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SEPARABLE DISCRETE FUNCTIONS

RRR 26-2009,

RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

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Endre Boros Ondrej Cepek Vladimir Gurvich Kazuhisa Makino
Igor E. Zverovich

[1]

SEPARABLE DISCRETE FUNCTIONS

ENDRE BOROS, ONDŘEJ ČEPEK, VLADIMIR A. GURVICH,
KAZUHISA MAKINO, AND IGOR E. ZVEROVICH

Abstract. An *assignable matrix* $A = (a_{ij})$ has the property that it is possible to assign elements r_i and c_j to all rows and columns in such a way that every a_{ij} is either r_i or c_j. A matrix is called *totally tight* if every 2×2 submatrix has at least one constant line. We show that every totally tight matrix is assignable.

A *strongly assignable matrix* has an assignment r_i, c_j such that such that every a_{ij} is exactly one of r_i or c_j. In other words, the set of row labels is disjoint from the set of column labels.

The classes $\mathcal{AM}$ and $\mathcal{SAM}$ of all assignable matrices and strongly assignable matrices are hereditary, that is they are closed under taking submatrices. There are infinitely many forbidden submatrices for $\mathcal{AM}$. However, we show that the class $\mathcal{SAM}$ admits a finite forbidden submatrix characterization. Moreover, we give explicitly such a characterization for the case of matrices over $\{a, b, c\}$, where a, b and c are pairwise distinct.

2000 Mathematics Subject Classification: 05B20 (Matrices).

[1] *Corresponding author*: Igor E. Zverovich: RUTCOR − Rutgers Center for Operations Research, Rutgers, The State University of New Jersey, 640 Bartholomew Road, Piscataway, NJ 08854-8003, USA, e-mail: igor@rutcor.rutgers.edu

Date: May 20, 2008.

Key words and phrases. Assignable matrices, totally tight matrices, hereditary classes of matrices, minimal forbidden submatrices.

1. Introduction

Let $A = (a_{ij})$ be an $m \times n$ matrix over a set S. An *assignment* to A consists of elements $r_1, r_2, \ldots, r_m \in S$ and $c_1, c_2, \ldots, c_n$ assigned to the rows and to the columns of A, respectively, in such a way that

$$a_{ij} \in \{r_j, c_j\}$$

for every entry a_{ij} of A. We write an assignment as $R = (r_1, r_2, \ldots, r_m)$ and $C = (c_1, c_2, \ldots, c_n)$. All elements r_i and c_j in R and C are called *labels* of the corresponding rows and columns. A label r_i (respectively, c_j) *satisfies* all entries r_i in row i (respectively, all entries c_j in row j).

Definition 1. *An* assignable matrix *is a matrix that admits an assignment.*

Every 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is assignable. Indeed, one can set $R = (a, d)$ and $C = (c, b)$, or $R = (c, b)$ and $C = (a, d)$.

Definition 2. *A* strongly assignable matrix *has an assignment r_i, c_j such that such that every a_{ij} is exactly one of r_i or c_j.*

The following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

has two different assignments, but it is not strongly assignable. The class $\mathcal{AM}$ of all assignable matrices is closed under independent permutations of rows and columns, that is if A is an assignable matrix, then $PAQ \in \mathcal{AM}$ for any permutations matrices P and Q of appropriate dimensions. Also, $\mathcal{AM}$ is closed under transposition, i.e., $A \in \mathcal{AM}$ implies that the transpose A^T of A is an assignable matrix.

The classes $\mathcal{AM}$ and $\mathcal{SAM}$ of all assignable matrices and strongly assignable matrices are hereditary, that they are closed under taking submatrices. There are infinitely many forbidden submatrices for $\mathcal{AM}$. Figure 1 shows an infinite series of minimal forbidden $n\times$ submatrices, where $n \geq 3$. The smallest among them is

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

Thus, the class of all assignable matrices does not admit a finite forbidden submatrix characterization.

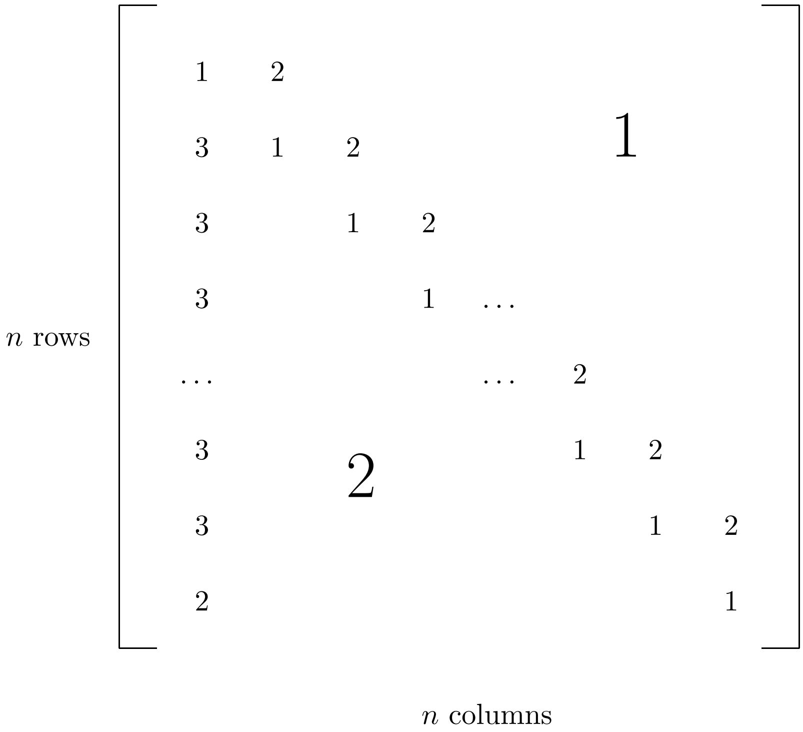

FIGURE 1. An infinite series of minimal forbidden submatrices, $n \geq 3$.

However, we show that the class $\mathcal{SAM}$ admits a finite forbidden submatrix characterization. Moreover, we give explicitly such a characterization for the case of matrices over $\{a, b, c\}$, where a, b, c are pairwise distinct.

A *submatrix* is obtained from a matrix by deleting some columns and/or rows, possibly none. A *line* in a matrix is either a row or a column.

2. **Assignability of totally tight matrices**

A matrix is called *totally tight* if every 2×2 submatrix has at least one constant line (a row or a column).

Theorem 1. *Every totally tight matrix is assignable.*

Proof. Let $A = (a_{ij})$ be a minimal non-assignable totally tight $m \times n$ matrix. Minimality of A means that every proper submatrix is assignable. Clearly, A does not have constant lines or a pair of identical rows or columns.

Consider a row of A that contains the largest number of equal identical entries, say row 1 has

$$a_{11} = a_{12} = \cdots = a_{1k} = a,$$

$a_{1j} \neq a$ for all $j > k$, and no row has more than k identical entries.

Claim 1. *There are no entries a in columns $k+1, k+1, \ldots, n$.*

Proof. Suppose $a_{ij} = a$ for some $j > k$. By maximality of k, $i \geq 2$. Let $a_{1j} = b \neq a$. For an arbitrary $j' \leq k$, rows $1, i$ and columns j', j produce the submatrix

$$\begin{pmatrix} a & b \\ a_{ij'} & a \end{pmatrix},$$

and therefore $a_{ij'} = a$. Thus, row i has at least $k+1$ entries a, a contradiction to maximality of k. □

Without loss of generality we may assume that column 1 has the largest number of entries a, say

$$a_{11} = a_{21} = \cdots = a_{l1} = a,$$

$a_{i1} \neq a$ for all $i > l$, and no column has more than l entries a.

