

R U T C O R  
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METRIC AND ULTRAMETRIC SPACES  
OF RESISTANCES<sup>a</sup>

Vladimir Gurvich<sup>b</sup>

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RUTCOR  
Rutgers Center for  
Operations Research  
Rutgers University  
640 Bartholomew Road  
Piscataway, New Jersey  
08854-8003  
Telephone: 732-445-3804  
Telefax: 732-445-5472  
Email: rrr@rutcor.rutgers.edu  
<http://rutcor.rutgers.edu/~rrr>

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<sup>b</sup>RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ 08854-8003; email: [gurvich@rutcor.rutgers.edu](mailto:gurvich@rutcor.rutgers.edu)

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METRIC AND ULTRAMETRIC SPACES OF  
RESISTANCES<sup>1</sup>

Vladimir Gurvich

**Abstract.** Given an electrical circuit each edge  $e$  of which is an isotropic conductor with a monomial conductivity function  $y_e^* = y_e^r/\mu_e^s$ . In this formula,  $y_e$  is the potential difference and  $y_e^*$  current in  $e$ , while  $\mu_e$  is the resistance of  $e$ ; furthermore,  $r$  and  $s$  are two strictly positive real parameters common for all edges. In particular, the case  $r = s = 1$  corresponds to the standard Ohm law.

In 1987, Gvishiani and Gurvich [Russ. Math. Surveys, 42:6(258) (1987) 235–236] proved that, for every two nodes  $a, b$  of the circuit, the effective resistance  $\mu_{a,b}$  is well-defined and for every three nodes  $a, b, c$  the inequality  $\mu_{a,b}^{s/r} \leq \mu_{a,c}^{s/r} + \mu_{c,b}^{s/r}$  holds. It obviously implies the standard triangle inequality  $\mu_{a,b} \leq \mu_{a,c} + \mu_{c,b}$  when  $s \geq r$ . For the case  $s = r = 1$ , these results were rediscovered in 1990s. Now, in 23 years, I venture to reproduce the proof of the original result for the following reasons:

- It is more general than just the case  $r = s = 1$  and one can get several interesting metric and ultrametric spaces playing with parameters  $r$  and  $s$ .

In particular, (i) the effective Ohm resistance, (ii) the length of a shortest path, (iii) the inverse width of a bottleneck path, and (iv) the inverse capacity (maximum flow per unit time) between any pair of terminals  $a$  and  $b$  provide four examples of the resistance distances  $\mu_{a,b}$  that can be obtained from the above model by the following limit transitions: (i)  $r(t) = s(t) \equiv 1$ ,

(ii)  $r(t) = s(t) \rightarrow \infty$ , (iii)  $r(t) \equiv 1, s(t) \rightarrow \infty$ , and (iv)  $r(t) \rightarrow 0, s(t) \equiv 1$ ,

as  $t \rightarrow \infty$ . In all four cases the limits  $\mu_{a,b} = \lim_{t \rightarrow \infty} \mu_{a,b}(t)$  exist for all pairs  $a, b$  and the metric inequality  $\mu_{a,b} \leq \mu_{a,c} + \mu_{c,b}$  holds for all triplets  $a, b, c$ , since  $s(t) \geq r(t)$  for any sufficiently large  $t$ . Moreover, the stronger ultrametric inequality  $\mu_{a,b} \leq \max(\mu_{a,c}, \mu_{c,b})$  holds for all triplets  $a, b, c$  in examples (iii) and (iv), since in these two cases  $s(t)/r(t) \rightarrow \infty$ , as  $t \rightarrow \infty$ .

- Communications of the Moscow Math. Soc. in Russ. Math. Surveys were (and still are) strictly limited to two pages; the present paper is much more detailed. Although translation in English of the Russ. Math. Surveys is available, it is not free in the web and not that easy to find out.
- The last but not least: priority.

**Key words:** distance, metric, ultrametric; potential, voltage, current; Ohm law, Joule-Lenz heat, Maxwell principle; maximum flow, shortest path, bottleneck path

# 1 Introduction

We consider an electrical circuit modeled by a (non-directed) *connected* graph  $G = (V, E)$  in which each edge  $e \in E$  is an isotropic conductor with the monomial conductivity law  $y_e^* = y_e^r / \mu_e^s$ . Here  $y_e$  is the voltage, or potential difference,  $y_e^*$  current, and  $\mu_e$  is the resistance of  $e$ , while  $r$  and  $s$  are two strictly positive real parameters independent of  $e \in E$ .

In particular, the case  $r = 1$  corresponds to Ohm's law, while  $r = 0.5$  is the standard square law of resistance typical for hydraulics or gas dynamics. Parameter  $s$ , in contrast to  $r$ , is redundant; yet, it will play an important role too.

Given a circuit  $G = (V, E)$ , let us fix two arbitrary nodes  $a, b \in V$ . It will be shown (Proposition 1) that the obtained two-pole circuit  $(G, a, b)$  satisfies the same monomial conductivity law  $y_{a,b}^* = y_{a,b}^r / \mu_{a,b}^s$ , where  $y_{a,b}^*$  is the total current and  $y_{a,b}$  voltage between  $a$  and  $b$ , while  $\mu_{a,b}$  is the *effective resistance* of  $(G, a, b)$ . In other words,  $(G, a, b)$  can be effectively replaced by a single edge  $e = (a, b)$  of resistance  $\mu_{a,b}$  with the same  $r$  and  $s$ . Obviously,  $\mu_{a,b} = \mu_{b,a}$ , due to symmetry (isotropy) of the conductivity functions; it is also clear that  $\mu_{a,b} > 0$  whenever  $a \neq b$ ; finally, by convention, we set  $\mu_{a,b} = 0$  for  $a = b$ .

