. By (3), this yields $\mathbf{w}'(S) = \mathbf{b}'$. Together with $w_1(S) = \beta_1$, this is clearly equivalent to $\mathbf{w}(S) = \mathbf{b}$.

We will show $w_1(S) = \beta_1$ in two steps. If $w_1(S) \leq \beta_1 - 1$, then $w(S) = Nw_1(S) + w_2(S) \leq N\beta_1 - N + w_2(S) < N\beta_1 \leq N\beta_1 + \beta_2 = \beta$, contrary to the assumption $w(S) = \beta$. Therefore $w_1(S) \geq \beta_1$. If $w_1(S) \geq \beta_1 + 1$, then $w(S) = Nw_1(S) + w_2(S) \geq N\beta_1 + N > N\beta_1 + \beta_2 = \beta$, which again contradicts the assumption $w(S) = \beta$. This shows that $w_1(S) = \beta_1$ which concludes the proof of the proposition. □

Corollary 2. *Vector-equistable graphs are precisely the equistable graphs.*

Now, let us consider the case of vector-interstable graphs. Let $d \in \mathbb{N}$. A graph $G = (V, E)$ is said to be *d-interstable* if there exists a mapping $\mathbf{w} : V \to \mathbb{N}^d$ and two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{N}^d$ such that for all $S \subseteq V$,

$$S \text{ is a maximal stable set of } G \iff \mathbf{a} \leq \mathbf{w}(S) := \sum_{v \in S} \mathbf{w}(v) \leq \mathbf{b}$$

(inequality interpreted componentwise). A graph G is said to be *vector interstable* if there exists a natural number $d \in \mathbb{N}$ such that G is d-interstable.

Proposition 3. *Every graph is vector interstable.*

Proof. Let $G = (V, E)$ be a graph. Consider the following system of linear inequalities:

$$\begin{array}{rcll} x_u + x_v & \leq & 1 & \forall \;\; uv \in E\,, \\ x_v + \sum_{u:uv \in E} x_u & \geq & 1 & \forall \;\; v \in V\,. \end{array}$$

Clearly, the only feasible solutions to the above system from the set $\{0,1\}^V$ are the characteristic vectors of the maximal stable sets of G. With respect to solutions from $\{0,1\}^V$, the above system can be rewritten equivalently in the following form:

$$\begin{array}{rcccll} 0 & \leq & x_u + x_v & \leq & 1 & \forall \;\; uv \in E\,, \\ 1 & \leq & x_v + \sum_{u:uv \in E} x_u & \leq & |V| & \forall \;\; v \in V\,. \end{array}$$

This shows that G is $(|V| + |E|)$-interstable. □

Since not every graph is interstable (for example, odd cycles of order at least 5, their complements and even paths P_{2k} for $k \geq 3$ are not interstable), vector-interstable graphs form a proper generalization of interstable graphs.

Proposition 3 motivates the following definition: The *interstable dimension* of a graph G is the minimum integer d such that G is a d-interstable graph. Geometrically, the interstable dimension of a graph is equal to the minimum number of pairs of parallel hyperplanes that separate the (characteristic vectors of) the maximal stable sets from the (characteristic vectors of) other subsets of vertices—in such a way that the characteristic vectors of the maximal stable sets are precisely those vectors in $\{0,1\}^n$ that lie between the two hyperplanes from each of the pairs. Notice the analogy with the *threshold dimension* of a graph [3, 4], which asks for the minimum number of inequalities needed to distinguish between (the characteristic vectors of) stable and nonstable sets of the given graph.

4 Graph transformations

In this section, we present several graph transformations that preserve the properties of being equistable (resp. interstable). We start by showing that equistable and interstable graphs are closed under addition and deletion of dominating vertices.

Theorem 1. *Let G be a graph and let G' be the graph obtained from G by introducing a new vertex and joining it to all vertices of G. Then G' is equistable (resp. interstable) if and only if G is equistable (resp. interstable).*

Proof. Let v^* be the added vertex. Clearly, $\mathcal{S}(G') = \mathcal{S}(G) \cup \{\{v^*\}\}$. Suppose that G is interstable, and let $(\mathbf{w}, a, b)$ be an interstable weight structure of G. Define $\mathbf{w}' : V(G') \to \mathbb{R}_+$ by:

$$w'(v) = \begin{cases} w(v), & \text{if } v \in V(G); \\ b, & \text{if } v = v^*. \end{cases}$$

Then $(\mathbf{w}', a, b)$ is an interstable weight structure of G'. In particular, if $a = b$ (that is, if G is equistable), then $(\mathbf{w}', b)$ is an equistable weight structure of G'.