Claim 2. *There are no entries a in rows $l+1, l+1, \ldots, m$.*

Proof. Suppose $a_{ij} = a$ for some $i > l$. According to Claim 1, $j \leq k$. By maximality of l, $j \geq 2$. Let $a_{i1} = b \neq a$. For an arbitrary $i' \leq l$, rows i', i and columns $1, j$ produce the submatrix

$$\begin{pmatrix} a & a_{i'j} \\ b & a \end{pmatrix},$$

and therefore $a_{i'j} = a$. Thus, column j has at least $l+1$ entries a, a contradiction to maximality of l. □

Now we consider those entries in row 1 and column 1 which are distinct from a.

Claim 3. $a_{1,k+1} = a_{1,k+2} = \cdots = a_{1,n} = c \neq a$, *and* $a_{l+1,1} = a_{l+2,1} = \cdots = a_{m,1} = c$.

Proof. Since column 1 is not constant, it contains an entry $b \neq a$. Similarly, row 1 contains an entry $c \neq a$. We show that $b = c$, which implies the statement. Suppose that $a_{l+1} = b \neq c = a_{1,k+1}$. The entry $a_{l+1,k+1}$ must be either b or c.

First, let $a_{l+1,k+1} = b$. For an arbitrary $j \leq k$, rows $1, l+1$ and columns $j, k+1$ produce the submatrix

$$\begin{pmatrix} a & c \\ a_{l+1,j} & b \end{pmatrix},$$

and therefore $a_{l+1,j}$ is either a or b. Claim 2 shows that $a_{l+1,j} \neq a$, therefore $a_{l+1,j} = b$. Thus, row $l+1$ has at least $k+1$ entries b, a contradiction to maximality of k.

Finally, let $a_{l+1,k+1} = c$. For an arbitrary $i \leq l$, rows $i, l+1$ and columns $1, k+1$ produce the submatrix

$$\begin{pmatrix} a & a_{i,k+1} \\ b & c \end{pmatrix},$$

and therefore $a_{i,k+1}$ is either a or c. Claim 1 shows that $a_{i,k+1} \neq a$, therefore $a_{i,k+1} = c$. Thus, $a_{1,k+1} = a_{2,k+1} = \cdots = a_{l+1,k+1} = c$. Column $k+1$ is not constant, therefore it contains some entry $d \neq c$. We may assume that $a_{l+2,k+1} = d$. By Claim 1, $d \neq a$. Considering rows 1 and $l+2$, one can easily show that $a_{l+2,1} = a_{l+2,2} = \cdots = a_{l+2,k+1} = d$, a contradiction to maximality of k. □

Claim 3 implies that $a_{ij} = c$ for all $i \in \{l+1, l+1, \dots, m\}$ and $j \in \{k+1, k+1, \dots, n\}$, and the matrix A has the form

$$A = \begin{pmatrix} a & a & \cdots & a & c & c & \cdots & c \\ a & & & & & & & \\ \cdots & & & & & & & \\ a & & & & & & & \\ c & & & & c & c & \cdots & c \\ c & & & & c & c & \cdots & c \\ \cdots & & & & \cdots & & & c \\ c & & & & c & c & \cdots & c \end{pmatrix}.$$

Claim 4. *There is a row without entries c.*

Proof. Suppose $a_{ij} = a$ for some $i > l$. According to Claim 1, $j \leq k$. By maximality of l, $j \geq 2$. Let $a_{i1} = b \neq a$. For an arbitrary $i' \leq l$, rows i', i and columns $1, j$ produce the submatrix

$$\begin{pmatrix} a & a_{i'j} \\ b & a \end{pmatrix},$$

and therefore $a_{i'j} = a$. Thus, column j has at least $l+1$ entries a, a contradiction to maximality of l. □

For a row i, let N_i denote the number of entries c in the columns $k+1, k+2, \dots, n$. Permute rows $2, 3, \dots, l$ so that $N_2 \leq N_3 \leq \cdots \leq N_l$.

Claim 5. *$N_2 = 0$, that is row 2 does not have entries c in the columns $k+1, k+2, \dots, n$.*

Proof. Suppose that $a_{2n} = c$. Since column n is not constant, we may assume that $a_{3n} = d \neq c$. The inequality $N_2 \leq N_3$ shows that row 2 contains entry c, say $a_{3,n-1} = c$, such that $a_{2,n-1} \neq c$. We obtain a contradiction in rows $2, 3$ and columns $n-1, n$:

$$\begin{pmatrix} \neq c & c \\ c & d \neq c \end{pmatrix}.$$

□

Row 2 has less entries a than row 1 has. Indeed, otherwise rows 1 and 2 are identical. Therefore we may assume that $a_{2k} = b \neq a$. Considering the submatrix in rows $1, 2$ and columns k, k', where $k' > k$:

$$\begin{pmatrix} a & c \\ b & d \neq c \end{pmatrix},$$

we see that $d = b$. It follows that rows 1 and 2 induce a submatrix [after a suitable permutation of columns $2, 3, \dots, k$]:

$$\begin{pmatrix} a & \cdots & a & a & \cdots & a & c & \cdots & c \\ a & \cdots & a & b & \cdots & b & b & \cdots & b \end{pmatrix}.$$

Here $a_{21} = a_{22} = \cdots = a_{2p} = a$ and $a_{2,p+1} = a_{2,p+2} = \cdots = a_{2n} = b$.

Now we consider row $l+1$. For $2 \leq j \leq p$, the submatrix in rows $2, l+1$ and columns j, n is

$$\begin{pmatrix} a & b \\ a_{l+1,j} & c \end{pmatrix},$$

which shows that $a_{l+1,j} \in \{a, c\}$. But row $l+1$ does not contain a, therefore $a_{l+1,j} = c$, $2 \leq j \leq p$. For $p+1 \leq j \leq k$, the submatrix in rows $2, l+1$ and columns $1, p+1$ is

$$\begin{pmatrix} a & b \\ c & a_{l+1,j} \end{pmatrix},$$

which shows that $a_{l+1,j} \in \{c, b\}$, $p+1 \leq j \leq k$. Here at least one $a_{l+1,j}$ is b, since row $l+1$ is not constant, but c is not necessary. It follows that rows $1, 2$ and $l+1$ induce the following submatrix [after a suitable permutation of columns]:

$$\begin{pmatrix} \mathbf{a} & \mathbf{a} & \mathbf{a} & \mathbf{c} \\ \mathbf{a} & \mathbf{b} & \mathbf{b} & \mathbf{b} \\ \mathbf{c} & \mathbf{c} & \mathbf{b} & \mathbf{c} \end{pmatrix},$$

where $\mathbf{x}$ represents a line $(x, x, \dots, x)$ of a suitable length. Column 2 here may be empty, while the other three columns are non-empty. It is convenient to rearrange the four columns as

$$D = \begin{pmatrix} \mathbf{a} & \mathbf{a} & \mathbf{c} & \mathbf{a} \\ \mathbf{a} & \mathbf{b} & \mathbf{b} & \mathbf{b} \\ \mathbf{c} & \mathbf{b} & \mathbf{c} & \mathbf{c} \end{pmatrix},$$

Rows $1, 2$ and 3 must have labels a, b and c, respectively, and we call them *a-row*, *b-row* and *c-row*. Similarly, columns $1, 2$ and 3 must have labels a, b and c, respectively, and we call them *a-part*, *b-part* and *c-part* of A. Column 4 does not get a specific label, and it is called $*$*-part* of A.