In [4], it was shown that for arbitrary three nodes  $a, b, c$  the following inequality holds.

$$\mu_{a,b}^{s/r} \leq \mu_{a,c}^{s/r} + \mu_{c,b}^{s/r} \quad (1)$$

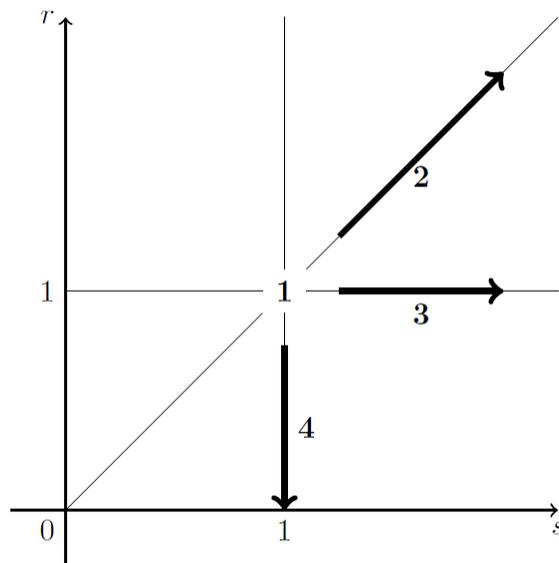
In [6], it was also shown that the equality in (1) holds if and only if node  $c$  belongs to every path between  $a$  and  $b$ . Clearly, if  $s \geq r$  then (1) implies the standard triangle inequality

$$\mu_{a,b} \leq \mu_{a,c} + \mu_{c,b} \quad (2)$$

Thus, a circuit can be viewed as a metric space in which the distance between any two nodes  $a$  and  $b$  is the effective resistance  $\mu_{a,b}$ . Playing with parameters  $r$  and  $s$ , one can get several interesting examples. Let  $r = r(t)$  and  $s = s(t)$  depend on a real parameter  $t$ ; in other words, these two functions define a curve in the positive quadrant  $r \geq 0, s \geq 0$ .

We will show that for the next four limit transitions, as  $t \rightarrow \infty$ , for all pairs of poles  $a, b \in V$ , the limits  $\mu_{a,b} = \lim_{t \rightarrow \infty} \mu_{a,b}(t)$  exist and can be interpreted as follows:

- (i) The effective Ohm resistance between poles  $a$  and  $b$ , when  $s(t) = r(t) \equiv 1$ , or more generally, whenever  $s(t) \rightarrow 1$  and  $r(t) \rightarrow 1$ .
- (ii) The standard length (travel time or cost) of a shortest route between terminals  $a$  and  $b$ , when  $s(t) = r(t) \rightarrow \infty$ , or more generally,  $s(t) \rightarrow \infty$  and  $s(t)/r(t) \rightarrow 1$ .
- (iii) The inverse width of a bottleneck path between terminals  $a$  and  $b$  when  $s(t) \rightarrow \infty$  and  $r(t) \equiv 1$ , or more generally,  $r(t) \leq \text{const}$ , or even more generally  $s(t)/r(t) \rightarrow \infty$ .
- (iv) The inverse capacity (maximum flow per unit time) between terminals  $a$  and  $b$ , when  $s(t) \equiv 1$  and  $r(t) \rightarrow 0$ ; or more generally, when  $s(t) \rightarrow 1$ , while  $r(t) \rightarrow 0$ .

Figure 1: Four types of limit transitions for  $s$  and  $r$ 

Obviously, all four examples define metric spaces, since in all cases  $s(t) \geq r(t)$  for any sufficiently large  $t$ . Moreover, for the last two examples the ultrametric inequality

$$\mu_{a,b} \leq \max(\mu_{a,c}, \mu_{c,b}) \quad (3)$$

holds for any nodes  $a, b, c$ , because  $s(t)/r(t) \rightarrow \infty$ , as  $t \rightarrow \infty$ , in the cases (iii) and (iv).

These examples allow us to interpret  $s$  and  $r$  as parameters of a transportation problem.

In particular,  $s$  can be viewed as a measure of divisibility of a transported material;  $s(t) \rightarrow 1$  in examples (i) and (iv), because liquid, gas, or electrical charge are fully divisible; in contrast,  $s(t) \rightarrow \infty$  for (ii) and (iii), because a car, ship, or individual transported from  $a$  to  $b$  are indivisible. Furthermore, the ratio  $s/r$  can be viewed as a measure of subadditivity of the transportation cost; so  $s(t)/r(t) \rightarrow 1$  in examples (i) and (ii), because in these cases the cost of transportation along a path is additive, i.e., is the sum of the costs of the edges that form this path; in contrast,  $s(t)/r(t) \rightarrow \infty$  for (iii) and (iv), because in these cases only edges of the maximum cost ("the width of a bottleneck") matter. Other values of parameters  $s$  and  $s/r$ , between 1 and  $\infty$ , correspond to an intermediate divisibility of the transported material and subadditivity of the transportation cost, respectively.

