It is a coNP-complete problem to decide whether a positive $\lor - \land$ formula of depth 3 defines a read-once or respectively quadratic Boolean function $a$

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Abstract. Let us consider the DNF \( D_0 = x_1 y_1 \lor \cdots \lor x_n y_n \) and let \( C \) be a positive CNF of the same \( 2n \) variables. It is a coNP-complete problem to verify the inequality \( D_0 \geq C \), which is obviously equivalent with the equality \( C \lor D_0 = D_0 \). Clearly, the problem is in coNP and its polynomial reduction from SAT is straightforward. Nevertheless, this trivial observation has many applications in enumeration, game, and graph theories, as Khachiyan and the author demonstrated in 1995. Since then several other interesting applications were found. In this paper, we show that it is coNP-complete to verify whether the expression \( C \lor D_0 \) defines a quadratic or, respectively, read-once Boolean function.

Key words: positive Boolean function or formula, read-once, linear, quadratic, disjunctive and conjunctive normal forms, DNF, CNF, coNP-complete problem
We will make use of many elementary concepts of Boolean function theory that can be found in any textbook; see for example [2], where quadratic and read-once Boolean functions are considered in Chapters 5 and 10, respectively.

Let us remark that the equality of two Boolean expression will mean that the corresponding two Boolean functions (rather than the expressions themselves) are equal.

For example, verifying the equality \( C = D \), where \( C \) and \( D \) are positive CNF and DNF, respectively, is the famous "dualization problem"; see Section 4.4 in [2]. Obviously, this problem belongs to coNP. Although no polynomial dualization algorithm is known, yet, a quasi-polynomial (\( N^{\log N} \)) one was given in [3]. Thus, dualization is not coNP-complete unless every problem from coNP can be solved in quasi-polynomial time, which is unlikely.

Let us consider a similar verification problem: whether equality \( C_0 = D_0 \) holds for a very special DNF \( D_0 = x_1y_1 \lor \ldots \lor x_ny_n \) and a CNF \( C \) (of \( m \) clauses and the same \( 2n \) variables). Obviously, the equality in question, \( C_0 \leq D_0 \), is equivalent with the inequality \( D_0 \geq C \). It is also clear that the problem is in coNP. Moreover, it is coNP-complete.

**Lemma 1** ([5]). It is a coNP-complete problem to verify the equality \( C_0 = D_0 \), or equivalently, the inequality \( C \leq D_0 \).

Moreover, the problem remains coNP-complete when CNF \( C \) satisfies the extra condition:

\[
(q_1): \text{no literal appears in all } m \text{ clauses of } C, \text{ that is, } C \text{ has no linear implicant.}
\]

**Proof.** Let us replace \( y_i \) by \( \bar{x}_i \) in \( C \) for all \( i \in [n] = \{1, \ldots, n\} \) and denote the obtained (non-positive) CNF by \( C' \). Assuming \((q_1)\), we can easily see that CNF \( C' \) is not satisfiable if and only if \( C \lor D_0 = D_0 \). Furthermore, condition \((q_1)\) can be directly checked in \( nm^2 \) time, and obviously, CNF \( C' \) is satisfiable whenever \((q_1)\) fails. Thus, verifying \( c \lor D_0 = D_0 \) is coNP-complete in both cases, with or without assumption \((q_1)\). \( \square \)

The above Lemma can be extended by the following fully similar claim.

**Lemma 2** Verifying equality \( C \lor D_0 = D_0 \) remains coNP-complete when CNF \( C \) satisfies the following (weaker than \((q_1)\)) condition

\[
(q_2): \text{CNF } C \text{ has no linear or quadratic implicants except } x_iy_i \text{ for } i \in [n].
\]

**Proof.** It is similar to the previous one. Clearly, \((q_2)\) can be verified in \( n^2m \) time; moreover CNF \( C' \) is obviously satisfiable when \((q_2)\) fails. Hence, both problems, the satisfiability of \( C' \) and verifying \( C \lor D_0 = D_0 \) remain coNP-complete in case when \((q_2)\) holds. \( \square \)

The following statement explains the role of the assumption \((q_1)\).