Now we consider an arbitrary row i of A which is not involved in D.

Claim 6. *Row i does not have an entry $d \notin \{a, b, c\}$ in a-part or b-part or c-part.*

Proof. Suppose that a-part contains $d \notin \{a, b, c\}$. Then b-row implies that all entries in b-part, c-part and $*$-part are either b or d. Also, all entries in c-part are either c or d due to the following submatrix

$$\begin{pmatrix} a & \mathbf{c} \\ d & . \end{pmatrix}.$$

Hence all entries in c-part must be d. Then a-row implies that all entries in a-part, b-part and $*$-part are either a or d. Therefore all entries in b-part and $*$-part must be d. Finally, the submatrix

$$\begin{pmatrix} \mathbf{c} & \mathbf{b} \\ . & \mathbf{d} \end{pmatrix}$$

shows that all entries in a-part must be d. Thus, row i is constant, a contradiction. □

Claim 7. (1a) *If row i contains b in its a-part, then*

- *its a-part consists of a and b only,*
- *its b-part consists of b only,*
- *its c-part consists of b only, and*
- *its $*$-part consists of b only.*

(1b) *If row i contains c in its b-part, then*

- *its a-part consists of c only,*
- *its b-part consists of b and c only,*
- *its c-part consists of c only, and*
- *its $*$-part consists of c only.*

(1c) *If row i contains a in its c-part, then*

- *its a-part consists of a only,*
- *its b-part consists of a only,*
- *its c-part consists of a and c only, and*
- *its $*$-part consists of a only.*

(2a) *If row i contains c in its a-part, but* (1b) *does not take place, then*

- *its a-part consists of a and c only,*
- *its b-part consists of b only,*
- *its c-part consists of c only, and*
- *its $*$-part consists of b and c only.*

(2b) *If row i contains a in its b-part, but* (1c) *does not take place, then*

- *its a-part consists of a only,*
- *its b-part consists of a and b only,*
- *its c-part consists of c only, and*
- *its $*$-part consists of a and c only.*

(2b) *If row i contains b in its c-part, but* (1a) *does not take place, then*

- *its a-part consists of a only,*
- *its b-part consists of b only,*
- *its c-part consists of b and c only, and*

- *its $*$-part consists of a and b only.*

Proof. First we prove (1a). b-row shows that all entries in b-part, c-part and $*$-part are b. The submatrix formed by a column in a-part, a column in c-part, rows 1 and i is

$$\begin{pmatrix} a & c \\ . & b \end{pmatrix},$$

and therefore a-part of row i consists of a and b only. The statements (1b) and (1c) are similar.

Now we prove (2a). b-row shows that all entries in b-part, c-part and $*$-part are either b or c. Since (1b) does not take place, there are no c in b-part, and therefore b-part consists of b only. The submatrix

$$\begin{pmatrix} a & c \\ c & . \end{pmatrix}$$

[formed by rows $1, i$ and columns from a-part and c-part] shows that c-part of row i consists of c only. The submatrix

$$\begin{pmatrix} a & b \\ . & c \end{pmatrix}$$

[formed by rows $2, i$ and columns from a-part and c-part] shows that a-part of row i consists of a and c only. The statements (2b) and (2c) are similar. □

If we have situation (1a) of Claim 7, then we delete row i [which is ($\mathbf{a/b}$, $\mathbf{b}$, $\mathbf{b}$,$\mathbf{b}$)]. The resulting submatrix is assignable by minimality of A. Moreover, rows $1, 2, 3$ force label a for a-part. Therefore we can assign b to row i, a contradiction. Situations (1b) and (1c) of Claim 7 are similar. If we have situation (2a) of Claim 7, then we delete row 3 [which is ($\mathbf{c}$, $\mathbf{b}$, $\mathbf{c}$,$\mathbf{c}$)]. The resulting submatrix is assignable by minimality of A. Moreover, rows $1, 2, i$ force label b for b-part. Therefore we can assign c to row 3, a contradiction. Situations (2b) and (2c) of Claim 7 are similar.

We have proved that the six situations of Claim 7 are impossible. Hence the matrix is

$$A = \begin{pmatrix} \mathbf{a} & \mathbf{a} & \mathbf{c} & \mathbf{a} \\ \mathbf{a} & \mathbf{b} & \mathbf{b} & \mathbf{b} \\ \mathbf{c} & \mathbf{b} & \mathbf{c} & \mathbf{c} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & A' \end{pmatrix},$$

where A' is a totally tight matrix. By minimality of A, A' is assignable, and so is A, a contradiction. □

3. **Strongly assignable matrices**

A *forbidden submatrix* for a hereditary class $\mathcal{P}$ of matrices is an arbitrary matrix that dies not belong to $\mathcal{P}$. A *minimal* forbidden submatrix for $\mathcal{P}$ is a forbidden submatrix A such that every proper submatrix of A is in $\mathcal{P}$. Here we prove that the set of all minimal forbidden submatrices for the class of all strongly assignable matrices is finite. Note that matrices are distinguished up to row/column permutations, up to transposition, and up to renaming of entries (without identification).

Let $A = (a_{ij})$ be a matrix for which we want to find a strong assignment. A *vertical a-forcing* in A is a submatrix

$$V(a,b,c) = \begin{pmatrix} a & b \\ a & c \end{pmatrix}, \tag{1}$$

or

$$V(a,b,c) = \begin{pmatrix} a & . & b \\ . & a & c \end{pmatrix}, \tag{2}$$

where a, b, c are pairwise distinct. Vertical a-forcings (1) and (2) produce label a for the (two) column(s) containing a. Similarly, a *horizontal a-forcing* in A is a submatrix

$$H(a,b,c) = \begin{pmatrix} a & a \\ b & c \end{pmatrix}, \tag{3}$$

or

$$H(a,b,c) = \begin{pmatrix} a & . \\ . & a \\ b & c \end{pmatrix}, \tag{4}$$

where a, b, c are pairwise distinct. Horizontal a-forcings (3) and (4) produce label a for the (two) row(s) containing a.

If some row (respectively, some column) is already assigned a label a, then *implied forcing* assign a to all rows (respectively, all columns) that contain a. Indeed, by the definition the sets of all row labels and of all column labels are disjoint. Based on this observation, it is easy to create some forbidden submatrices.

Definition 3. *An* orthogonal a-forcing *in A is a submatrix containing a vertical a-forcing $V(a,b,c)$ and a horizontal a-forcing $H(a,b',c')$ with the same a.*

The largest size of an orthogonal a-forcing without redundant lines is 5×5:

$$\begin{pmatrix} . & . & a & . & b \\ . & . & . & a & c \\ a & . & . & . & . \\ . & a & . & . & . \\ b' & c' & . & . & . \end{pmatrix},$$

but of course can be smaller, like

$$\begin{pmatrix} a & b & a & a \\ a & c & b & c \end{pmatrix}.$$

Thus, all minimal forbidden submatrices with an orthogonal a-forcing are of bounded dimensions.