We conjecture that the limits  $s = \lim_{t \rightarrow \infty} s(t)$  and  $p = \lim_{t \rightarrow \infty} s(t)/r(t)$ , when exist, fully define the model, that is, then the limits  $\mu_{a,b} = \lim_{t \rightarrow \infty} \mu_{a,b}(t)$  also exist for all  $a, b \in V$  and depend only on  $s$  and  $p$ . This conjecture holds when  $s$  and  $p$  are strictly positive and finite,  $0 < s < \infty$  and  $0 < p < \infty$ , as, for example, in case (i). Furthermore, we will show that it holds for examples (ii, iii, iv), too, and also for the series-parallel circuits.

**Remark 1** *The above approach can be developed not only for the circuits but for continuum as well; inequality (1) and its corollaries still hold. However, this should be the subject of a separate research.*

For the case  $s = r = 1$ , inequality (1) was rediscovered in 1993 by Klein and Randić [10]. Then, several interesting related results were obtained in [1, 9, 11, 13, 16, 18] and surveyed in [3, 19, 20]. In this paper, we reproduce the original proof of (1) and several its corollaries, for the reasons listed in the Abstract.

Recently, these results were presented as a sequence of problems and exercises for high-school students in the Russian journal "Matematicheskoe Prosvetshenie" ("Mathematical Education") [7]. Here, these problems and exercises are given with solutions and in English.

## 2 Two-pole circuits and their effective resistances

### 2.1 Conductivity law

Let  $e$  be an electrical conductor with the monomial conductivity law

$$y_e^* = f_e(y_e) = \lambda_e^s |y_e|^r \text{sign}(y_e) = \frac{|y_e|^r}{\mu_e^s} \text{sign}(y_e), \quad (4)$$

where  $y_e$  is the *voltage or potential difference*,  $y_e^*$  *current*  $\lambda_e$  *conductance* and  $\mu_e = \lambda_e^{-1}$  *resistance* of  $e$ ; furthermore,  $r$  and  $s$  are two strictly positive real parameters. Obviously, the monomial function  $f_e$  is

- continuous, strictly monotone increasing, and taking all real values;
- symmetric (odd or isotropic), that is,  $f_e(-y_e) = -f_e(y_e)$ ;
- the inverse function  $f_e^{-1}$  is also monomial with parameters  $r' = r^{-1}$  and  $s' = s^{-1}$ .

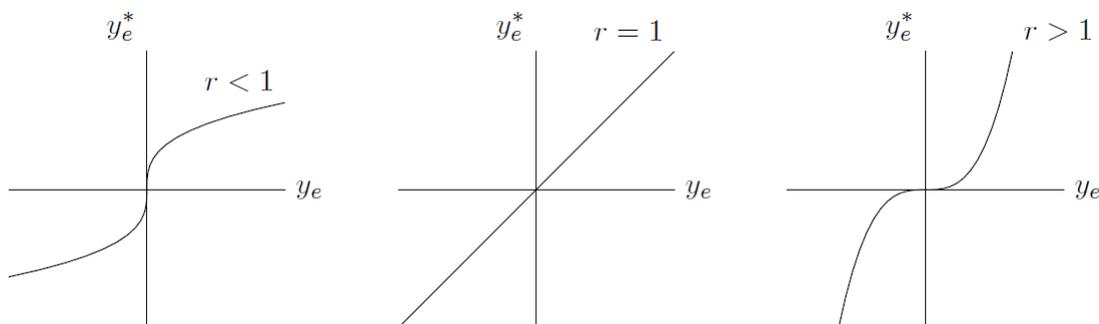


Figure 2: Monomial conductivity law.

## 2.2 Main variables and related equations

An electrical circuit is modeled by a *connected* weighted non-directed graph  $G = (V, E, \mu)$  in which weights of the edges are *positive* resistances  $\mu_e, e \in E$ . Let us introduce the following four groups of real variables; two for each node  $v \in V$  and edge  $e \in E$ :

*potential*  $x_v$ ; *difference of potentials, or voltage*  $y_e$ ; *current*  $y_e^*$ ; *sum of currents, or flux*  $x_v^*$ .

We say that the first Kirchhoff law holds for a node  $v$  whenever  $x_v^* = 0$ .

The above variables are not independent. By (4), the current  $y_e^*$  depends on voltage  $y_e$ . Furthermore, the voltage (respectively, flux) is a linear function of potentials (respectively, of currents). To define these linear functions, let us fix an arbitrary orientation of edges and introduce the node-edge incidence function:

$$\text{inc}(v, e) = \begin{cases} +1, & \text{if node } v \text{ is the beginning of } e; \\ -1, & \text{if node } v \text{ is the end of } e; \\ 0, & \text{in every other case.} \end{cases} \quad (5)$$

We shall assume that the next two systems of linear equations always hold:

$$y_e = \sum_{v \in V} \text{inc}(v, e) x_v; \quad (6)$$

$$x_v^* = \sum_{e \in E} \text{inc}(v, e) y_e^*. \quad (7)$$

Equation (6) for edge  $e = (v', v'')$  can be reduced to  $y_e = \text{inc}(e, v') x_{v'} + \text{inc}(e, v'') x_{v''}$  and further to  $y_e = x_{v'} - x_{v''}$ ; for the latter we should assume that  $e$  is directed from  $v'$  to  $v''$ .