**Lemma 3** If CNF \( C \) satisfies condition \((q_1)\) and expression \( C \lor D_0 \) defines a read-once function \( f \) then \( f \) is quadratic, that is, all its prime implicants are quadratic.

**Proof.** First, let us notice, that \( x_iy_i \) is a prime implicant of \( C \lor D_0 \) for each \( i \in [n] \). Indeed, \( D_0 = h \lor x_1y_1 \lor \ldots \lor x_ny_n \) and \( C \) contains no linear implicants, by \((q_1)\).

Furthermore, function \( f = C \lor D_0 \) is read-once. Let \( \psi \) be a read-once expression of \( f \) and \( T \) be the corresponding tree. By definitions, the leaves of \( T \) are labeled by the variables...
\[x_1, y_1, \ldots, x_n, y_n, \text{ each of which appears at most once; in fact, exactly once, since } x_i y_i \text{ is a prime implicant of } f \text{ for each } i \in [n].\]

Furthermore, all other nodes of \(T\) are labeled by \(\lor\) and \(\land\). For each \(i \in [n]\), let us consider in \(T\) two paths \(p_i\) and \(r_i\) from the root \(v_0\) to the leaves labeled by \(x_i\) and \(y_i\), respectively, and denote by \(v_i\) the last common vertex of these two paths. Obviously, \(v_i\) is a \(\land\)-vertex, since \(x_i y_i\) is a prime implicant of \(f\). For the same reason, vertex \(v_i\) is of degree 3 in \(T\): the corresponding 3 edges lead towards \(x_i, y_i\), and \(v_0\). Moreover, for the same reason, paths \(p_i\) and \(r_i\) have no other \(\land\)-vertices. Since \(i \in [n]\) was chosen arbitrary, we conclude that every path in \(T\) from the root to a leaf has exactly one \(\land\)-vertex and this vertex is of degree 3. This easily implies every prime implicant of \(f\) is quadratic.

Moreover, it is easy to see that we can strengthen the above lemma and completely characterize the obtained read-once functions as follows.

**Theorem 1** If CNF \(C\) satisfies condition \((q_1)\) and expression \(C \lor D_0\) defines a read-once function \(f\) then there is a partition \([n] = I_1 \cup \ldots \cup I_k\) such that \(f = \bigvee_{j=1}^{k} \mu_j \nu_j\), where \(\mu_j\) and \(\nu_j\) for all \(j \in [k] = \{1, \ldots, k\}\) are elementary disjunctions each of which contains exactly one of two variables \(x_i, y_i\) for each \(i \in I_j\) and no other variables.

**Proof.** In fact, it is already given in the proof of the previous lemma.

Let us consider several examples:

\[
\begin{align*}
(x_1 \lor y_2)(y_1 \lor x_2) & \lor (x_1y_1 \lor x_2y_2 \lor x_3y_3 \lor x_4y_4) = (x_1 \lor y_2)(y_1 \lor x_2) \lor x_3y_3 \lor x_4y_4; \\
(x_1 \lor y_2 \lor x_3 \lor x_4)(x_1 \lor y_2 \lor y_3 \lor y_4)(x_1 \lor x_2 \lor x_3 \lor x_4)(y_1 \lor x_2 \lor y_3 \lor y_4) & \lor \\
(x_1y_1 \lor x_2y_2 \lor x_3y_3 \lor x_4y_4 \lor x_5y_5) = (x_1 \lor y_2)(y_1 \lor x_2) \lor (x_3 \lor x_4)(y_3 \lor y_4) \lor x_5y_5.
\end{align*}
\]

**Remark 1** It is also easy to demonstrate that condition \((q_1)\) is essential in the above Lemma and Theorem. Let us consider, for example, the CNF

\[C = (x_1 \lor x_2 \lor \ldots \lor x_n \lor y_1)(x_1 \lor x_2 \lor \ldots \lor x_n \lor y_2) \ldots (x_1 \lor x_2 \lor \ldots \lor x_n \lor y_n).
\]

Obviously, the corresponding function \(f = C \lor D_0 = x_1 \lor x_2 \lor \ldots \lor x_n \lor (y_1 y_2 \ldots y_n)\) is read-once but it contains the prime implicant \((y_1 y_2 \ldots y_n)\) which is not quadratic when \(n > 2\). Yet, in this example \(C\) does not satisfy condition \((q_1)\).