Definition 4. *A* row obstruction *in A is a submatrix $RO(a,b,c;a',b',c')$ that contains a horizontal a-forcing $H(a,b,c)$, a horizontal a'-forcing $H(a',b',c')$, with $a \neq a'$, and a row containing both a and a'.*

The forcings $H(a, b, c)$ and $H(a', b', c')$ make both a and a' to be row labels, and the row containing both a and a' produces a contradiction, since it may have just one label, but implied forcing produces two distinct labels a and a'. The largest size of a row obstruction without redundant lines is 7×6:

$$\begin{pmatrix} a & . & . & . & . & . \\ . & a & . & . & . & . \\ b & c & . & . & . & . \\ . & . & a' & . & . & . \\ . & . & . & a' & . & . \\ . & . & b' & c' & . & . \\ . & . & . & . & a & a' \end{pmatrix}.$$

A *column obstruction* is defined similarly: there are vertical a-forcing $V(a, b, c)$ and a'-forcing $V(a', b', c')$ with $a \neq a'$, and and a column containing both a and a'. The largest size of a column obstruction without redundant lines is 6×7.

Theorem 2. *The class $\mathcal{SAM}$ can be characterized by finitely many forbidden submatrices.*

Proof. Let $A = (a_{ij})$ be a minimal forbidden submatrix for the class $\mathcal{SAM}$. Suppose that A has size $m \times n$, where $m \geq 8$ and $n \geq 8$. Denote by $C(x)$ the set of all entries $a_{ij} = x$ in A.

Case 1. Every non-empty set $C(a)$ contains a pair of entries involved in a horizontal a-forcing $H(a, b, c)$ or in a vertical a-forcing $V(a, b, c)$.

By minimality, A does not contain an orthogonal a-forcing for every a. First we assign multiple labels to the lines of A in the following way. Consider an arbitrary class $C(x)$. Since there are no orthogonal x-forcings, we may assume that $C(x)$ is involved in horizontal x-forcings only. Using implied forcing, we assign label x to every row that contains x. Thus, every class $C(x)$ produces $|C(x)|$ row labels or $|C(x)|$ column labels. As a result, every entry of A is covered by a row label or by a column label.

Suppose that some line has at least to labels, say row i has distinct labels a and a'. It means that A contains a row obstruction $RO(a, b, c; a', b', c')$, which is impossible by minimality.

We have constructed a legal assignment for A, and it satisfies all entries, since every entry produces a label in its row or in its column. Thus, A is strongly assignable, a contradiction.

Case 2. A has an entry $a_{ij} = a$ such that the set $C(a)$ does not contain a pair of entries involved in a horizontal a-forcing or in a vertical a-forcing.

Let $R_a \neq \emptyset$ be the set of all rows that contain a. For simplicity, R_a contains rows $1, 2, \ldots, k$. Consider an arbitrary truncated column j:

$$c'_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{kj} \end{pmatrix}.$$

If c'_j contains some $b \neq c$, both distinct from a, then a is involved in a horizontal a-forcing $H(a, b, c)$, a contradiction to the condition of Case 2. Therefore c'_j is column over $\{a, b_j\}$, where $b_j \neq a$.

To get a contradiction we try to construct a strong assignment for A. First we assign labels a to all rows in R_a. As a result every column j such that

$$c'_j \neq \begin{pmatrix} a \\ a \\ \cdot \\ \cdot \\ \cdot \\ a \end{pmatrix}$$

is forced to have label b_j.

Claim 8. *Every column, except at most one, obtain a label.*

Proof. Suppose that distinct columns p and q do not have labels. It means that $a_{1p} = a_{2p} = \cdots = a_{kp} = a$ and $a_{1q} = a_{2q} = \cdots = a_{kq} = a$. It is easy to see that a minimal forbidden submatrix may not have a pair of identical columns. Therefore $a_{ip} \neq a_{iq}$ for some $i \geq k + 1$. Since $R_a = \{1, 2, \ldots, k\}$, both a_{ip} and a_{iq} are distinct from a. Thus, $a_{1p}, a_{1q}, a_{ip}, a_{iq}$ form a horizontal a-forcing $H(a, a_{ip}, a_{iq})$, a contradiction to the condition of Case 2. ☐

The only possible column that does not have label (Claim 8) will be called *exceptional*. Here is an example

$$\begin{pmatrix} a & b_1 & b_2 & a \\ a & a & b_2 & b_1 \\ a & a & b_2 & b_1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where $k = 3$, column 1 is exceptional, both b_1 and b_2 are distinct from a, and therefore column $1, 2, 3$ obtain labels b_1, b_2, b_1, respectively. Now we extend the current partial assignment, enumerating all arising contradictions.

Contradiction 1. Suppose that column j with label b_j contains some column label $b_l \neq b_j$ as an entry. Then b_l becomes a row label, which is impossible. The largest size of such a contradiction is 3×4:

$$\begin{pmatrix} a & . & b_j & . \\ . & a & . & b_l \\ . & . & b_l & . \end{pmatrix}.$$

Contradiction 2. Suppose that the exceptional column 1 contains some distinct column labels b_j and b_l as entries. Then at least one of b_j, b_l becomes a row label, which is impossible. The largest size of such a contradiction is 4×3:

$$\begin{pmatrix} a & b_j & . \\ a & . & b_l \\ b_j & . & . \\ b_l & . & . \end{pmatrix}.$$

If the exceptional column 1 contains just one column label, say b_j, as an entry, then b_j becomes a label of column 1. If the exceptional column 1 does not contain column labels as an entries, then it remains unlabelled. In all cases, every entry that coincide with some column label is currently covered by its column label. All currently uncovered entries must become either row labels or a label of the exceptional column.

Contradiction 3. Suppose that distinct non-exceptional columns labelled b_j and b_l (possibly, $b_j = b_l$) contain entries $c \neq b_j$ and $d \neq b_l$, $c \neq d$, in the same row. Then the row obtains two labels c and d, which is impossible. The largest size of such a contradiction is 3×4:

$$\begin{pmatrix} a & . & b_j & . \\ . & a & . & b_l \\ . & . & c & d \end{pmatrix}.$$

Contradiction 4. Let the exceptional column 1 have label b_j (since b_j occurs as a label of another column and as an entry of column 1). Suppose that column 1 and another column labelled b_l (possibly, $b_j = b_l$) contain entries $c \neq b_j$ and $d \neq b_l$, $c \neq d$, in the same row. Then the row obtains two labels c and d, which is impossible. The largest size of such a contradiction is 4×3:

$$\begin{pmatrix} a & b_j & . \\ a & . & b_l \\ b_j & . & . \\ c & . & d \end{pmatrix}.$$

Contradiction 5. Let the exceptional column 1 do not have label. Suppose that a column labelled b_j forces a label c for row i, that is $c \neq b_j$ is the ith entry of that column. Suppose that another column labelled b_l forces a label d for row $i' \neq i$. If $a_{i1} = e \neq c$, $a_{i'1} = f \neq d$ and $e \neq f$, then we cannot cover both e and f, since the only possibility is to assign one label to the exceptional column 1. The largest size of such a contradiction is 4×3:

$$\begin{pmatrix} a & b_j & . \\ a & . & b_l \\ e & c & . \\ f & . & d \end{pmatrix}.$$

Now consider an arbitrary uncovered entry, say c, in row i. Clearly, $i \geq k+1$. Suppose that the entry c belongs to a column labelled $b_j \neq c$. Then row i obtains label c, and the three possible contradictions (Contradictions 3, 4, 5) arising with such assignment were already considered. If all columns have labels, then every uncovered row is just the sequence of column labels. Since a minimal forbidden submatrix may not have a pair of identical rows, there is at most one uncovered row, and we can assign label a to it, thus obtaining a strong assignment for A. (In this situation, a is just a formal label that does not cover any entry). The only unlabelled column may be the exceptional column 1, in which case every uncovered row has the form

$$a_{i1}, b_{i_2}, b_{i_3}, \ldots, b_{i_n},$$

where a_{i1} is distinct from all row labels (otherwise the exceptional column would be assigned) and b_{i_j} is the label of column $j \geq 2$. We assign label a_{i1} to every uncovered row i, thus obtaining a strong assignment for A.