Let us introduce four vectors, one for each group of variables:

$$x = (x_v \mid v \in V), \quad x^* = (x_v^* \mid v \in V), \quad y = (y_e \mid e \in E), \quad y^* = (y_e^* \mid e \in E), \quad x, x^* \in \mathbb{R}^n; y, y^* \in \mathbb{R}^m,$$

where  $n = |V|$  and  $m = |E|$  are the numbers of nodes and edges of the graph  $G = (V, E)$ . Let  $A = A_G$  be the edge-node  $m \times n$  incidence matrix of graph  $G$ , that is,  $A(v, e) = \text{inc}(v, e)$  for all  $v \in V$  and  $e \in E$ . Equations (6) and (7) can be rewritten in this matrix notation as  $y = Ax$  and  $x^* = A^T y^*$ , respectively.

It is both obvious and well known that these two equations imply the identity

$$(x, x^*) = \sum_{v \in V} x_v x_v^* = \sum_{e \in E} y_e y_e^* = (y, y^*).$$

Let us also recall that vectors  $y$  and  $y^*$  uniquely define each other, by (4). Thus, given  $x$ , the remaining three vectors  $y$ ,  $y^*$ , and  $x^*$  are uniquely defined by (6), (4), and (7).

**Lemma 1** *For a positive constant  $c$ , two quadruples  $(x, y, y^*, x^*)$  and  $(cx, cy, c^r y^*, c^r x^*)$  can satisfy all equations of (6), (7), and (4) only simultaneously.*

**Proof** is straightforward. □

## 2.3 Existence and uniqueness of a solution

Let us fix two distinct nodes  $a, b \in V$  and call them the *poles*; then, fix potentials

$$x_a = x_a^0, \quad x_b = x_b^0, \quad (8)$$

in both poles and add to equations (6), (7), (4), and (8) also the first Kirchhoff law

$$x_v^* = 0, \quad \text{for all } v \in V \setminus \{a, b\}. \quad (9)$$

**Lemma 2** *The obtained system of equations (4)- (9) has a unique solution.*

Respectively, we will say that the corresponding unique potential vector  $x = x(G, a, b)$  solves the circuit  $(G, a, b)$  for  $x_a = x_a^0$  and  $x_b = x_b^0$ .

**Proof of existence.** Given  $x_a^0$  and  $x_b^0$ , let us assume without any loss of generality that  $x_a^0 \geq x_b^0$  and apply the method of successive approximations to compute  $x_v$  for all remaining nodes  $v \in V \setminus \{a, b\}$ . To do so, let us order these nodes and initialize  $x_v = x_a^0$  for all  $v \in V \setminus \{b\}$ . Then, obviously,

$$x_v^* \geq 0 \quad \text{for all } v \in V \setminus \{b\}. \quad (10)$$

Moreover, the inequality is strict whenever  $v$  is adjacent to  $b$  and  $x_a^0 > x_b^0$ . In this case, there is a unique potential  $x'_v$  such that the corresponding flux  $x'_v$  becomes equal to 0 after we replace  $x_v$  with  $x'_v$  leaving all other potentials unchanged. Finally, it is clear that (10) still holds and, moreover,

$$x_a^0 \geq x_v \geq x'_v \geq x_b^0 \quad \text{for all } v \in V. \quad (11)$$

We shall consider the nodes of  $V \setminus \{a, b\}$  one by one in the defined (cyclical) order and apply in turn the above transformation to each node. Obviously, equations (10) and (11) hold all time. In particular,  $x_a \equiv x_a^0$ ,  $x_b \equiv x_b^0$ , and  $x_v$ , for each  $v \in V \setminus \{a, b\}$ , is a monotone non-increasing sequence bounded by  $x_b^0$  from below. Hence, it has a limit  $x_v^0 \in [x_a^0, x_b^0]$ .

As we know, values of potentials uniquely define values of all other variables. Let us show that the limit values obtained above satisfy all equations (4)-(9).

To do so, we shall watch  $x_v^*$  for all  $v \in V$ . First, let us notice that  $x_a^*$  is non-negative and monotone non-decreasing, while  $x_b^*$  is non-positive, and monotone non-increasing.

[Moreover, the voltage  $y_e$  and current  $y_e^*$  are non-negative and monotone non-decreasing for each  $e = (a, v)$  and non-positive and monotone non-increasing for each  $e = (v, b)$ .]