Conversely, if $(\mathbf{w}, t)$ is an equistable weight structure of G', then $(\mathbf{w}\big|_{V(G)}, t)$ is an equistable weight structure of G, and if $(\mathbf{w}, a, b)$ is an interstable weight structure of G', then $(\mathbf{w}\big|_{V(G)}, a, b)$ is an interstable weight structure of G. □

Our next result deals with another basic operation, the disjoint union of two graphs.

Theorem 2. *Let G be equistable, and let H be equistable (resp. interstable). Then the disjoint union of G and H is equistable (resp. interstable). In particular, equistable graphs are closed under disjoint union.*

Proof. Let G' be the disjoint union of G and H. Then $\mathcal{S}(G') = \{S_1 \cup S_2 : S_1 \in \mathcal{S}(G), S_2 \in \mathcal{S}(H)\}$. Let $(\mathbf{w}^G, t^G)$ be an equistable weight structure of G.

Suppose first that H is equistable, with an equistable weight structure $(\mathbf{w}^H, t^H)$. Define $\mathbf{w} : V(G') \to \mathbb{R}^2_+$ by:

$$w(v) = \begin{cases} [\ w^G(v) \quad 0\ ]^T, & \text{if } v \in V(G); \\ [\ 0 \quad w^H(v)\ ]^T, & \text{if } v \in V(H). \end{cases}$$

Then, it is easy to see that $(\mathbf{w}, (t^G, t^H)^T)$ is a vector-equistable weight structure of G'. It follows from Corollary 2 that G' is equistable.

Suppose that H is interstable, with an interstable weight structure $(\mathbf{w}^H, a^H, b^H)$. We can define an interstable weight structure of G', as follows. Let $N = \max\{w^H(V(H)), b^H\} + 1$. Define $a = Nt^G + a^H$, $b = Nt^G + b^H$ and define $\mathbf{w} : V(G') \to \mathbb{R}_+$ by:

$$w(v) = \begin{cases} Nw^G(v), & \text{if } v \in V(G); \\ w^H(v), & \text{if } v \in V(H). \end{cases}$$

It is a matter of routine verification to check that $(\mathbf{w}, a, b)$ is an interstable weight structure of G'. This shows that G' is interstable and the theorem is proved. □

For equistable graphs, it is easy to see that the converse of Theorem 2 holds, that is, equistable graphs are closed under taking connected components. More generally, we show now that equistable and interstable graphs are closed under taking modules. A *module* in a graph is a nonempty subset of vertices such that every vertex outside the set is either adjacent to all the vertices in the set, or to none of them. We also show that equistable and interstable graphs are closed under contractions of modules. A *neighbor* of a module M is a vertex outside M that is adjacent to every vertex in M.

Theorem 3. *Let G be equistable (resp. interstable), and let M be a module of G. Then, the graph induced by M is equistable (resp. interstable). Furthermore, the graph obtained from G by replacing M with a single vertex adjacent to the neighbors of M is equistable (resp. interstable).*

Proof. It is easy to see that the intersection of every maximal stable set of G with M is either empty, or a maximal stable set of $G[M]$. This observation, together with Lemma 1 allows us to show that every equistable (resp. interstable) weight function of G naturally defines an equistable (resp. interstable) weight function of the subgraph induced by the module. More precisely, we have the following claim.

Claim. Let $\mathbf{w} : V(G) \to \mathbb{R}_+$, and let $\mathbf{w}_M$ denote the restriction of $\mathbf{w}$ to M. If $\mathbf{w}$ is an equistable weight function of G, then $\mathbf{w}_M$ is an equistable weight function of $G[M]$. If $\mathbf{w}$ is an interstable weight function of G, then $\mathbf{w}_M$ is an interstable weight function of $G[M]$.

Clearly, the claim implies the first statement of the theorem.