Since A is a forbidden submatrix, it does not admit a strong assignment, therefore one of the Contradictions 1, 2, 3, 4, 5 must take place.

Claim 9. *Assuming that all entries a are covered by row labels, the largest size of a contradiction is* 3×4 *or* 4×3.

By symmetry, an analogue of Claim 9 holds if we cover all entries a by column labels: a contradiction of size at most 3×4 or 4×3 arises. Thus, a total contradiction for row a-assignment and for column a-assignment is of size at most 6×8, 7×7 or 8×6. Recall that A was assumed to be an $m \times n$ matrix with $m \geq 8$ and $n \geq 8$. Thus, A is not minimal, a final contradiction. Note that renaming of entries guarantees that the number of matrices of bounded size is finite. □

4. **A particular characterization**

Here we give explicitly such a finite forbidden submatrix characterization of strongly assignable matrices for the case of matrices over $\{a, b, c\}$, where a, b, c are pairwise distinct. This class will be denoted by the class $\mathcal{SAM}(3)$. Note that every matrix over $\{a, b\}$ is strongly assignable: one can assign a to all rows and b to all columns, or conversely.

We distinguish matrices up to row/column permutations, up to transposition, and up to renaming of entries (without identification). For example, the smallest forbidden submatrix

$$F_1 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

represents twelve 2×3 matrices

$$\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, \begin{pmatrix} a & c & b \\ b & a & c \end{pmatrix}, \begin{pmatrix} a & c & b \\ c & b & a \end{pmatrix}, \begin{pmatrix} b & a & c \\ a & c & b \end{pmatrix}, \begin{pmatrix} b & a & c \\ c & b & a \end{pmatrix},$$

$$\begin{pmatrix} b & c & a \\ a & b & c \end{pmatrix}, \begin{pmatrix} b & c & a \\ c & a & b \end{pmatrix}, \begin{pmatrix} c & a & b \\ a & b & c \end{pmatrix}, \begin{pmatrix} c & a & b \\ b & c & a \end{pmatrix}, \begin{pmatrix} c & b & a \\ a & c & b \end{pmatrix}, \begin{pmatrix} c & b & a \\ b & a & c \end{pmatrix},$$

and their transposes.

Below $A = a_{ij}$ is an arbitrary minimal forbidden submatrix for the class $\mathcal{SAM}(3)$. We assume that all entries of A belong to the set $\{a, b, c\}$, where a, b, c are pairwise distinct. Minimality of A means that deleting at least one line (a row or a column) produces a strongly assignable matrix. Every matrix in $\mathcal{SAM}(3)$ admits a *semi-constant* assignment, that is an assignment where all row labels are the same or all column labels are the same. Assigning a to all rows of A cannot be extended to a strong assignment, therefore some column j_{bc} of A contains both b and c. Similarly, there is a column j_{ac} containing both a and c, and there is a column j_{ab} containing both a and b. One possibility is that some column contains a, b and c, and we may assume that $j_{bc} = j_{ac} = j_{ab}$. If it is not the case, then the three columns j_{bc}, j_{ac} and j_{ab} are pairwise distinct. A similar situation takes place for rows. There must be rows i_{bc}, i_{ac} and i_{ab} containing both b and c, both a and c, and both a and b, respectively. One may happen that $i_{bc} = i_{ac} = i_{ab}$, or the three rows are pairwise distinct. Accordingly, we consider four cases.

Case 1. There is a row containing a, b, c, and there is a column containing a, b, c.

Without loss of generality we may assume that $a_{11} = a$, $a_{12} = b$, and $a_{13} = c$. Let column j contains a, b, c. By symmetry, we may assume that $j = 1$ or $j = 4$. If $j = 1$ then

$$A = \begin{pmatrix} a & b & c & . \\ b & . & . & . \\ c & . & . & . \\ . & . & . & . \end{pmatrix},$$

since the variant

$$A = \begin{pmatrix} a & b & c & . \\ a & . & . & . \\ b & . & . & . \\ c & . & . & . \\ . & . & . & . \end{pmatrix}$$

is redundant (one can delete row 2). Now let $j = 4$. We have two possibilities:

$$A = \begin{pmatrix} a & b & c & a & . \\ . & . & . & b & . \\ . & . & . & c & . \\ . & . & . & . & . \end{pmatrix}$$

where column 1 is redundant, and

$$A = \begin{pmatrix} a & b & c & . & . \\ . & . & . & a & . \\ . & . & . & b & . \\ . & . & . & c & . \\ . & . & . & . & . \end{pmatrix}$$

which is also redundant (if $a_{14} = a$ then both row 2 and column 1 can be deleted, etc.). Thus,

$$A = \begin{pmatrix} a & b & c \\ b & . & . \\ c & . & . \end{pmatrix}.$$

There are $3^4 = 81$ possibilities to specify the missing entries a_{22}, a_{23}, a_{32} and a_{33}, but some of them contain the forbidden submatrix F_1, namely

$$P_1 = \begin{pmatrix} a & b & c \\ b & c & a \\ c & . & . \end{pmatrix},$$

where row 3 is redundant,

$$P_2 = \begin{pmatrix} a & b & c \\ b & . & . \\ c & a & b \end{pmatrix},$$

where row 2 is redundant,

$$P_3 = \begin{pmatrix} a & b & c \\ b & a & c \\ c & b & a \end{pmatrix},$$

where row 1 is redundant,

$$P_4 = \begin{pmatrix} a & b & c \\ b & c & . \\ c & a & . \end{pmatrix},$$

where column 3 is redundant,

$$P_5 = \begin{pmatrix} a & b & c \\ b & . & a \\ c & . & b \end{pmatrix},$$

where column 2 is redundant, and

$$P_6 = \begin{pmatrix} a & b & c \\ b & a & b \\ c & c & a \end{pmatrix}.$$

where column 1 is redundant.

One can directly check that there are 54 matrices avoiding the patterns P_1, P_2, P_3, P_4, P_5 and P_6.

We classify them according to their cardinality sets $S = \{N_a, N_b, N_c\}$, where N_x is the number of entries equal to x.

