RUTCOR
RESEARCH
REPORT

Uniform Quasi-Concavity in Probabilistic Constrained Stochastic Programming

András Prékopa[a] Kunikazu Yoda[b]
Munevver Mine Subasi[c]

RRR 6-2010, April, 2010

RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

[a]RUTCOR, Rutgers Center for Operations Research, 640 Bartholomew Road, Piscataway, NJ 08854-8003; prekopa@rutcor.rutgers.edu
[b]RUTCOR, Rutgers Center for Operations Research, 640 Bartholomew Road, Piscataway, NJ 08854-8003; kyoda@rutcor.rutgers.edu
[c]RUTCOR, Rutgers Center for Operations Research, 640 Bartholomew Road, Piscataway, NJ 08854-8003; msub@rutcor.rutgers.edu

Rutcor Research Report
RRR 6-2010, April, 2010

Uniform Quasi-Concavity in Probabilistic Constrained Stochastic Programming

András Prékopa Kunikazu Yoda Munevver Mine Subasi

Abstract. A probabilistic constrained stochastic programming problem is considered, where the underlying problem has linear constraints with random technology matrix. The rows of the matrix are assumed to be stochastically independent and normally distributed. For the convexity of the problem the quasi-concavity of the constraining function is needed that is ensured if the factors are uniformly quasi-concave. In the paper a necessary and sufficient condition is given for that property to hold. It is also shown, through numerical examples, that such a special problem still has practical application in optimal portfolio construction.

Acknowledgements: This research was supported by National Science Foundation Grant CMMI-0856663.

1 Introduction

The stochastic programming problem, termed programming under probabilistic constraint can be formulated in the following way:

$$\begin{aligned} &\text{minimize} \;\; f(x) \\ &\text{subject to}\; h_0(x) = \mathbb{P}\left(g_i(x,\xi) \geq 0,\; i = 1,\dots,r\right) \geq p \\ &\qquad\qquad h_i(x) \geq 0,\; i = 1,\dots,m, \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^q$, $f(x), g_i(x,y), i = 1,\dots,r$, $h_i(x), i = 1,\dots,m$ are some functions and p is a fixed large probability, e.g., $p = 0.8, 0.9, 0.95, 0.99$. In many application the stochastic constraints have the form $\xi - Tx \geq 0$ and the probabilistic constraint specializes as

$$h_0(x) = \mathbb{P}(Tx \leq \xi) \geq p. \tag{1.2}$$

For the case of continuously distributed random vector ξ general theorems are available to ensure the convexity of the set determined by the probabilistic constraint in (1.1). For example, if $g_i,\; i = 1,\dots,r$ are concave or at least quasi-concave in all variables and ξ has a logconcave p.d.f., then the function $h_0(x)$ is logconcave and the set $\{x \mid h_0(x) \geq p\}$ is convex (see, e.g. [12, 14]). This implies that if ξ has the above-mentioned property, then the set determined by the constraint (1.2) is convex. Many applications of the model with probabilistic constraint (1.2) have been carried out, for the cases of some special continuous multivariate distributions such as normal, gamma, and Dirichlet, and problem solving packages have been developed [14, 3, 6, 15].

The solution of problems where ξ in (1.2) is a discrete random vector is more recent. The key concept here is that of a p-efficient point, introduced in [11] and further developed and used in [2, 4, 8, 13].

For the case of a random T in the constraint (1.2), few results are known. The earliest papers dealing with random matrix T in probabilistic constraint are [7, 9]. In these papers, however, there is only one stochastic constraint and to establish the concavity of the set $\{x \mid \mathbb{P}(Tx \leq \xi) \geq p\}$ is relatively easy (see, the proof of Lemma 2.2).

The first paper where convexity theorems are presented for the set of feasible solution and random matrix T has more than one row, is [10]. If T has more than one row, then even if they are independent, it is not easy to ensure the convexity of the set $\{x \mid \mathbb{P}(Tx \leq d) \geq p\}$. The papers [12] and [5] can be mentioned, where progress in this direction has been made. The problem is that the products or sums of quasi-concave functions are not quasi-concave, in general. We briefly recall the main results of the paper [10] (see also [12] pp.312–314).

