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It is a coNP-complete problem to decide whether a positive $\vee$-$\wedge$ formula of depth 3 defines a read-once or respectively quadratic Boolean function [a]

Vladimir Gurvich [b]

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RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

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[b]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ 08854-8003; email: gurvich@rutcor.rutgers.edu

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It is a coNP-complete problem to decide whether a positive ∨-∧ formula of depth 3 defines a read-once or respectively quadratic Boolean function [1]

Vladimir Gurvich

Abstract. Let us consider the DNF $D_0 = x_1y_1 \vee \ldots \vee x_ny_n$ and let C be a positive CNF of the same $2n$ variables. It is a coNP-complete problem to verify the inequality $D_0 \geq C$, which is obviously equivalent with the equality $C \vee D_0 = D_0$. Clearly, the problem is in coNP and its polynomial reduction from SAT is straightforward. Nevertheless, this trivial observation has many applications in enumeration, game, and graph theories, as Khachiyan and the author demonstrated in 1995. Since then several other interesting applications were found. In this paper, we show that it is coNP-complete to verify whether the expression $C \vee D_0$ defines a quadratic or, respectively, read-once Boolean function.
Key words: positive Boolean function or formula, read-once, linear, quadratic, disjunctive and conjunctive normal forms, DNF, CNF, coNP-complete problem

We will make use of many elementary concepts of Boolean function theory that can be found in any textbook; see for example [2], where quadratic and read-once Boolean functions are considered in Chapters 5 and 10, respectively.

Let us remark that the equality of two Boolean expression will mean that the corresponding two Boolean functions (rather than the expressions themselves) are equal.

For example, verifying the equality $C = D$, where C and D are positive CNF and DNF, respectively, is the famous "dualization problem"; see Section 4.4 in [2]. Obviously, this problem belongs to coNP. Although no polynomial dualization algorithm is known, yet, a quasi-polynomial ($N^{\log N}$) one was given in [3]. Thus, dualization is not coNP-complete unless every problem from coNP can be solved in quasi-polynomial time, which is unlikely.

Let us consider a similar verification problem: whether equality $C \vee D_0 = D_0$ holds for a very special DNF $D_0 = x_1y_1 \vee \ldots \vee x_ny_n$ and a CNF C (of m clauses and the same $2n$ variables). Obviously, the equality in question, $C \vee D_0 = D_0$, is equivalent with the inequality $D_0 \geq C$. It is also clear that the problem is in coNP. Moreover, it is coNP-complete.

Lemma 1 *([5]). It is a coNP-complete problem to verify the the equality $C \vee D_0 = D_0$, or equivalently, the inequality $C \leq D_0$.*

Moreover, the problem remains coNP-complete when CNF C satisfies the extra condition:

$(q_1):$ *no literal appears in all m clauses of C, that is, C has no linear implicant.*

Proof. Let us replace y_i by $\bar{x}_i$ in C for all $i \in [n] = \{1, \ldots, n\}$ and denote the obtained (non-positive) CNF by C'. Assuming (q_1), we can easily see that CNF C' is *not* satisfiable if and only if $C \vee D_0 = D_0$. Furthermore, condition (q_1) can be directly checked in nm time, and obviously, CNF C' is satisfiable whenever (q_1) fails. Thus, verifying $c \vee D_0 = D_0$ is coNP-complete in both cases, with or without assumption (q_1). □

The above Lemma can be extended by the following fully similar claim.

Lemma 2 *Verifying equality $C \vee D_0 = D_0$ remains coNP-complete when C satisfies the following (weaker than (q_1)) condition*

$(q_2):$ *CNF C has no linear or quadratic implicants except x_iy_i for $i \in [n]$.*

Proof. It is similar to the previous one. Clearly, (q_2) can be verified in n^2m time; moreover CNF C' is obviously satisfiable when (q_2) fails. Hence, both problems, the satisfiability of C' and verifying $C \vee D_0 = D_0$ remain coNP-complete in case when (q_2) holds. □

The following statement explains the role of the assumption (q_1).

Lemma 3 *If CNF C satisfies condition (q_1) and expression $C \vee D_0$ defines a read-once function f then f is quadratic, that is, all its prime implicants are quadratic.*

Proof. First, let us notice, that x_iy_i is a prime implicant of $C \vee D_0$ for each $i \in [n]$. Indeed, $D_0 = h \vee x_1y_1 \vee \ldots \vee x_ny_n$ and C contains no linear implicants, by (q_1).

Furthermore, function $f = C \vee D_0$ is read-once. Let ψ be a read-once expression of f and T be the corresponding tree. By definitions, the leaves of T are labeled by the variables

$x_1, y_1, \ldots, x_n, y_n$, each of which appears at most once; in fact, exactly once, since $x_i y_i$ is a prime implicant of f for each $i \in [n]$.

Furthermore, all other nodes of T are labeled by $\vee$ and $\wedge$. For each $i \in [n]$, let us consider in T two paths p_i and r_i from the root v_0 to the leaves labeled by x_i and y_i, respectively, and denote by v_i the last common vertex of these two paths. Obviously, v_i is a $\wedge$-vertex, since $x_i y_i$ is a prime implicant of f. For the same reason, vertex v_i is of degree 3 in T: the corresponding 3 edges lead towards x_i, y_i, and v_0. Moreover, for the same reason, paths p_i and r_i have no other $\wedge$-vertices. Since $i \in [n]$ was chosen arbitrary, we conclude that every path in T from the root to a leaf has exactly one $\wedge$-vertex and this vertex is of degree 3. This easily implies every prime implicant of f is quadratic. □

Moreover, it is easy to see that we can strengthen the above lemma and completely characterize the obtained read-once functions as follows.