S	Additional property	Number of matrices
$\{6,2,1\}$		2 matrices
$\{5,3,1\}$	no constant line	2 matrices
$\{5,3,1\}$	a constant line	6 matrices
$\{5,2,2\}$	two constant lines	2 matrices
$\{5,2,2\}$	no constant line	3 matrices
$\{4,4,1\}$	two constant lines	2 matrices
$\{4,4,1\}$	no constant line	4 matrices
$\{4,3,2\}$	a constant line	12 matrices
$\{4,3,2\}$	no constant line	16 matrices
$\{3,3,3\}$		5 matrices

The two matrices with $S = \{6,2,1\}$ are

$$F_2 = \begin{pmatrix} a & b & c \\ b & c & c \\ c & c & c \end{pmatrix}$$

and $\begin{pmatrix} a & b & c \\ b & b & b \\ c & b & b \end{pmatrix}$, which is essentially the same as F_2 (permute rows 2 and 3, permute columns 2 and 3, and rename b as c and c as b).

The 2 matrices with $S = \{5,3,1\}$ and without constant line are

$$F_3 = \begin{pmatrix} a & b & c \\ b & c & c \\ c & c & b \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & c & b \\ c & b & b \end{pmatrix}$, which is essentially the same as F_3 (permute rows 2 and 3, permute columns 2 and 3, and rename b as c and c as b).

The 6 matrices having $S = \{5,3,1\}$ and a constant line are

$$F_4 = \begin{pmatrix} a & b & c \\ b & b & c \\ c & c & c \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & b & b \\ c & b & c \end{pmatrix}$, which is essentially the same as F_4 (permute rows 2 and 3, permute columns 2 and 3, and rename b as c and c as b),

$$F_5 = \begin{pmatrix} a & b & c \\ b & c & b \\ c & c & c \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & c & c \\ c & b & c \end{pmatrix} = F_5^T$, the transpose of F_5, $\begin{pmatrix} a & b & c \\ b & b & b \\ c & c & b \end{pmatrix}$, which is essentially the same as F_5 (permute rows 2 and 3, permute columns 2 and 3, and rename b as c and c as b), and $\begin{pmatrix} a & b & c \\ b & b & c \\ c & b & b \end{pmatrix}$, which is the transpose of previous matrix. The matrices F_4 and F_5 are distinct, since b in F_4 does not appear in a row and in a column, while b in F_5 appears in all columns.

The 2 matrices with $S = \{5, 2, 2\}$ and two constant lines are

$$F_6 = \begin{pmatrix} a & b & c \\ b & a & c \\ c & c & c \end{pmatrix}$$

and $\begin{pmatrix} a & b & c \\ b & b & b \\ c & b & a \end{pmatrix}$, which is essentially the same as F_6 (permute rows 2 and 3, permute columns 2 and 3, and rename b as c and c as b),

The 3 matrices with $S = \{5, 2, 2\}$ and without constant lines are

$$F_7 = \begin{pmatrix} a & b & c \\ b & a & a \\ c & a & a \end{pmatrix},$$

$$F_8 = \begin{pmatrix} a & b & c \\ b & a & b \\ c & b & b \end{pmatrix},$$

and $\begin{pmatrix} a & b & c \\ b & c & c \\ c & c & a \end{pmatrix}$, which is essentially the same as F_8 (permute rows 2 and 3, permute columns 2 and 3, and rename b as c and c as b). The matrices F_7 and F_8 are distinct, since b and c in F_7 occupy just one row and and one column, while the corresponding elements a and c in F_8 appear in all rows and columns.

The 2 matrices with $S = \{4, 4, 1\}$ and with two constant lines are

$$F_9 = \begin{pmatrix} a & b & c \\ b & b & b \\ c & c & c \end{pmatrix}$$

and $\begin{pmatrix} a & b & c \\ b & b & c \\ c & b & c \end{pmatrix} = F_9^T$.

The 4 matrices with $S = \{4, 4, 1\}$ and without constant lines are

$$F_{10} = \begin{pmatrix} a & b & c \\ b & c & c \\ c & b & b \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & c & b \\ c & c & b \end{pmatrix} = F_{10}^T$,

$$F_{11} = \begin{pmatrix} a & b & c \\ b & b & c \\ c & c & b \end{pmatrix},$$

and $\begin{pmatrix} a & b & c \\ b & c & b \\ c & b & c \end{pmatrix}$, which is essentially the same as F_{11} (permute rows 2 and 3, permute columns 2 and 3, and rename b as c and c as b). The matrices F_{10} and F_{11} are distinct, since c in F_{11} can be covered by one row and and one column, while neither b nor c in F_{10} do not have this property.

The 12 matrices with $S = \{4, 3, 2\}$ and a constant line are

$$F_{12} = \begin{pmatrix} a & b & c \\ b & b & a \\ c & c & c \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & b & c \\ c & a & c \end{pmatrix} = F_{12}^T$, $\begin{pmatrix} a & b & c \\ b & b & b \\ c & a & c \end{pmatrix}$, which is essentially the same as F_{12} (permute rows 2 and 3, permute columns 2 and 3, and rename b as c and c as b), $\begin{pmatrix} a & b & c \\ b & b & a \\ c & b & c \end{pmatrix}$, which is the transpose of the previous matrix,

$$F_{13} = \begin{pmatrix} a & b & c \\ b & a & b \\ c & c & c \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & a & c \\ c & b & c \end{pmatrix} = F_{13}^T$, $\begin{pmatrix} a & b & c \\ b & b & b \\ c & a & a \end{pmatrix}$, which is essentially the same as F_{13} (permute rows 2 and 3, arrange columns as $3, 1, 2$, and rename a as b, b as c and c as a), $\begin{pmatrix} a & b & c \\ b & b & a \\ c & b & a \end{pmatrix}$, which is the transpose of the previous matrix, $\begin{pmatrix} a & b & c \\ b & a & a \\ c & c & c \end{pmatrix}$, which is essentially the same as F_{13} (permute columns 1 and 2 and rename a as b and b as a), $\begin{pmatrix} a & b & c \\ b & a & c \\ c & a & c \end{pmatrix}$, which is the transpose of the previous matrix, $\begin{pmatrix} a & b & c \\ b & b & b \\ c & c & a \end{pmatrix}$, which is essentially the same as F_{13}

(permute rows 1 and 2, columns 1 and 2 and rename b as c and c as b), and $\begin{pmatrix} a & b & c \\ b & b & c \\ c & b & a \end{pmatrix}$, which is the transpose of the previous matrix. The difference between F_{12} and F_{13} is that b in F_{12} can be covered by 2 rows or by 2 columns, while b in F_{12} does not have this property.