Theorem 1.1. *Let ξ be constant and T a random matrix with independent, normally distributed rows (or columns) such that their covariance matrices are constant multiples of each other. Then $h(x) = \mathbb{P}(Tx \leq \xi)$ is a quasi-concave function on the set $\{x \mid h(x) \geq 1/2\}$.*

We introduce a special class of quasi-concave functions.

Definition 1.1 (Uniformly quasi-concave functions)**.** *Let $h_1(x), \dots, h_r(x)$ be quasi-concave functions on a convex set $E \in \mathbb{R}^n$. We say that they are uniformly quasi-concave functions if for any $x, y \in E$ either*

$$\min(h_i(x), h_i(y)) = h_i(x),\ \ i = 1, \dots, r$$

or

$$\min(h_i(x), h_i(y)) = h_i(y),\ \ i = 1, \dots, r.$$

Obviously, the sum of uniformly quasi-concave functions, on the same set, is also quasi-concave and if the functions are also nonnegative, then the same holds for their product as well. The latter property is used in the next section, where we prove our main result.

In this paper we look at probabilistic constraints of the type

$$\mathbb{P}(Tx \leq b) \geq p, \tag{1.3}$$

where T is a random matrix that has independent, normally distributed rows and b is a constant vector. The constraining function in (1.3) is the product of special quasi-concave functions and we show that the uniform quasi-concavity of the factors implies that the covariance matrices of the rows are constant multiples of each other. Section 2 and 3 are devoted to this. In section 4 we show that this very special type of probabilistic constraint is still applicable to solve portfolio optimization problems. We present some numerical results in this respect.

2 Preliminary Results

First we provide a necessary condition for continuously differentiable and uniformly quasi-concave functions $h_1(x), \dots, h_r(x)$ on an open convex set.

Lemma 2.1. *If $h_1(x), \dots, h_r(x)$ are continuously differentiable and uniformly quasi-concave on an open convex set E, then any nonzero gradients $\nabla h_i(x), \nabla h_j(x)$ are positive multiples of each other, i.e., for any $i, j \in \{1, \dots, r\}$, there exists a positive-valued function $\alpha_{ij}(x) = 1/\alpha_{ji}(x) > 0$ defined on $E_{ij} = \{x \in E \mid \nabla h_i(x) \neq 0, \nabla h_j(x) \neq 0\} = E_{ji}$ such that for all $x \in E_{ij}$ we have*

$$\nabla h_i(x) = \alpha_{ij}(x) \nabla h_j(x) \tag{2.1}$$

Proof. We show that (2.1) holds for all $x \in E_{ij}$ by contradiction. Suppose that for some $x \in E_{ij}$ we cannot find an $\alpha_{ij}(x) > 0$ satisfying (2.1). Without loss of generality we assume that $i = 1, j = 2$.

From Farkas Lemma, either one of the following two systems has a solution

(i) $\nabla h_2(x)^T d \leq 0,\ \nabla h_1(x)^T d > 0$

(ii) $\nabla h_1(x) = \lambda \nabla h_2(x),\ \lambda \geq 0$

First, note that since $\nabla h_1(x) \neq 0$ and $\nabla h_2(x) \neq 0$, $\lambda = 0$ cannot be a solution of (ii). Also, $\lambda > 0$ cannot be a solution of (ii), otherwise we can define $\alpha_{12}(x) = \lambda > 0$. Hence, (i) has a solution d_1. Similarly, since $\nabla h_2(x) = \alpha_{21}(x)\nabla h_1(x)$ does not hold for any defined value of $\alpha_{21}(x) = 1/\alpha_{12}(x) > 0$ by the assumption, (i) with 1 and 2 interchanged has a solution d_2. So we have

$$\nabla h_2(x)^T d_1 \leq 0,\ \nabla h_1(x)^T d_1 > 0,$$
$$\nabla h_1(x)^T d_2 \leq 0,\ \nabla h_2(x)^T d_2 > 0.$$