The following statement explains the role of assumption \((q_2)\).

**Lemma 4** If CNF \(C\) satisfies \((q_2)\) then the corresponding Boolean function \(f = C \lor D_0\) is read-once if and only if \(C \lor D_0 = D_0\).

**Proof.** The "if part" is obvious, since function \(D_0\) is read-once, while the "only if part" follows immediately from the previous lemma.

**Remark 2** It is easy to demonstrate that assumption \((q_2)\) is essential. Indeed, the above examples show CNFs \(C\) such that \(f = C \lor D_0\) is a read-once function distinct from \(D_0\). However, in this case, by Theorem 1, all implicants of \(f\) are quadratic; hence, \((q_2)\) fails.
Now we are ready to prove the first desired result.

**Theorem 2** Let a Boolean function $f$ be given by a positive $\lor\land$ formula $C \lor D_0$, of depth 3, the decision problem $Q$ ”whether $f$ is read-once” is coNP-complete.

**Proof.** CoNP-hardness of $Q$ immediately follows from Lemmas 2 and 4.

It remains to show that $Q$ is in coNP. This will follow from an old characterization: $f$ is read once if and only if every two prime implicants, $P$ of $f$ and $D$ of $f^d$, have exactly one variable in common; [4], see also [6]. Hence, to disprove that $f$ is read-once, it is sufficient to demonstrate dual prime implicants $P_0$ and $D_0$ with at least two common variables. Furthermore, to verify that $P_0$ is a prime implicant of $f$ it is sufficient to check that

(i) $f$ is true if all variables of $P_0$ are true, while all others are false, and

(ii) $f$ is false if all variables of $P_0$ but one are true, while all others are false.

Similarly, we can check that $D_0$ is a prime implicant of $f^d$. To do so, it is enough to dualize the expression $\psi$ by swap of $\lor$ and $\land$ using de Morgan’s rules. \hfill \-box

**Remark 3** The problem remains in coNP when a positive Boolean function $f$ is given by any (not necessarily $\lor\land$) Boolean formula or by a polynomial oracle. Indeed, the prime implicants of dual functions $f$ and $f^d$ are in one-to-one correspondence with the minimal true and maximal false vectors of $f$ and every such vector can be standardly tested in at most $n$ questions to the oracle. Moreover, it has been shown in [1] that the problem remains in coNP even without assumption of the positivity of $f$.

Interestingly, the similar result for quadratic (rather than read-once) Boolean functions we obtain for free, since it is already proven by the above arguments.

**Theorem 3** Let a Boolean function $f$ be given by a positive $\lor\land$ formula $C \lor D_0$, of depth 3, the decision problem $Q_2$ ”whether $f$ is quadratic” is coNP-complete.

**Proof.** Obviously, function $f = C \lor D_0$ is quadratic if and only if $C \lor D_0 = D_0$, provided condition $(q_2)$ holds for $C$. Yet, in the latter case the satisfyability of $C'$ remains coNP-complete. \hfill \-box

**Remark 4** Finally, let us notice that the Theorem 3 will still hold if we replace $Q_2$ by a more general decision problem $Q_k$ ”whether $\dim(f) \leq k$”, or in other words, ”whether each prime implicant of $f$ contains at most $k$ variables”; problem $Q_k$ is coNP-complete for every given parameter $k \geq 2$; obviously, $Q_1$ is trivial.

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References