Proof of Claim. Suppose that $\mathbf{w}_M$ is not an equistable weight function of $G[M]$. Then, by Lemma 1 either $w(S_1) \neq w(S_2)$ for some $S_1, S_2 \in \mathcal{S}(G[M])$, or $w(S) = w(T)$ for some $S \in \mathcal{S}(G[M])$ and $T \in \mathcal{T}(G[M])$.

Let S' be an arbitrary maximal stable set of $G - (M \cup N_G(M))$.

If $w(S_1) \neq w(S_2)$ for some $S_1, S_2 \in \mathcal{S}(G[M])$, then let $S_i' = S_i \cup S'$ for $i \in \{1, 2\}$. Then, the sets S_1' and S_2' are maximal stable sets of G with $w(S_1') \neq w(S_2')$. By part 2 of Lemma 1, this implies that $\mathbf{w}$ is not an equistable weight function of G.

If $w(S) = w(T)$ for some $S \in \mathcal{S}(G[M])$ and $T \in \mathcal{T}(G[M])$, then we extend S and T to $S \cup S'$ and $T \cup S'$ respectively; the sets $S \cup S'$ and $T \cup S'$ then satisfy $S \cup S' \in \mathcal{S}(G)$ and $T \cup S' \in \mathcal{T}(G)$, and $w(S \cup S') = w(T \cup S')$, which again, by Lemma 1, implies that $\mathbf{w}$ is not an equistable weight function of G.

The second statement of the claim can be proved analogously, using part 1 of Lemma 1.

It remains to show that the graph G' obtained from G by replacing M with a single vertex (call it x) adjacent to the neighbors of M is equistable (resp. interstable). First, assume that G is equistable, and let $(\mathbf{w}, t)$ be an equistable weight structure of G. Let S be a maximal stable set in $G[M]$, and let $w'(x) = \mathbf{w}_M(S)$, the $\mathbf{w}_M$-weight of S. By the above claim, $\mathbf{w}_M$ is an equistable weight function of $G[M]$, so the value of $w'(x)$ does not depend on the choice of S. Moreover, for every $v \in V(G)\backslash M$, let $w'(v) = w(v)$. Using the fact that $\mathcal{S}(G') = \{S : S \in \mathcal{S}(G),\, S \cap M = \emptyset\} \cup \{(S\backslash M) \cup \{x\} : S \in \mathcal{S}(G),\, S \cap M \neq \emptyset\}$ it is not hard to verify that $(\mathbf{w}', t)$ is an equistable weight structure of G'.

Similarly, if $(\mathbf{w}, a, b)$ is an interstable weight structure of G, then $(\mathbf{w}', a, b)$ with

$$w'(v) = \begin{cases} w(S), & \text{if } v = x; \\ w(v), & \text{otherwise.} \end{cases}$$

(where S is an arbitrary maximal stable set of $G[M]$), is an interstable weight structure of G'. This completes the proof. □

Closely related to the notion of modules are graph substitutions. Recall that, for two disjoint graphs G_1, G_2 and a vertex x of G_1, the substitution $H = G_1(x \to G_2)$ is defined as the graph obtained by deleting x and joining each vertex of G_2 to each neighbor of x in G_1. (It is clear that $V(G_2)$ is a module of H.) Determining whether equistable graphs are closed under substitution is an open question. According to Theorem 5.4 of [13], equistable graphs are closed under substitution with an edgeless graph. We extend this result as follows.

Theorem 4. *Let G_1 and G_2 be disjoint equistable graphs, and x a vertex of G_1. Then the graph $H = G_1(x \to (K_1 \cup G_2))$ is equistable.*

Proof. For simplicity, let $V(H) = V(G_1) \cup V(G_2)$ (where the vertex x corresponds to the K_1). Let (w_1, t_1) and (w_2, t_2) be *integer* equistable weight structures of G_1 and G_2, respectively. We define an equistable weight structure $(\mathbf{w}, t)$ of H, as follows:

Fix an irrational number $\varepsilon \in (0,1)$ (say $\varepsilon = \frac{1}{\sqrt{2}}$); the weight function $\mathbf{w} : V(H) \to \mathbb{R}_+$ is given by

$$w(v) = \begin{cases} w_1(v), & \text{if } v \in V(G_1)\backslash\{x\}; \\ \varepsilon w_1(x), & \text{if } v = x; \\ \frac{(1-\varepsilon)w_1(x)}{t_2} w_2(v), & \text{if } v \in V(G_2). \end{cases},$$

The target weight t is given by $t = t_1$.