Let $d := d_1 - d_2$. Then it follows that

$$\nabla h_1(x)^T d > 0,\ \nabla h_2(x)^T d < 0. \tag{2.2}$$

Note that $d \neq 0$. By the use of finite Taylor series expansions we can write:

$$h_1(x + \varepsilon d) = h_1(x) + (\nabla h_1(x)^T d)\varepsilon + o(\varepsilon), \tag{2.3}$$
$$h_2(x + \varepsilon d) = h_2(x) + (\nabla h_2(x)^T d)\varepsilon + o(\varepsilon). \tag{2.4}$$

Since E_{12} is an open set, we can select $\varepsilon > 0$ small enough so that

$$\exists y := x + \varepsilon d \in E_{12},\ y \neq x, \quad h_1(y) > h_1(x), \quad h_2(y) < h_2(x)$$

Hence $h_1(x), \dots, h_r(x)$ are not uniformly quasi-concave, which is a contradiction. □

For $r = 1$, let us consider the function

$$h(x) = \mathbb{P}(Tx \leq b), \tag{2.5}$$

where T is a random row vector and b is a constant. The following lemma was first proved by Kataoka [7] and van de Panne and Popp [9]. See also Prékopa [12].

Lemma 2.2 ([7, 9]). *If T has normal distribution, then the function $h(x)$ is quasi-concave on the set*

$$\left\{ x \mid \mathbb{P}(Tx \leq b) \geq \frac{1}{2} \right\}.$$

Proof (from [14], pp 284-285). We prove the equivalent statement: for any $p \geq 1/2$ the set

$$\{x \mid \mathbb{P}(Tx \leq b) \geq p\} \tag{2.6}$$

is convex.

Let $\Phi(t)$ denote the c.d.f. of the one-dimensional standard normal distribution and let $\mu = \mathbb{E}(T^T)$ and $C = \mathbb{E}((T^T - \mu)(T^T - \mu)^T)$ denote the mean vector and the covariance matrix of T, respectively. For any x such that $x^T Cx > 0$,

$$\begin{aligned} h(x) &= \mathbb{P}(Tx \le b) \\ &= \mathbb{P}\left(\frac{(T-\mu^T)x}{\sqrt{x^T Cx}} \le \frac{b-\mu^T x}{\sqrt{x^T Cx}}\right) \\ &= \Phi\left(\frac{b-\mu^T x}{\sqrt{x^T Cx}}\right) \ge p \end{aligned}$$

is equivalent to

$$\mu^T x + \Phi^{-1}(p)\sqrt{x^T Cx} \le b. \tag{2.7}$$

Since $\Phi^{-1}(p) \ge 0$ and $\sqrt{x^T Cx}$ is a convex function of x on $\mathbb{R}^n$, it follows that inequality (2.7) determines a convex set.

For any x such that $x^T Cx = 0$, $Tx = \mu^T x$ with probability 1. Since $p > 0$ it follows that

$$h(x) = \mathbb{P}(Tx \le b) = \mathbb{P}(\mu^T x \le b) \ge p$$

is equivalent to

$$\mu^T x \le b. \tag{2.8}$$

The set of x determined by (2.8) is convex. □

Let r be an arbitrary positive integer and introduce the function:

$$h_i(x) = \mathbb{P}(T_i x \le b_i), \ i = 1, \dots, r, \tag{2.9}$$

where each row vector $T_i, \ i = 1, \dots, r$ has normal distribution with mean vector $\mu_i = \mathbb{E}(T_i^T)$ and covariance matrix $C_i = \mathbb{E}((T_i^T - \mu_i)(T_i^T - \mu_i)^T)$, and $b = (b_1, \dots, b_r)^T$ is constant.