The 16 matrices with $S = \{4, 3, 2\}$ and without constant lines are

$$F_{14} = \begin{pmatrix} a & b & c \\ b & c & c \\ c & b & a \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & c & b \\ c & c & a \end{pmatrix} = F_{14}^T$, $\begin{pmatrix} a & b & c \\ b & a & c \\ c & b & b \end{pmatrix}$, which is essentially the same as F_{14} (permute rows 1 and 2, columns 1 and 2 and rename b as c and c as b), $\begin{pmatrix} a & b & c \\ b & a & b \\ c & c & b \end{pmatrix}$, which is the transpose of the previous matrix, $\begin{pmatrix} a & b & c \\ b & a & a \\ c & b & a \end{pmatrix}$, which is essentially the same as F_{14} (permute rows 1 and 3, and rename a as c and c as a), $\begin{pmatrix} a & b & c \\ b & a & b \\ c & a & a \end{pmatrix}$, which is the transpose of the previous matrix, $\begin{pmatrix} a & b & c \\ b & a & c \\ c & a & a \end{pmatrix}$, which is essentially the same as F_{14} (arrange rows as $2, 3, 1$, permute columns 2 and 3 and rename a as c, b as a and c as b), $\begin{pmatrix} a & b & c \\ b & a & a \\ c & c & a \end{pmatrix}$, which is the transpose of the previous matrix,

$$F_{15} = \begin{pmatrix} a & b & c \\ b & b & c \\ c & c & a \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & a & b \\ c & b & c \end{pmatrix}$, which is essentially the same as F_{15} (permute rows 2 and 3, columns 2 and 3 and rename b as c and c as b),

$$F_{16} = \begin{pmatrix} a & b & c \\ b & b & a \\ c & a & a \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & a & a \\ c & a & c \end{pmatrix}$, which is essentially the same as F_{17} (permute rows 2 and 3, columns 2 and 3, and rename b as c and c as b).

$$F_{17} = \begin{pmatrix} a & b & c \\ b & a & c \\ c & c & b \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & a & b \\ c & b & a \end{pmatrix}$, which is essentially the same as F_{17} (arrange rows as $3, 1, 2$, permute columns 2 and 3 and rename a as b, b as c and c as a), $\begin{pmatrix} a & b & c \\ b & c & b \\ c & b & a \end{pmatrix}$, which is essentially the same as F_{17} (permute rows 2 and 3 columns 2 and 3 and rename b as c and c as b), and $\begin{pmatrix} a & b & c \\ b & a & c \\ c & c & a \end{pmatrix}$, which is essentially the same as F_{17} (permute rows 1 and 2, columns 1 and 2, and rename a as b and b as a).

To see the difference between the matrices F_{14}, F_{15}, F_{14} and F_{17} we look at the pattern of the element that occurs exactly twice (a in F_{14}, F_{15}, F_{17}, and c in F_{16}). For every line which contains this element, we specify cardinalities of the other two entries. For example, row 1 in F_{14} is a, b, c, and the elements occur 2, 3 and 4, respectively, therefore row 1 will contribute $\{3, 4\}$ in the pattern of F_{14}. The whole pattern of F_{14} is

$$\{\{3, 4\}, \{3, 4\}\} \text{ and } \{\{3, 4\}, \{4, 4\}\}.$$

Similarly, the patterns of F_{15}, F_{16} and F_{17} are

$$\{\{3, 4\}, \{3, 4\}\} \text{ and } \{\{4, 4\}, \{4, 4\}\},$$

$$\{\{3, 4\}, \{4, 4\}\} \text{ and } \{\{3, 4\}, \{4, 4\}\},$$

$$\{\{3, 4\}, \{3, 4\}\} \text{ and } \{\{3, 4\}, \{3, 4\}\},$$

respectively. Since the patterns are pairwise distinct, the matrices are also pairwise distinct.

The 5 matrices with $S = \{3, 3, 3\}$ are

$$F_{18} = \begin{pmatrix} a & b & c \\ b & b & a \\ c & a & c \end{pmatrix},$$

$$F_{19} = \begin{pmatrix} a & b & c \\ b & a & b \\ c & a & c \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & a & a \\ c & b & c \end{pmatrix} = F_{19}^T$, $\begin{pmatrix} a & b & c \\ b & b & a \\ c & c & a \end{pmatrix}$, which is essentially the same as F_{19} (permute rows 2 and 3, columns 2 and 3, and rename b as c and c as b), and $\begin{pmatrix} a & b & c \\ b & b & c \\ c & a & a \end{pmatrix}$, which is the transpose of the previous matrix.

The element a in F_{18} occurs in all lines, while F_{19} does not have such an element, therefore F_{18} and F_{19} are distinct.

Case 2. There is a row containing a, b, c, and there are three columns containing $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$, respectively.

Without loss of generality, the matrix has the form

$$A = \begin{pmatrix} a & b & c & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

A *binary line* is a non-constant line over a 2-element set, say $\{x, y\}$. We may assume that every column is either constant or binary, otherwise we have Case 1. We refer to such a line as *xy-line*. There must be *ab*-column, *ac*-column and *bc*-column.

Subcase 2.1. *ab*-column, *ac*-column and *bc*-column are columns 1, 2 and 3.

Clearly, A has exactly three columns. Let $a_{21} = b$, and consider all possible variants for a_{22} and a_{23}:

$$A_1 = \begin{pmatrix} a & b & c \\ b & b & a \\ \cdot & \cdot & \cdot \end{pmatrix},$$

$$A_2 = \begin{pmatrix} a & b & c \\ b & b & b \\ \cdot & \cdot & \cdot \end{pmatrix},$$

$$A_3 = \begin{pmatrix} a & b & c \\ b & b & c \\ \cdot & \cdot & \cdot \end{pmatrix},$$

$$A_4 = \begin{pmatrix} a & b & c \\ b & c & c \\ \cdot & \cdot & \cdot \end{pmatrix},$$

$$A_5 = \begin{pmatrix} a & b & c \\ b & c & a \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

In A_1, column 2 is a *bc*-column, and we may assume that $a_{32} = c$. Depending on $a_{31} \in \{a, b\}$ and $a_{33} \in \{a, c\}$, there are four variants:

$$F_{20} = \begin{pmatrix} a & b & c \\ b & b & a \\ a & c & a \end{pmatrix},$$

$$F_{21} = \begin{pmatrix} a & b & c \\ b & b & a \\ a & c & c \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & b & a \\ b & c & a \end{pmatrix}$, which contains F_1 in rows 1 and 3, and $\begin{pmatrix} a & b & c \\ b & b & a \\ b & c & c \end{pmatrix}$, which is essentially the same as F_{20} (permute rows 2 and 3, arrange columns as $2, 3, 1$, and rename a as c, b as a and c as b). The matrices F_{20} and F_{21} are distinct, since they have different cardinality sets: $\{4, 3, 2\}$ and $\{3, 3, 3\}$.

The variant A_2 is impossible, since there is no ac-column. among columns $1, 2, 3$.

In A_3 column 1 is a bc-column, and column 3 is an ac-column. We may assume that $a_{32} = c$, and either $a_{33} = b$ or $a_{33} = c$ and $a_{43} = b$. Here are all possible variants:

$$F_{22} = \begin{pmatrix} a & b & c \\ b & b & c \\ a & c & b \end{pmatrix},$$

$$F_{23} = \begin{pmatrix} a & b & c \\ b & b & c \\ b & c & b \end{pmatrix},$$

$\begin{pmatrix} a & b & c \\ b & b & c \\ . & c & c \\ . & c & b \end{pmatrix}$, which is redundant (row 3 can be deleted), $\begin{pmatrix} a & b & c \\ b & b & c \\ . & c & c \\ . & b & b \end{pmatrix}$, which is redundant unless $a_{31} = a_{41} = a$ (row 2 can be deleted), and

$$F_{24} = \begin{pmatrix} a & b & c \\ b & b & c \\ a & c & c \\ a & b & b \end{pmatrix}.$$

The matrices F_{21} and F_{22} are distinct, since they have different cardinality sets: $\{3, 3, 3\}$ and $\{4, 3, 2\}$. The matrices F_{20} and F_{22} are distinct, since c in F_{20} are not in the same line, while the corresponding element a in F_{22} occurs twice in a column. F_{23} has cardinality set $\{5, 3, 1\}$, and therefore differs from F_{20}, F_{21} and F_{22}.