Then,  $x_v^* \geq 0$  all time for all  $v \in V \setminus \{b\}$ . Yet, the value of  $x_v^*$  is not monotone in time: it becomes zero when we treat  $v$  and then it monotone increases, while we treat other nodes of  $V \setminus \{a, b\}$ . Finally,  $\sum_{v \in V} x_v^* = 0$  all time, by the conservation of electric charge. If the first Kirchhoff law holds, that is,  $x_v^* = 0$  for all  $V \setminus \{a, b\}$ , then  $x_a^* + x_b^* = 0$ , all equations are satisfied, and we stop. Yet otherwise, we obviously can proceed with the potential reduction. Thus, the limit values of  $x$  solve  $(G, a, b)$  for  $x_a = x_a^0$  and  $x_b = x_b^0$ .  $\square$

**Remark 2** *A very similar monotone potential reduction, or pumping, algorithm for stochastic games with perfect information was recently suggested in [2].*

**Remark 3** *The connectivity of  $G$  is an essential assumption. Indeed, let us assume that  $G$  is not connected. If  $a$  and  $b$  are in one connected component then, obviously, all potentials of any other component must be equal. Yet, the corresponding constants might be arbitrary. If  $a$  and  $b$  are in two distinct connected components then, obviously, all potentials in these two components must be equal to  $x_a^0$  and  $x_b^0$ , respectively, and to an arbitrary constant for another component, if any. Clearly, in this case  $x_v^* = 0$  for all  $v \in V$ .*

Let us note also that the above successive approximation method does not prove the uniqueness of a solution. For example, it is not clear why the limit potential values do not depend on the cyclic order of nodes fixed above. Moreover, even if they do not, it is still not clear whether one can get another solution by a different method. Unfortunately, we have no elementary proof for uniqueness.

Of course, both existence and uniqueness are well known; see, for example, [15, 17, 5, 6]. For example, uniqueness results from the following famous Maxwell principle of the minimum dissipation of energy: the potential vector  $x$  that solves the two-pole circuit  $(G, a, b)$  must minimize the generalized Joule-Lenz heat

$$F(y) = \sum_{e \in E} F_e(y_e) = \sum_{e \in E} \int f_e(y_e) dy_e, \quad (12)$$

where,  $x_a = x_a^0$ ,  $x_b = x_b^0$ , by (8),  $y = A_G x$ , by (6), and  $f_e$  is the conductivity function of edge  $e$ . Obviously,  $F_e$  is (strictly) convex if and only if  $f_e$  is (strictly) monotone increasing. In particular, strict monotonicity and convexity hold when  $f_e$  is defined by (4). In this case

$$F_e(y_e) = \int f_e(y_e) dy_e = \frac{|y_e|^{r+1}}{(r+1)\mu_e^s}. \quad (13)$$

Let us notice that (13) turns into the standard Joule-Lenz formula when  $r = s = 1$ .

Clearly,  $F(A_G x)$  is a strictly convex function of  $x$ , since  $r > 0$ . It remains to recall from calculus that if a strictly convex function reaches a minimum then - in a unique vector.  $\square$

## 2.4 Effective resistances

The difference  $y_{a,b} = x_a - x_b$  is called the *voltage* (or potential difference) and the value  $y_{a,b}^* = x_a^* - x_b^*$  is called the *current in the two-pole circuit*  $(G, a, b)$ . Lemmas 1 and 2 immediately imply the next statement.

**Proposition 1** *The current  $y_{a,b}^*$  and voltage  $y_{a,b}$  are still related by a monomial conductivity law with the same parameters  $r$  and  $s$ :*

$$y_{a,b}^* = f_{a,b}(y_{a,b}) = \lambda_{a,b}^s |y_{a,b}|^r \text{sign}(y_{a,b}) = \frac{|y_{a,b}|^r}{|\mu_{a,b}|^s} \text{sign}(y_{a,b}). \quad (14)$$

□

The values  $\lambda_{a,b}$  and  $\mu_{a,b} = \lambda_{a,b}^{-1}$  are called respectively the (effective) conductance and resistance of the two-pole circuit  $(G, a, b)$ .

**Remark 4** *We restricted ourselves by the monomial conductivity law (4), because Proposition 1 cannot be extended to any other family of continuous monotone non-decreasing functions, as it was shown in [6].*

**Remark 5** *Again, the connectivity of  $G$  is an essential assumption. Indeed, if graph  $G$  is not connected and poles  $a$  and  $b$  belong to distinct connected components then  $y_{a,b}^* \equiv 0$ .*

## 2.5 On a monotone property of effective resistances

Given a two-pole circuit  $(G, a, b)$ , where  $G = (V, E, \mu)$ , let us fix an edge  $e_0 \in E$ , replace the resistance  $\mu_{e_0}$  by a larger one  $\mu'_{e_0} \geq \mu_{e_0}$ , and denote by  $G' = (V, E, \mu')$  the obtained circuit.

Of course, the total resistance will not decrease either, that is,  $\mu'_{a,b} \geq \mu_{a,b}$ . Yet, how to prove this "intuitively obvious" statement? Somewhat surprisingly, according to [12], the simplest way is to apply again the Maxwell principle of the minimum energy dissipation. Let  $x$  and  $x'$  be unique potential vectors that solve  $(G, a, b)$  and  $(G', a, b)$ , respectively, while  $y$  and  $y'$  be the corresponding voltage vectors defined by (6). Let us consider  $G'$  and vector  $x$ , instead of  $x'$ . Since  $\mu_{e_0} \leq \mu'_{e_0}$ , inequality  $F'_e(y_{e_0}) \leq F_e(y_{e_0})$  is implied by (13). Furthermore,  $F'_e(y_e) = F_e(y_e)$  for all other  $e \in E$  and, hence,  $F'(y) \leq F(y)$ . In addition,  $F'(y') \leq F'(y)$ , by the Maxwell principle. Thus,  $F'(y') \leq F(y)$  and, by (13),  $\mu'_{a,b} \geq \mu_{a,b}$ .