We need to show that $(\mathbf{w}, t)$ is an equistable weight structure of H, that is, that $S \in \mathcal{S}(H)$ if and only if $w(S) = t$. First, we express the set of all maximal stable sets of H in terms of the maximal stable sets of G_1 and G_2:

$$\mathcal{S}(H) = \{S \in \mathcal{S}(G_1) : x \notin S\} \cup \{S \cup S' : S \in \mathcal{S}(G_1) \text{ such that } x \in S, \text{ and } S' \in \mathcal{S}(G_2)\}. \tag{6}$$

Now, let us show that $w(S) = t$ for every $S \in \mathcal{S}(H)$. Let S be a maximal stable set S of H. We consider two cases.

Case 1. $x \notin S$. Then $S \subseteq V(G_1)$ and moreover $S \in \mathcal{S}(G_1)$. Therefore, the total weight of S satisfies $w(S) = w_1(S) = t_1 = t$.

Case 2. $x \in S$. For $i \in \{1,2\}$, let $S_i = S \cap V(G_i)$. Then, $S_1 \in \mathcal{S}(G_1)$ and $S_2 \in \mathcal{S}(G_2)$, and we can compute the total cots of S as

$$w(S) = w(S_1\backslash\{x\}) + w(x) + w(S_2) = w_1(S_1\backslash\{x\}) + \varepsilon w_1(x) + \frac{(1-\varepsilon)w_1(x)}{t_2} w_2(S_2) =$$

$$= \Big(t_1 - w_1(x)\Big) + \varepsilon w_1(x) + \frac{(1-\varepsilon)w_1(x)}{t_2} \cdot t_2 = t_1 = t\,.$$

For the converse direction, let $S \subseteq V(H)$ be an arbitrary set such that $w(S) = t$. (We need to show that $S \in \mathcal{S}(H)$.) Again, we consider two cases.

Case 1. $S \subseteq V(G_1)$. Suppose that $x \in S$. Then $t = w(S) = w_1(S\backslash\{x\}) + \varepsilon w_1(x) \notin \mathbb{Z}$, contradicting the fact that $t = t_1$ is integer. Thus, x does not belong to S. Therefore $w(S) = w_1(S) = t_1$, which implies that $S \in \mathcal{S}(G_1)$. By (6), S is also a maximal stable set of H.

Case 2. $S \nsubseteq V(G_1)$. That is, $S \cap (\{x\} \cup V(G_2)) \neq \emptyset$. For $i \in \{1,2\}$, let $S_i = S \cap V(G_i)$.

Suppose that $x \notin S$. Then $S_2 \neq \emptyset$, and the weight of S satisfies

$$t = w(S) = w(S_1) + w(S_2) = w_1(S_1) + \frac{(1-\varepsilon)w_1(x)}{t_2} w_2(S_2)\,.$$

However, since $w_2(S_2) \neq 0$, this last expression is an irrational number, contradicting the fact that $t = t_1$ is integer. Therefore, $x \in S$.

Once again, we expand the expression for the weight of S, this time using the fact that $x \in S$:

$$t = w(S) = w(S_1 \backslash \{x\}) + w(x) + w(S_2) = w_1(S_1 \backslash \{x\}) + \varepsilon w_1(x) + \frac{(1-\varepsilon)w_1(x)}{t_2} w_2(S_2)\,.$$

Rearranging the terms, we obtain

$$\left(w_1(S_1 \backslash \{x\}) + \frac{w_1(x)}{t_2} w_2(S_2)\right) + \varepsilon\left(w_1(x) - \frac{w_1(x)}{t_2} w_2(S_2)\right) = 0\,.$$

Since ε is irrational, both coefficients must vanish:

$$w_1(S_1 \backslash \{x\}) + \frac{w_1(x)}{t_2} w_2(S_2) = 0 \tag{7}$$

and

$$w_1(x) - \frac{w_1(x)}{t_2} w_2(S_2) = 0\,. \tag{8}$$

Equation (8) yields $w_2(S_2) = t_2$, which, together with the equation (7), implies that $w_1(S_1) = t_1$. Since (w_1, t_1) and (w_2, t_2) are equistable weight structures of G_1 and G_2 respectively, the sets S_1 and S_2 are maximal stable sets od G_1 and G_2, respectively. It now follows from (6) that $S = S_1 \cup S_2$ is a maximal stable set of H. The proof is complete. □