Suppose $b_i > 0, \ i = 1, \dots, r$. Let us define set E as follows:

$$\begin{aligned} &E \text{ is convex.} \\ &E \supset B \supset \{0\} \text{ for some open set } B. \\ &\text{Each } h_i(x), \ i = 1, \dots, r \text{ is quasi-concave on } E. \end{aligned} \tag{2.10}$$

One example of such E is

$$E = \bigcap_{i=1}^{r} \left\{ x \mid h_i(x) \ge \frac{1}{2} \right\}. \tag{2.11}$$

Note that by lemma 2.2, $h_i(x)$ is quasi-concave on the convex set $E_i = \{x \mid h_i(x) \ge 1/2\}$ and that for sufficiently small open ball $B_\epsilon(0) = \{x \mid \|x\| < \epsilon\}$ of the origin, $h_i(x) \ge 1/2, \ \forall x \in B_\epsilon(0)$, thus $E_i \supset B_\epsilon(0)$. Also note that the intersection of convex sets is a convex set. If rows $T_1, \ldots, T_r$ of T are independent and $h_1(x), \ldots, h_r(x)$ are uniformly quasi-concave, then $h(x) = \mathbb{P}(T_i x \le b_i,\ i = 1, \ldots, r) = \prod_{i=1}^r \mathbb{P}(T_i \le b_i) = h_1(x) \cdots h_r(x)$ is quasi-concave on E.

Suppose $b_i > 0$ and C_i is positive definite for all $i \in \{1, \ldots, r\}$.

$$h_i(x) = \begin{cases} \Phi\left(\dfrac{b_i - \mu_i^T x}{\sqrt{x^T C_i x}}\right) & \text{for } x \neq 0, \\ \mathbb{P}(0 \le b_i) = 1 & \text{for } x = 0. \end{cases} \tag{2.12}$$

Since

$$\lim_{x \to 0} h_i(x) = \lim_{t \to \infty} \Phi(t) = 1 = h_i(0),$$

$h_i(x)$ is continuous at $x = 0$. Let us calculate the gradient of $h_i(x)$ for $x \in \operatorname{int}(E) \setminus \{0\}$.

$$\begin{aligned} \nabla h_i(x) &= \nabla \Phi\left(\frac{b_i - \mu_i^T x}{\sqrt{x^T C_i x}}\right) \\ &= \phi\left(\frac{b_i - \mu_i^T x}{\sqrt{x^T C_i x}}\right) \nabla \frac{b_i - \mu_i^T x}{\sqrt{x^T C_i x}} \\ &= \phi\left(\frac{b_i - \mu_i^T x}{\sqrt{x^T C_i x}}\right) \frac{-\sqrt{x^T C_i x}\mu_i - (b_i - \mu_i^T x) C_i x / \sqrt{x^T C_i x}}{x^T C_i x} \\ &= -\phi\left(\frac{b_i - \mu_i^T x}{\sqrt{x^T C_i x}}\right) \frac{(x^T C_i x)\mu_i + (b_i - \mu_i^T x) C_i x}{(x^T C_i x)^{3/2}}, \end{aligned} \tag{2.13}$$

where $\phi(t)$ is the p.d.f. of the one-dimensional standard normal distribution.

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} t^2\right).$$

For any fixed $x \neq 0$, we have

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \nabla h_i(\varepsilon x) &= -\lim_{\varepsilon \downarrow 0} \phi\left(\frac{b_i}{\varepsilon\sqrt{x^T C_i x}} - \frac{\mu_i^T x}{\sqrt{x^T C_i x}}\right) \left\{\frac{(x^T C_i x)\mu_i - (\mu_i^T x) C_i x}{\varepsilon\sqrt{x^T C_i x}} + \frac{b_i C_i x}{\varepsilon^2 (x^T C_i x)^{3/2}}\right\} \\ &= 0. \end{aligned}$$

Hence $\lim_{x \to 0} \nabla h_i(x) = 0$ and $\nabla h_i(x)$ is continuous at $x = 0$. Therefore $h_1(x), \ldots, h_r(x)$ are continuously differentiable on the open convex set $\operatorname{int}(E)$.