## 2.6 Potential reduction along paths

Let us say that a node  $v$  is *between*  $a$  and  $b$  if  $v \neq a, v \neq b$ , and  $v$  belongs to a simple (without self-intersections) path between  $a$  and  $b$ . Then, Lemma 2 can be extended as follows.

**Lemma 3** (o) *If  $x_a^0 = x_b^0$  then  $x_v^0 = x_a^0 = x_b^0$  for all  $v \in V$ ;*

*Otherwise, let us assume without any loss of generality that  $x_a^0 > x_b^0$ . Then*

- (i) *Inequalities  $x_a^0 \geq x_v^0 \geq x_b^0$  holds for all  $v \in V$ ;*
- (i') *If  $v$  is between  $a$  and  $b$  then  $x_a^0 > x_v^0 > x_b^0$ .*
- (ii) *The voltage  $y_e$  and current  $y_e^*$  are non-negative whenever  $e = (a, v)$  or  $e = (v, b)$ .*
- (ii') *Moreover, they are strictly positive if also  $v$  is between  $a$  and  $b$ .*

**Proof** Claim (i), (ii), and (o) result immediately from Lemma 2, yet, connectivity is essential. In fact, the same is true for (i') and (ii'). Indeed, let us recall the successive approximations, which were instrumental in the proof of Lemma 2, then, fix a simple path between  $a$  and  $b$  and any node  $v$  in it, distinct from  $a$  and  $b$ . Obviously, potential  $x_v$  will be strictly reduced from its original value  $x_a$  but it will never reach  $x_b$ .  $\square$

**Remark 6** *If  $v$  is not between  $a$  and  $b$  then inequalities in the above Lemma might be still strict, yet, they might be not strict, too.*

### 3 Proof of the main inequality and related claims

**Theorem 1** *Given an electrical circuit, that is, a connected graph  $G = (V, E, \mu)$  with strictly positive weights-resistances ( $\mu_e | e \in E$ ), three arbitrary nodes  $a, b, c \in V$ , and strictly positive real parameters  $r$  and  $s$ , then inequality (1) holds:  $\mu_{a,b}^{s/r} \leq \mu_{a,c}^{s/r} + \mu_{c,b}^{s/r}$ . It holds with equality if and only if node  $c$  belongs to every path between  $a$  and  $b$  in  $G$ .*

**Remark 7** *The proof of the first statement was sketched in [4]; see also [7]. Both claims were proven in [6]. Here we shall follow the plan suggested in [4] but give more details.*

**Proof** Let us fix arbitrary potentials  $x_a^0$  and  $x_b^0$  in nodes  $a$  and  $b$ . Then, by Proposition 1, all variables, and in particular all remaining potentials, are uniquely defined by equations (4)-(8). Let  $x_c^0$  denote the potential in  $c$ . Without any loss of generality, let us assume that  $x_a^0 \geq x_b^0$ . Then,  $x_a^0 \geq x_c^0 \geq x_b^0$ , by Lemma 2. Let us consider the two-pole circuit  $(G, a, c)$  and fix in it  $x_a = x_a^0$  and  $x_c = x_c^0$ .

**Lemma 4** *The currents in the circuits  $(G, a, b)$  and  $(G, a, c)$  satisfy inequality  $y_{a,b}^* \geq y_{a,c}^*$ . Moreover, the equality holds if and only if  $c$  belongs to every path between  $a$  and  $b$ .*

**Proof** As in the proof of Lemma 2, we will apply successive approximations to compute a (unique) potential vector  $\bar{x} = x(G, a, c)$  that solves the circuit  $(G, a, c)$  for  $\bar{x}_a = x_a^0$  and  $\bar{x}_c = x_c^0$ . Yet, as an initial approximation, we shall now take the unique potential vector  $x = x(G, a, b)$  that solves the circuit  $(G, a, b)$  for  $x_a = x_a^0$  and  $x_b = x_b^0$ . As we know,  $x$  uniquely defines all other variables, in particular,  $x^* = x^*(G, a, b)$ . Obviously, for  $x^*$  the first Kirchhoff law holds for all nodes of  $V \setminus \{a, b\}$ . Yet, for  $b$ , it does not hold:  $x_b^* < 0$ . Let us replace the current potential  $x_b$  by  $x'_b$  to get  $x'_b = 0$ . Obviously, there is a unique such  $x'_b$  and  $x'_b > x_b$ . Yet, after this, the value  $x_v^*$  will become negative for some  $v \in V \setminus \{a, c\}$ . Let us order the nodes of  $V \setminus \{a, c\}$  and repeat the same iterations as in the proof of Lemma 2. By the same arguments, we conclude that in each  $v \in V \setminus \{a, c\}$ , the potentials  $x_v$  form a monotone non-decreasing sequence that converges to a unique solution  $\bar{x}_v = x_v(G, a, c)$ . By construction, potentials  $\bar{x}_a = x_a^0$  and  $\bar{x}_c = x_c^0$  remain constant.

Thus, the value  $x_a^*$  is monotone non-increasing and the inequality  $y_{a,b}^* \geq y_{a,c}^*$  follows.