In A_4, we may assume that $a_{33} = a$, and we have the following variants: $\begin{pmatrix} a & b & c \\ b & c & c \\ a & b & a \end{pmatrix}$, which is essentially the same as F_{21} (arrange columns as $2, 3, 1$, and rename a as c, b as a and c as b), $\begin{pmatrix} a & b & c \\ b & c & c \\ a & c & a \end{pmatrix}$, which is essentially the same as F_{20} (permute rows 2 and 3, arrange columns as $3, 1, 2$, and rename a as b, b as c and c as a), $\begin{pmatrix} a & b & c \\ b & c & c \\ b & b & a \end{pmatrix}$, which is essentially the same as F_{20} (arrange columns as $2, 3, 2$, and rename a as c, b as a and c as b), and $\begin{pmatrix} a & b & c \\ b & c & c \\ b & c & a \end{pmatrix}$, which contains F_1 in rows 1 and 3.

Finally, A_5 contains F_1 in rows 1 and 2.

Subcase 2.2. *ab*-column and *ac*-column are among columns 1, 2 and 3, while *bc*-column is column 4.

Let columns 1 and 3 be an *ab*-column and an *ac*-column, respectively. We may assume that $a_{21} = b$, and either $a_{23} = a$ or $a_{23} = c$ and $a_{33} = a$. If $a_{14} = b$ then column 2 is redundant, therefore $a_{14} = c$. The are no c in column 2 (this situation was considered in Subcase 2.1). Now we specify all possibilities:

$$F_{25} = \begin{pmatrix} a & b & c & c \\ b & a & a & b \end{pmatrix},$$

$$F_{26} = \begin{pmatrix} a & b & c & c \\ b & b & a & b \end{pmatrix},$$

$\begin{pmatrix} a & b & c & c \\ b & . & a & c \\ . & . & . & b \end{pmatrix}$, which is redundant (column 2 can be deleted), $\begin{pmatrix} a & b & c & c \\ b & a & c & b \\ . & . & a & . \end{pmatrix}$, which is redundant (column 1 can be deleted), $\begin{pmatrix} a & b & c & c \\ b & b & c & b \\ b & . & a & . \end{pmatrix}$, which is redundant (column 2 can be deleted), $\begin{pmatrix} a & b & c & c \\ b & b & c & b \\ a & . & a & b \end{pmatrix}$, which is redundant (column 2 can be deleted), $\begin{pmatrix} a & b & c & c \\ b & b & c & b \\ a & b & a & c \end{pmatrix}$, which is redundant (row 1 can be deleted),

$$F_{27} = \begin{pmatrix} a & b & c & c \\ b & b & c & b \\ a & a & a & c \end{pmatrix},$$

$\begin{pmatrix} a & b & c & c \\ b & . & c & c \\ . & . & a & b \end{pmatrix}$, which is redundant (column 2 can be deleted), $\begin{pmatrix} a & b & c & c \\ b & . & c & c \\ . & . & a & c \\ . & . & . & b \end{pmatrix}$, which is redundant unless $a_{31} = a_{41} = a$ and $a_{32} = a_{42} = b$ (row 2 can be deleted), and $\begin{pmatrix} a & b & c & c \\ b & . & c & c \\ a & b & a & c \\ a & b & . & b \end{pmatrix}$, which is redundant (row 1 can be deleted).

Subcase 2.3. *ab*-column is column 1, while, *ac*-column and *bc*-column are columns 4 and 5.

Since a_{15} is either b or c, one of the columns $2, 3$ can be deleted, and therefore the matrix is redundant.

Subcase 2.4. ab-column, ac-column and bc-column are columns 4, 5 and 6, respectively.

We may assume that columns 1, 2 and 3 are constant, otherwise we have one of the previous subcases. By symmetry, let $a_{14} = a$ and $a_{24} = b$:

$$A = \begin{pmatrix} a & b & c & a & . \\ a & b & c & b & . \\ . & . & . & . & . \end{pmatrix}.$$

We can delete column 1, obtaining one of the previous subcases.

Case 3. There is a column containing a, b, c, and there are three rows containing $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$, respectively.

This case is essentially the same as Case 2, and it does not produce new forbidden submatrices.

Case 4. There are three rows containing $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$, respectively, and there are three columns containing $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$, respectively.

Let rows $1, 2, 3$ be an ab-row, an ac-row, and a bc-row, respectively. Similarly, let columns $1, 2, 3$ be an ab-column, an ac-column, and a bc-column, respectively. Therefore $a_{12} = a$, $a_{13} = b$, $a_{21} = a$, $a_{23} = c$, $a_{31} = b$, $a_{32} = c$, and

$$A = \begin{pmatrix} . & a & b \\ a & . & c \\ b & c & . \end{pmatrix}.$$

Thus, we obtain $2^3 = 8$ variants:

$$F_{28} = \begin{pmatrix} a & a & b \\ a & a & c \\ b & c & b \end{pmatrix},$$

$\begin{pmatrix} a & a & b \\ a & a & c \\ b & c & c \end{pmatrix}$, which is essentially the same as F_{28} (permute rows 1 and 2, columns 1 and 2, rename b as c and c as b),

$$F_{29} = \begin{pmatrix} a & a & b \\ a & c & c \\ b & c & b \end{pmatrix},$$

$\begin{pmatrix} a & a & b \\ a & c & c \\ b & c & c \end{pmatrix}$, which is essentially the same as F_{28} (arrange rows as $2, 3, 1$ columns as $2, 3, 1$, and rename a as b, b as c and c as a), $\begin{pmatrix} b & a & b \\ a & a & c \\ b & c & b \end{pmatrix}$, which is essentially the same as F_{28} (permute rows 2 and 3, columns 2 and 3, and rename a as b and b as a), $\begin{pmatrix} b & a & b \\ a & a & c \\ b & c & c \end{pmatrix}$, which is essentially the same as F_{29} (permute rows 1 and 2, columns 1 and 2, rename b as c and

c as b), $\begin{pmatrix} b & a & b \\ a & c & c \\ b & c & b \end{pmatrix}$, which is essentially the same as F_{28} (arrange rows as $3, 1, 2$ columns as $3, 1, 2$, and rename a as c, b as a and c as b), and $\begin{pmatrix} b & a & b \\ a & c & c \\ b & c & c \end{pmatrix}$, which is essentially the same as F_{28} (arrange rows as $3, 1, 2$ columns as $3, 1, 2$, and rename a as c and c as a).

The matrices F_{28} and F_{29} are distinct, since they have different cardinality sets.

Thus, all cases are considered, and we can formulate the result.

Theorem 3. *The class $\mathcal{SAM}(3)$ is characterized by the set $\{F_1, F_2, \ldots, F_{29}\}$ of minimal forbidden submatrices.*

Open Problem 1. *Find all minimal forbidden submatrices for the class $\mathcal{SAM}$.*

Open Problem 2. *Find forbidden submatrix characterization of the class $\mathcal{AM}$.*

RUTCOR – Rutgers Center for Operations Research, Rutgers, The State University of New Jersey, 640 Bartholomew Road, Piscataway, NJ 08854-8003, USA

E-mail address: boros@rutcor.rutgers.edu
E-mail address: cepek@rutcor.rutgers.edu
E-mail address: gurvich@rutcor.rutgers.edu
E-mail address: makino@sys.es.osaka-u.ac.jp
E-mail address: igor@rutcor.rutgers.edu