Corollary 3. *Let G_1 and G_2 be disjoint graphs, and x a vertex of G_1. Then $H = G_1(x \to (K_1 \cup G_2))$ is equistable if and only if both G_1 and G_2 are equistable.*

Proof. Suppose that $H = G_1(x \to (K_1 \cup G_2))$ is equistable. The graph G_1 can be obtained from H by contracting the module corresponding to $K_1 \cup G_2$ to a single vertex. The graph G_2 is a subgraph of H induced by a module of it. Hence, by Theorem 3, both G_1 and G_2 are equistable.

The converse direction follows directly from Theorem 4. □

As an application of the above structural results, we now show that all (P_4, $\overline{2P_3}$)-free graphs are equistable. It is known that all P_4-free graphs are equistable [13]. However, the proof of this fact relies, among other results, on the following two results: (i) the strongly equistable graphs are closed under disjoint union and join, and (ii) the strongly equistable graphs are equistable (for the definition of strongly equistable graphs, see Section 5). Here, we propose a direct proof of equistability for a subclass of P_4-free graphs defined by one more forbidden induced subgraph, the complement of $2P_3 = P_3 \cup P_3$. The resulting class, that of all (P_4, $\overline{2P_3}$)-free graphs, properly contains the threshold and the domishold graphs (see [4, 1]).[2]

[2] A direct proof of the fact that P_4-free graphs are equistable would follow if one could show that equistable graphs are closed under join. However, to the best of our knowledge, this is an open problem.

We first recall the recursive definition of quasi-threshold graphs [19]. A graph G is said to be *quasi-threshold* if either $G = K_1$, or G equals to the join of another quasi-threshold graph G' and K_1, or G equals to the disjoint union of two quasi-threshold graphs. Theorems 1-2 immediately show that quasi-threshold graphs are equistable. A particular case of Theorem 4 (as well as a result from [13]) shows that equistable graphs are closed under substitution with an edgeless graph. Therefore, every graph obtained from a quasi-threshold graph by a sequence of substitutions with an edgeless graph, is equistable.

Below we show that these are precisely the (P_4, $\overline{2P_3}$)-free graphs. The proof relies on the characterization of the quasi-threshold graphs as the graphs that contain no induced P_4 or C_4 [19].

Proposition 4. *The class of graphs, obtained from a quasi-threshold graph by a sequence of substitutions with an edgeless graph, equals to the class of* (P_4, $\overline{2P_3}$)*-free graphs.*

Proof. First, we show that every graph obtained from a quasi-threshold graph by a sequence of substitutions with an edgeless graph, is (P_4, $\overline{2P_3}$)-free. Since quasi-threshold graphs are (P_4, C_4)-free and thus also (P_4, $\overline{2P_3}$)-free, it is enough to show that (P_4, $\overline{2P_3}$)-free graphs are closed under substitution of a vertex with an edgeless graph. Let $G = H(x \to K)$ where $x \in V(H)$, K is an edgeless graph and H is (P_4,$\overline{2P_3}$)-free. Thus, K is a stable set in G all the elements of which have the same neighborhood. Therefore, every induced copy H_0 of a P_4 or a $\overline{2P_3}$ in G contains at most one vertex in K. In particular, this implies that G is (P_4,$\overline{2P_3}$)-free, for otherwise H would not be (P_4,$\overline{2P_3}$)-free.