Let us show that it is strict whenever there is a path  $P$  between  $a$  and  $b$  that does not contain  $c$ . Without loss of generality, we can assume that path  $P$  is simple, that is, it has no self-intersections. Also without loss of generality, we can order  $V \setminus \{a, c\}$ , so that nodes of  $V(P) \setminus \{a\}$  go first in order from  $b$  towards  $a$ . Obviously, after the first  $|P|$  successive approximations, potentials will strictly increase in all nodes of  $P$ , except  $a$ . Thus, the value  $x_a^*$  will be strictly reduced. Let us remark, however, that the above arguments do not work when  $c$  belongs to  $P$ , since potential  $x_c = x_c^0$  cannot be changed.

Moreover, if  $c$  belongs to every path between  $a$  and  $b$  then clearly  $y_{a,b}^* = y_{a,c}^* = y_{c,b}^*$ .  $\square$

**Remark 8** *The same arguments prove that inequality  $y_{a,b}^* \geq y_{a,c}^*$  holds not only for monomial but for arbitrary monotone non-decreasing conductivity functions.*

Furthermore, by symmetry, we conclude that  $y_{a,b}^* \geq y_{c,b}^*$  holds, too, and obtain

$$y_{a,b}^* = \frac{(x_a^0 - x_b^0)^r}{\mu_{a,b}^s} \geq \frac{(x_a^0 - x_c^0)^r}{\mu_{a,c}^s} = y_{a,c}^*; \quad y_{a,b}^* = \frac{(x_a^0 - x_b^0)^r}{\mu_{a,b}^s} \geq \frac{(x_c^0 - x_b^0)^r}{\mu_{c,b}^s} = y_{c,b}^*, \quad (15)$$

which can be obviously rewritten as follows

$$\left(\frac{\mu_{a,c}}{\mu_{a,b}}\right)^{s/r} \geq \frac{x_a^0 - x_c^0}{x_a^0 - x_b^0}; \quad \left(\frac{\mu_{c,b}}{\mu_{a,b}}\right)^{s/r} \geq \frac{x_c^0 - x_b^0}{x_a^0 - x_b^0} \quad (16)$$

Summing up these two inequalities we obtain (1).

Obviously, (1) holds with equality if and only if  $y_{a,b}^* = y_{a,c}^* = y_{c,b}^*$ , which, by Lemma 4, happens if and only if  $c$  belongs to every path between  $a$  and  $b$ .  $\square$

**Remark 9** *As a corollary, we obtain that  $y_{a,b}^* = y_{a,c}^*$  if and only if  $y_{a,b}^* = y_{c,b}^*$ .*

Let us also note that  $\mu_{a,b} = \mu_{b,a}$  for all  $a, b \in V$ . This easily follows from the fact that conductivity functions  $f_e$  are odd for all  $e \in E$ . Furthermore, obviously,  $\mu_{a,b} > 0$  whenever nodes  $a$  and  $b$  are distinct. By definition, let us set  $\mu_{a,b} = 0$  whenever  $a = b$ . As we already mentioned, (1) obviously implies the triangle inequality (2) whenever  $s \geq r$ . Thus, in this case, the effective resistances form a metric space. In the next section we consider two examples in which (1) turns into the ultrametric inequality (3).

## 4 Examples and interpretations

### 4.1 Parallel and series connection of edges

Let us consider two simplest two-pole circuits given in Figure 3.

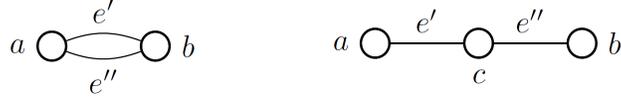


Figure 3: Parallel and series connection

**Lemma 5** *The resistances of these two circuits can be determined, respectively, from*

$$\mu_{a,b}^{-s} = (\mu_{e'}^{-s} + \mu_{e''}^{-s}) \quad \text{and} \quad \mu_{a,b}^{s/r} = (\mu_{e'}^{s/r} + \mu_{e''}^{s/r}). \quad (17)$$

**Proof** If  $r = s = 1$  then (17) turns into familiar high-school formulas. The general case is just a little more difficult. Without loss of generality let us assume that  $y_{a,b} = x_a - x_b \geq 0$ .

In case of the parallel connection we obtain the following chain of equalities.

$$y_{a,b}^* = f_{a,b}(y_{a,b}) = \frac{y_{a,b}^r}{\mu_{a,b}^s} = f_{e'}(y_{a,b}) + f_{e''}(y_{a,b}) = \frac{y_{e'}^r}{\mu_{e'}^s} + \frac{y_{e''}^r}{\mu_{e''}^s} = \frac{y_{a,b}^r}{\mu_{e'}^s} + \frac{y_{a,b}^r}{\mu_{e''}^s}.$$

Let us compare the third and the last terms to arrive at (17).