Theorem 1 *If CNF C satisfies condition (q_1) and expression $C \vee D_0$ defines a read-once function f then there is a partition $[n] = I_1 \cup \ldots \cup I_k$ such that $f = \bigvee_{j=1}^{k} \mu_j \nu_j$, where μ_j and ν_j for all $j \in [k] = \{1, \ldots, k\}$ are elementary disjunctions each of which contains exactly one of two variables x_i, y_i for each $i \in I_j$ and no other variables.*

Proof. In fact, it is already given in the proof of the previous lemma. □

Let us consider several examples:

$(x_1 \vee y_2)(y_1 \vee x_2) \vee (x_1 y_1 \vee x_2 y_2 \vee x_3 y_3 \vee x_4 y_4) = (x_1 \vee y_2)(y_1 \vee x_2) \vee x_3 y_3 \vee x_4 y_4$;

$(x_1 \vee y_2 \vee x_3 \vee x_4)(x_1 \vee y_2 \vee y_3 \vee y_4)(y_1 \vee x_2 \vee x_3 \vee x_4)(y_1 \vee x_2 \vee y_3 \vee y_4) \vee$
$(x_1 y_1 \vee x_2 y_2 \vee x_3 y_3 \vee x_4 y_4 \vee x_5 y_5) = (x_1 \vee y_2)(y_1 \vee x_2) \vee (x_3 \vee x_4)(y_3 \vee y_4) \vee x_5 y_5$.

Remark 1 *It is also easy to demonstrate that condition (q_1) is essential in the above Lemma and Theorem. Let us consider, for example, the CNF*

$C = (x_1 \vee x_2 \vee \ldots \vee x_n \vee y_1)(x_1 \vee x_2 \vee \ldots \vee x_n \vee y_2) \ldots (x_1 \vee x_2 \vee \ldots \vee x_n \vee y_n)$.

Obviously, the corresponding function $f = C \vee D_0 = x_1 \vee x_2 \vee \ldots \vee x_n \vee (y_1 y_2 \ldots y_n)$ is read-once but it contains the prime implicant $(y_1 y_2 \ldots y_n)$ which is not quadratic when $n > 2$. Yet, in this example C does not satisfy condition (q_1).

The following statement explains the role of assumption $(q2)$.

Lemma 4 *If CNF C satisfies (q_2) then the corresponding Boolean function $f = C \vee D_0$ is read-once if and only if $C \vee D_0 = D_0$.*

Proof. The "if part" is obvious, since function D_0 *is* read-once, while the "only if part" follows immediately from the previous lemma. □

Remark 2 *It is easy to demonstrate that assumption (q_2) is essential. Indeed, the above examples show CNFs C such that $f = C \vee D_0$ is a read-once function distinct from D_0. However, in this case, by Theorem 1, all implicants of f are quadratic; hence, (q_2) fails.*

Now we are ready to prove the first desired result.

Theorem 2 *Let a Boolean function f be given by a positive $\vee$-$\wedge$ formula $C \vee D_0$, of depth 3, the decision problem Q "whether f is read-once" is coNP-complete.*

Proof. CoNP-hardness of Q immediately follows from Lemmas 2 and 4.

It remains to show that Q is in coNP. This will follow from an old characterization: f is read once if and only if every two prime implicants, P of f and D of f^d, have exactly one variable in common; [4], see also [6]. Hence, to disprove that f is read-once, it is sufficient to demonstrate dual prime implicants P_0 and D_0 with at least two common variables. Furthermore, to verify that P_0 is a prime implicant of f it is sufficient to check that

(i) f is true if all variables of P_0 are true, while all others are false, and

(ii) f is false if all variables of P_0 but one are true, while all others are false.

Similarly, we can check that D_0 is a prime implicant of f^d. To do so, it is enough to dualize the expression ψ by swap of $\vee$ and $\wedge$ using de Morgan's rules. □

Remark 3 *The problem remains in coNP when a positive Boolean function f is given by any (not necessarily $\vee$-$\wedge$) Boolean formula or by a polynomial oracle. Indeed, the prime implicants of dual functions f and f^d are in one-to-one correspondence with the minimal true and maximal false vectors of f and every such vector can be standardly tested in at most n questions to the oracle. Moreover, it has been shown in [1] that the problem remains in coNP even without assumption of the positivity of f.*

Interestingly, the similar result for quadratic (rather than read-once) Boolean functions we obtain for free, since it is already proven by the above arguments.

Theorem 3 *Let a Boolean function f be given by a positive $\vee$-$\wedge$ formula $C \vee D_0$, of depth 3, the decision problem Q_2 "whether f is quadratic" is coNP-complete.*

Proof. Obviously, function $f = C \vee D_0$ is quadratic if and only if $C \vee D_0 = D_0$, provided condition (q_2) holds for C. Yet, in the latter case the satisfyability of C' remains coNP-complete. □

Remark 4 *Finally, let us notice that the Theorem 3 will still hold if we replace Q_2 by a more general decision problem Q_k "whether $dim(f) \leq k$", or in other words, "whether each prime implicant of f contains at most k variables"; problem Q_k is coNP-complete for every given parameter $k \geq 2$; obviously, Q_1 is trivial.*

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