For the converse direction, let G be a (P_4, $\overline{2P_3}$)-free graph. Let us partition the vertex set of G according to the equivalence relation $u \sim v$ if and only if $N(u) = N(v)$. Let H be the graph whose vertices are the equivalence classes of this relation, and where two classes C and C' are joined by an edge if and only if there is an edge in G of the form $\{v, v'\}$ with $v \in C$, $v' \in C'$. It is clear that G is obtained from H by a sequence of substitutions with an edgeless graph. It remains to show that H is (P_4, C_4)-free. Since G contains an induced subgraph isomorphic to H and G is P_4-free, H must be P_4-free too. Now suppose that H contains an induced C_4, with vertices $ABCD$ in the cyclic order. Since $\{A, C\}$ is a stable set in H and A and C are different equivalence classes, we may assume without loss of generality that there is a vertex in H, say X, that is adjacent to A but not to C. Then, we conclude that X is adjacent to B (to avoid an induced P_4 on $\{X, A, B, C\}$) and to D (to avoid an induced P_4 on $\{X, B, C, D\}$). A similar argument applied to the pair $\{B, D\}$ shows that there is a vertex in H, say Y, that is adjacent to A, B, C, but not to D. Then, Y is also adjacent to X, since otherwise an induced P_4 arises on $\{X, D, C, Y\}$. But now, the subgraph of H induced by $\{A, B, C, D, X, Y\}$ is isomorphic to a $\overline{2P_3}$, contrary to the assumption that G was $\overline{2P_3}$-free. This contradiction completes the proof of the proposition. □

Corollary 4. *Every* (P_4, $\overline{2P_3}$)*-free graph is equistable.*

We now present another graph transformation that preserves equistability. In particular, this transformation provides a new way of building bigger equistable graphs from smaller ones. For example, let A denote the graph obtained from a path P_4 by introducing a new vertex and joining it to the two middle vertices of the path. It is easy to see that A is equistable, yet A does not seem to be decomposable into smaller equistable graphs by means of the known results. Note that A can be built up from two copies of a one-vertex graph, by introducing a new triangle and joining each of the two initial vertices to a distinct vertex of the triangle. Surprisingly enough, it turns out that for equistability, this is a meaningful (de)composition.

To describe the decomposition in general, we introduce the notion of a central clique. A clique $K = \{v_0, v_1, \ldots, v_k\}$ with $k \geq 1$ in a graph G is said to be *central* if $N_G(v_0) = \{v_1, \ldots, v_k\}$ and the graph $G - K$ consists of k connected components $G_1, \ldots, G_k$ such that for all $i \in [k]$, it holds that

$$N_G(v_i) = (K - \{v_i\}) \cup V(G_i)\,.$$

The components $G_1, \ldots, G_k$ are called the *leaves* of the clique K.

Theorem 5. *Let $K = \{v_0, v_1, \ldots, v_k\}$ be a central clique in a graph G, with leaves $G_1, \ldots, G_k$. Then G is equistable if and only if each of $G_1, \ldots, G_k$ is equistable.*

Proof. One direction is immediate: If G is equistable, then each of the G_i's is also equistable, by Theorem 3.

Suppose now that each of $G_1, \ldots, G_k$ is equistable, and let $(\mathbf{w}^i, t_i)$ denote an *integer* equistable weight structure of G_i, for $i \in [k]$. We will construct an equistable weight structure $(\mathbf{w}, t)$ of G.

Let $\alpha_0, \alpha_1 \ldots, \alpha_k$ be $k+1$ real numbers algebraically independent over the rational numbers, which are also algebraically independent of all the numbers that occur in the weight structures of the G_i's. Let us define a weight structure $(\mathbf{w}, t)$ of G as follows:

$$w(v) = \begin{cases} \alpha_i w^i(v), & \text{if } v \in V(G_i); \\ \alpha_0, & \text{if } v = v_0; \\ \alpha_0 + \alpha_i t_i, & \text{if } v = v_i \text{ for some } i \in [k]. \end{cases}$$

Moreover, let $t = \alpha_0 + \sum_{i=1}^{k} \alpha_i t_i$.

First, let S be a maximal stable set of G. Then, S contains precisely one vertex of K. Suppose first that $S \cap K = \{v_i\}$ for some $i \in [k]$. Then, $S \cap V(G_i) = \emptyset$, and $S \cap V(G_j)$ is a maximal stable set of G_j, for each $j \neq i$. Therefore, $w^j(S \cap V(G_j)) = t^j$. It follows that

$$w(S) = w(v_i) + \sum_{j \in [k] \setminus \{i\}} w(S \cap V(G_j)) = \alpha_0 + \alpha_i t_i + \sum_{j \in [k] \setminus \{i\}} \sum_{v \in S \cap V(G_j)} \alpha^j w^j(v)$$

$$= \alpha_0 + \alpha_i t_i + \sum_{j \in [k] \setminus \{i\}} \alpha^j \sum_{v \in S \cap V(G_j)} w^j(v) = \alpha_0 + \alpha_i t_i + \sum_{j \in [k] \setminus \{i\}} \alpha^j t^j = t\,.$$

If $S \cap K = \{v_0\}$, then $S \cap V(G_i)$ is a maximal stable set of G_i, for each $i \in [k]$. A similar calculation as above shows that $w(S) = t$ in this case too.