3 The Main Result

In what follows we make use of the following theorem [1] from linear algebra:

Theorem 3.1 (Simultaneous Diagonalization of Two Matrices)**.** *Given two real symmetric matrices, A and B, with A positive-valued definite, there exists a nonsingular matrix U such* *that*

$$U^T AU = I, \quad U^T BU = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{pmatrix} \tag{3.1}$$

In the next theorem we present our main result.

Theorem 3.2. *Suppose $b_i > 0$ and C_i is positive definite for $i \in \{1, \dots, r\}$. The functions $h_1(x), \dots, h_r(x)$ defined by* (2.9) *(in this case,* (2.12)*) are uniformly quasi-concave on a convex set E satisfying* (2.10) *if and only if each C_i is a constant multiple of a covariance matrix C, and*

$$\frac{\mu_1}{b_1} = \dots = \frac{\mu_r}{b_r}.$$

Proof. Sufficiency ($\Leftarrow$) is obvious, so we only show necessity ($\Rightarrow$). It is enough to show that C_1, C_2 are constant multiples of each other and that $\mu_1/b_1 = \mu_2/b_2$ for $r \geq 2$. $h_i(x)$ is continuously differentiable on the open convex set $\operatorname{int}(E)$. From (2.13) we have for $x \neq 0$

$$x^T \nabla h_i(x) = -\phi\left(\frac{b_i - \mu_i^T x}{\sqrt{x^T C_i x}}\right) \frac{b_i}{\sqrt{x^T C_i x}} < 0.$$

Thus $\nabla h_i(x) \neq 0$ for $x \neq 0$. We know $\lim_{x \to 0} \nabla h_i(x) = 0$. Let $E' := \operatorname{int}(E) \setminus \{0\}$. Then $E' = \{x \in \operatorname{int}(E) \mid \nabla h_i(x) \neq 0,\ i \in \{1, \dots, r\}\}$. From Lemma 2.1 and (2.13), there is a positive function $\alpha_{12}(x) > 0$ such that for all $x \in E'$ we have

$$(x^T C_1 x)\mu_1 + (b_1 - \mu_1^T x) C_1 x = \alpha_{12}(x) \left\{(x^T C_2 x)\mu_2 + (b_2 - \mu_2^T x) C_2 x\right\} \tag{3.2}$$

For small $\varepsilon > 0$ and $x \in E'$, let us replace x with $\varepsilon x \in E'$ in (3.2) and divide by ε for both sides of the equation.

$$\varepsilon(x^T C_1 x)\mu_1 + (b_1 - \varepsilon\mu_1^T x) C_1 x = \alpha_{12}(\varepsilon x) \left\{\varepsilon(x^T C_2 x)\mu_2 + (b_2 - \varepsilon\mu_2^T x) C_2 x\right\} \tag{3.3}$$

Taking the limit of the both sides of (3.3) as $\varepsilon \to 0$ we obtain

$$b_1 C_1 x = (\lim_{\varepsilon \to 0} \alpha_{12}(\varepsilon x)) b_2 C_2 x. \tag{3.4}$$

Since $0 < x^T C_1 x < \infty,\ 0 < x^T C_2 x < \infty$ for $x \in E'$, the limit

$$\lim_{\varepsilon \to 0} \alpha_{12}(\varepsilon x) = \frac{b_1 x^T C_1 x}{b_2 x^T C_2 x} =: \alpha'_{12}(x) \tag{3.5}$$

exists and $0 < \alpha'_{12}(x) < \infty$. Thus we have

$$b_1 C_1 x = \alpha'_{12}(x) b_2 C_2 x \quad \text{for } x \in E'. \tag{3.6}$$

Since C_1 and C_2 are symmetric and C_2 is positive definite, from Theorem 3.1 there is a nonsingular matrix U such that

$$U^T C_1 U = D, \quad U^T C_2 U = I,$$

where $D = \mathrm{diag}(\lambda_1, \dots, \lambda_r)$ is a diagonal matrix. Let $y := U^{-1}x$ and $F := \{U^{-1}x \mid x \in E'\}$. Since U is nonsingular, F is a neighborhood of the origin 0, and $0 \notin F$.

For all $y \in F$ we have by multiplying U^T from left to (3.6)

$$\begin{aligned} & b_1 D y = \alpha'_{12}(Uy) b_2 y \\ \Rightarrow\ & b_1 \begin{bmatrix} \lambda_1 y_1 \\ \vdots \\ \lambda_r y_r \end{bmatrix} = \alpha'_{12}(Uy) b_2 \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix} \end{aligned}$$

which implies that

$$0 < \alpha'_{12}(x) = \alpha'_{12}(Uy) = \frac{b_1 \lambda_1}{b_2} = \cdots = \frac{b_1 \lambda_r}{b_2} =: \alpha'_{12}$$

is constant. Therefore we have from (3.6)

$$C_1 = \alpha'_{12} \frac{b_2}{b_1} C_2. \tag{3.7}$$

Let us plug (3.7) into (3.2).

$$\begin{aligned} & x^T C_2 x \left(\alpha'_{12} b_2 \mu_1 - \alpha_{12}(x) b_1 \mu_2 \right) \\ & + \left\{ (\alpha'_{12} - \alpha_{12}(x)) b_1 b_2 - \left(\alpha'_{12} b_2 \mu_1 - \alpha_{12}(x) b_1 \mu_2 \right)^T x \right\} C_2 x = 0. \end{aligned} \tag{3.8}$$

Multiplying (3.8) by x^T from left we obtain

$$\left\{ \alpha'_{12} - \alpha_{12}(x) \right\} b_1 b_2 x^T C_2 x = 0 \;\Rightarrow\; \alpha_{12}(x) = \alpha'_{12}. \tag{3.9}$$

If we substitute (3.9) into (3.8), we get

$$x^T C_2 x (b_2 \mu_1 - b_1 \mu_2) = x^T (b_2 \mu_1 - b_1 \mu_2) C_2 x. \tag{3.10}$$

Let us introduce $w := U^T (b_2 \mu_1 - b_1 \mu_2)$. Since $x = Uy$ we have

$$\begin{aligned} & (y^T y) w = (y^T w) y \\ \Rightarrow\ & \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix} = \frac{y_1 w_1 + \cdots + y_r w_r}{y_1^2 + \cdots + y_r^2} \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix} \end{aligned} \tag{3.11}$$

Since (3.11) holds for $y = \varepsilon[0, 1, \dots, 1]^T, \dots, y = \varepsilon[1, \dots, 1, 0]^T \in F$ for some small $\varepsilon > 0$, it follows that

$$w_1 = 0, \dots, w_r = 0 \;\Rightarrow w = 0 \;\Rightarrow\; \frac{\mu_1}{b_1} = \frac{\mu_2}{b_2}. \tag{3.12}$$

□

4 Application in Portfolio Optimization

In this section we look at a probabilistic constrained stochastic programming problem, where the probabilistic constraint is of type (1.2). We assume that T has independent, normally distributed rows and the factors in the product $\prod_{k=1}^{K} \mathbb{P}(T_k x \leq b_k)$ are uniformly quasi concave. The problem is special, but still can be applied, e.g., in portfolio optimization.

Consider n assets and K consecutive periods. Let us introduce the following notations: for $k = 1, ..., K$

$$\begin{aligned} T_k &: \text{random loss of the assets during the } k\text{-th period} \\ \mu_k = \mathbb{E}[T_k^T] &: \text{expected loss} \\ C_k = \mathbb{E}[(T_k^T - \mu_k)(T_k^T - \mu_k)^T] &: \text{covariance matrix of } T_k \ . \end{aligned}$$

We assume that $T_k, k = 1, .., K$, are independent and normally distributed random vectors and $\mu_k \leq 0,\ k = 1, ..., K$. We also assume that the time window of the K periods is relatively short and a linear trend for the expectations prevails. Formally, our assumptions are:

$$\mu_1 = \mu \quad \text{and} \quad \mu_{k+1} = \alpha\mu_k, \quad k = 1, \dots, K-1 \tag{4.1}$$

$$C_1 = C \quad \text{and} \quad C_{k+1} = \alpha^2 C_k, \quad k = 1, \dots, K-1. \tag{4.2}$$