Conversely, let $S \subseteq V(G)$ satisfy $w(S) = t = \alpha_0 + \sum_{i=1}^{k} \alpha_i t_i$. Since the α_i's are algebraically independent, it follows that $|S \cap K| = 1$. If $S \cap K = \{v_i\}$ for some $i \in [k]$, then $w(S \backslash K) = \sum_{j \in [k] \backslash \{i\}} \alpha_j t_j$. Using the algebraic independence of the α_i's, we conclude that $w(S \cap V(G_i)) = 0$ and $w(S \cap V(G_j)) = \alpha_j t_j$ for $j \neq i$. Clearly, this implies that $S \cap V(G_i) = \emptyset$ and $w^j(S \cap V(G_j)) = t_j$. Since each G_j is an equistable graph, the intersection $S \cap V(G_j)$ is a maximal stable set of G_j. It follows that S is a maximal stable set of G. The case $S \cap K = \{v_0\}$ is handled analogously. This completes the proof. □

We conclude this section with two more simple transformations that preserve equistability (resp. interstability).

Proposition 5. *Let G be equistable (resp. interstable), and let S be a stable set of G. Then the graph, obtained from G by removing all the vertices in $S \cup N(S)$, is also equistable (resp. interstable).*

Proof. By induction, it suffices to show the proposition for the case when S consists of a single vertex, say $S = \{v\}$. Let $(\mathbf{w}, t)$ be an equistable weight structure of G, and let $\mathbf{w}'$ denote the restriction of $\mathbf{w}$ to $V(G) - N[v]$. There is a one-to-one correspondence between the maximal stable sets of $G - N[v]$ and the maximal stable sets of G containing v. It is then straightforward to verify that $(\mathbf{w}', t - w(v))$ is an equistable weight structure of $G - N[v]$. Similarly, if $(\mathbf{w}, a, b)$ is an interstable weight structure of G, then $(\mathbf{w}', a - w(v), b - w(v))$, where $\mathbf{w}'$ is the restriction of $\mathbf{w}$ to $V(G) - N[v]$, is an interstable weight structure of $G - N[v]$. □

Proposition 6. *Let G be equistable (resp. interstable), and let x be a vertex of G such that $N(x)$ intersects every maximal stable set of $G - x$. Then $G - x$ is also equistable (resp. interstable).*

Proof. We have $\mathcal{S}(G - x) = \{S \in \mathcal{S}(G) : x \notin S\}$. Therefore, it follows immediately from Lemma 1 that the restriction of an equistable (resp. interstable) weight function of G to $V(G)\backslash\{x\}$ yields an equistable (resp. interstable) weight function of $G - x$. □

5 Concluding remarks

In [13], Mahadev et al. defined a graph $G = (V, E)$ to be *strongly equistable* if for each $T \in \mathcal{T}(G)$ and each $\gamma \leq 1$ there exists a weight function $\mathbf{w} : V \to \mathbb{R}_+$ such that $w(S) = 1$ for all $S \in \mathcal{S}$, and $w(T) \neq \gamma$. They showed that every strongly equistable graph is equistable, and conjectured that the converse assertion is valid. To the best of our knowledge, the conjecture is still open.[3] In view of this conjecture, a natural direction for further research would be to try to establish results analogous to those of Section 4 also for the strongly equistable graphs. Analogs of Theorems 1 and 2 for the strongly equistable graphs were already obtained in [13].

Other topics for future investigation include determining whether the equistable and/or interstable graphs are closed under join and substitution, and to try to identify other graph transformations that preserve equistability—with the goal of making further progress toward eventual combinatorial characterizations of equistable and related graphs. We also feel that it would be interesting to investigate the basic properties of the interstable dimension of a graph.

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