For the first period, we consider the portfolio optimization problem formulated by Kataoka [7]:

$$\begin{aligned} \text{(Problem 1):} \quad & \text{minimize} \quad b \\ & \text{subject to } \Phi\left(\frac{b - \mu^T x}{\sqrt{x^T C x}}\right) \geq p \\ & \qquad \sum_{j=1}^{n} x_j = 1 \\ & \qquad x_j \geq 0 \quad \text{for } j = 1, \dots, n. \end{aligned}$$

For the k-th period ($k \in \{2, \dots, K\}$), we consider the following problem.

$$\begin{aligned} \text{(Problem } k\text{):} \quad & \text{minimize} \quad b_1 \\ & \text{subject to } \prod_{i=1}^{k} \Phi\left(\frac{b_i - \mu_i^T x}{\sqrt{x^T C_i x}}\right) \geq p \\ & \qquad \sum_{j=1}^{n} x_j = 1 \\ & \qquad b_{i+1} = \alpha\, b_i \quad \text{for } i = 1, \dots, k-1 \\ & \qquad x_j \geq 0 \quad \text{for } j = 1, \dots, n \\ & \qquad b_1 \geq 0 \ . \end{aligned}$$

A related model is presented in [17], where individual probabilistic constraints are taken for more than one part of the distribution.

By Theorem 3.2 the functions $h_1(x), \ldots, h_K(x)$ defined by (2.12) are uniformly quasi-concave on the convex set $E := \cap_{k=1}^{K}\{x \mid h_k(x) \geq 1/2\} = \cap_{k=1}^{K}\{x \mid b_k \geq \mu_k^T x\} = \{x \mid b_K \geq \mu_K^T x\}$, and hence $h(x) = \prod_{k=1}^{K} h_k(x)$ is quasi-concave on E. Since set $\{x \mid h(x) \geq p,\ x \in E\}$ is convex, the set of feasible solutions of (Problem k) is convex.

Below we present a numerical example for the application of the above model. We take the initial expectations and covariance matrix from past history data but then proceed to obtain those values in accordance with the assumption formulated in the model.

Numerical Example.

Assets "Dow, S&P500, Nasdaq, NYSECI, 10YrBond" are obtained from Yahoo! Finance (`http://finance.yahoo.com`) and assets "Oil, Gold, Silver, EUR/USD" are obtained from Dukascopy (`http://www.dukascopy.com`).

We consider the expected values and the covariance matrix of the daily losses of the nine assets in May 2009. The data is shown in Table 1 and Table 2.

Gold	Silver	Nasdaq	S&P500	Oil	EUR/USD	10YrBond	Dow	NYSECI
-1.253	-3.008	-0.149	-0.711	-1.379	-0.82	-1.052	-0.546	-1.069

Table 1: Expected losses in May 2009

	Gold	Silver	Nasdaq	S&P500	Oil	EUR/USD	10YrBond	Dow	NYSECI
Gold	5.159	7.228	-1.437	1.492	3.989	2.764	-5.25	1.198	2.231
Silver	7.228	19.441	-0.785	6.454	10.143	4.94	-5.198	5.343	9.061
Nasdaq	-1.437	-0.785	15.084	11.202	1.562	0.974	-0.767	9.754	12.424
S&P500	1.492	6.454	11.202	16.238	10.709	4.223	-4.735	14.794	20.058
Oil	3.989	10.143	1.562	10.709	21.249	4.087	-5.719	10.043	15.451
EUR/USD	2.764	4.94	0.974	4.223	4.087	4.375	-3.255	3.764	5.996
10YrBond	-5.25	-5.198	-0.767	-4.735	-5.719	-3.255	38.003	-4.564	-4.928
Dow	1.198	5.343	9.754	14.794	10.043	3.764	-4.564	13.981	18.446
NYSECI	2.231	9.061	12.424	20.058	15.451	5.996	-4.928	18.446	25.706

Table 2: Covariance Matrix in May 2009

We assume that in the consecutive periods the expected returns are increased by $\alpha = 1.01$ (1%) and the covariance matrix is increased by $\alpha^2 = (1.01)^2$. The values of the nine assets obtained by the use of (Problem k), $k = 1, ..., 5$ are given in Table 3.

	Gold	Silver	Nasdaq	S&P500	Oil	EUR/USD	10YrBond	Dow	NYSECI
(Problem 1)	0.5246	0.0422	0.1267	0	0.0102	0.1433	0.1531	0	0
(Problem 2)	0.5350	0	0.1342	0	0.0103	0.1718	0.1487	0	0
(Problem 3)	0.5266	0	0.1379	0	0.0076	0.1804	0.1475	0	0
(Problem 4)	0.5218	0	0.1399	0	0.0063	0.1849	0.1471	0	0
(Problem 5)	0.5188	0	0.1414	0	0.0053	0.1878	0.1467	0	0

Table 3: Values of nine assets, May 2009

References

[1] R. E. Bellman, Introduction to matrix analysis, Society for Industrial Mathematics (1987) 58–59.

[2] P. Beraldi and A. Ruszczyński, A Branch and Bound Method for Stochastic Integer Problems under Probabilistic Constraints, Optimization Methods and Software, 17, 3 (2002) 359–382.

[3] I. Deák, Solving stochastic programming problems by successive regression approximations – numerical results, Dynamic stochastic optimization, Lecture notes in economics and mathematical systems, Springer 2003, 209–224.

[4] D. Dentcheva and A. Prékopa and A. Ruszczyński, Concavity and efficient points of discrete distributions in probabilistic programming, Mathematical Programming, 89, 1 (2000) 55–77.

[5] H. Henrion and C. Strugarek, Convexity of chance constraints with independent random variables, Computational Optimization and Applications, 41, 2, (2008) 263–276.

[6] P. Kall and J. Mayer, Stochastic Linear Programming: Models, Theory, and Computation, Springer 2005.

[7] S. Kataoka, A Stochastic Programming Model, Econometrica 31 (1963), 181–196.

[8] J. Luedtke and S. Ahmed and G. L. Nemhauser, An integer programming approach for linear programs with probabilistic constraints, Mathematical Programming, 122, 2 (2010) 247–272.

[9] C. van de Panne, W. Popp, Minimum Cost Cattle Feed under probabilistic protein constraints, Management Sciences, 9, 3 (1963) 405–430.

[10] A. Prékopa, Programming under Probabilistic Constraints with a Random Technology Matrix, Mathematische Operationsforschung und Statistik, Ser. Optimization 5 (1974), 109–116.

[11] A. Prékopa, Dual method for the solution of a one-stage stochastic programming problem with random RHS obeying a discrete probability distribution, ZOR - Methods and Models of Operations Research, 34 (1990) 441–461.

[12] A. Prékopa, Stochastic Programming, Kluwer Academic Publishers, Dordtecht, Boston, 1995.

[13] A. Prékopa and B. Vizvári and T. Badics, Programming under probabilistic constraint wit discrete random variable, New trends in mathematical programming, F. Giannessi and T. Rapcsák and S. Komlósi, Kluwer Academic Publishers, (1998) 235–255.

[14] A. Prékopa, Probabilistic Programming, Stochastic Programming, Handbooks in Operations Research and Management Science, A. Ruszczyński and A. Shapiro, Elsevier, 10 (2003) 207–351.

[15] T. Szántai, A computer code for solution of probabilistic constrained stochastic programming problems, Numerical Techniques for Stochastic Optimization, Y. Ermoliev and R. J.-B. Wets, Springer, New York (1988) 229–235.

[16] S. Uryasev, Probabilistic Constrained Optimization: Methodology and Applications, Kluwer Academic Publishers, Boston, 2000.

[17] K. Yoda and A. Prékopa, Optimal portfolio selection based on multiple Value at Risk constraints, to appear, RUTCOR research report, Rutgers Center for Operations Research, 2010.