In case of series connection, let us start with determining  $x_c$  from the first Kirchhoff law:

$$y_{a,b}^* = f_{a,b}(y_{a,b}) = \frac{y_{a,b}^r}{\mu_{a,b}^s} = \frac{(x_a - x_b)^r}{\mu_{a,b}^s} =$$

$$y_{e'}^* = f_{e'}(y_{e'}) = f_{e'}(x_a - x_c) = \frac{(x_a - x_c)^r}{\mu_{e'}^s} = y_{e''}^* = f_{e''}(y_{e''}) = f_{e''}(x_c - x_b) = \frac{(x_c - x_b)^r}{\mu_{e''}^s}.$$

It is sufficient to compare the last and eighth terms to get

$$x_c = \frac{x_b \mu_{e'}^{s/r} + x_a \mu_{e''}^{s/r}}{\mu_{e'}^{s/r} + \mu_{e''}^{s/r}}.$$

Then, let compare the last and fourth terms, substitute the obtained  $x_c$ , and get (17).  $\square$

Now, let us consider the convolution  $\mu(t) = (\mu_{e'}^t + \mu_{e''}^t)^{1/t}$ ; obviously,

$$\mu(t) \rightarrow \max(\mu_{e'}, \mu_{e''}), \text{ as } t \rightarrow +\infty, \quad \text{and} \quad \mu(t) \rightarrow \min(\mu_{e'}, \mu_{e''}), \text{ as } t \rightarrow -\infty. \quad (18)$$

## 4.2 Main four examples of resistance distances

Let us fix a weighted non-directed connected graph  $G = (V, E, \mu)$  and two strictly positive real parameters  $r$  and  $s$ . As we proved, the obtained circuit can be viewed as a metric space in which the distance between any two nodes  $a, b \in V$  is defined as the effective resistance  $\mu_{a,b}$ . As announced in Introduction, this model results in several interesting examples of metric and ultrametric spaces. Yet, to arrive to them we should allow for  $r$  and  $s$  to take values 0 and  $+\infty$ . More accurately, let  $r = r(t)$  and  $s = s(t)$  depend on a real parameter  $t$ , or in other words, these two functions define a curve in the positive quadrant  $s \geq 0, r \geq 0$ .

By Proposition 1, the resistances  $\mu_{a,b}(t)$  are well-defined for every two nodes  $a, b \in V$  and each  $t$ . Moreover, we will show that, for the four limit transitions listed below, limits  $\mu_{a,b} = \lim_{t \rightarrow \infty} \mu_{a,b}(t)$ , exist for all  $a, b \in V$  and can be interpreted as follows:

### Example 1: the effective Ohm resistance of an electrical circuit.

Let a weighted graph  $G = (V, E, \mu)$  model an electrical circuit in which  $\mu_e$  is the resistance of edge  $e$  and  $r(t) = s(t) \equiv 1$ , or more generally,  $s(t) \rightarrow 1$  and  $r(t) \rightarrow 1$ . Then,  $\mu_{a,b}$  is the effective Ohm resistance between poles  $a$  and  $b$ . For parallel and series connection of two edges  $e'$  and  $e''$ , as in Figure 3, we obtain, respectively,  $\mu_{a,b}^{-1} = \mu_{e'}^{-1} + \mu_{e''}^{-1}$  and  $\mu_{a,b} = \mu_{e'} + \mu_{e''}$ , which is known from the high school.

### Example 2: the length of a shortest route.

Let a weighted graph  $G = (V, E, \mu)$  model a road network in which  $\mu_e$  is the length (milage, traveling time, or gas consumption) of a road  $e$ . Then,  $\mu_{a,b}$  can be viewed as the distance between terminals  $a$  and  $b$ , that is, the length of a shortest path between them. In this case, for parallel and series connection of  $e'$  and  $e''$ , we obtain, respectively,  $\mu_{a,b} = \min(\mu_{e'}, \mu_{e''})$  and  $\mu_{a,b} = \mu_{e'} + \mu_{e''}$ . Hence, by (18),  $-s(t) \rightarrow -\infty$  and  $s(t) \equiv r(t)$  for all  $t$ , as in Figure 1; or more generally,  $s(t) \rightarrow \infty$  and  $s(t)/r(t) \rightarrow 1$ , as  $t \rightarrow \infty$ .

### Example 3: the inverse width of a bottleneck route.

Now, let  $G = (V, E, \mu)$  model a system of passages (rivers, canals, bridges, etc.), where the conductance  $\lambda_e = \mu_e^{-1}$  is the "width" of a passage  $e$ , that is, the maximum size (or tonnage) of a ship or a car that can pass  $e$ , yet. Then, the effective conductance  $\lambda_{a,b} = \mu_{a,b}^{-1}$  is interpreted as the maximum width of a (bottleneck) path between  $a$  and  $b$ , that is, the maximum size (or tonnage) of a ship or a car that can still pass between terminals  $a$  and  $b$ . In this case,  $\lambda_{a,b} = \max(\lambda_{e'}, \lambda_{e''})$  for the parallel connection and  $\lambda_{a,b} = \min(\lambda_{e'}, \lambda_{e''})$  for the series connection. Hence,  $s(t) \rightarrow \infty$  and  $s(t)/r(t) \rightarrow \infty$ , as  $t \rightarrow \infty$ ; in particular,  $r$  might be bounded by a constant,  $r(t) \leq \text{const}$ , or just  $r(t) \equiv 1$  for all  $t$ , as in Figure 1.

### Example 4: the inverse value of a maximal flow.

Finally, let  $G = (V, E, \mu)$  model a pipeline or transportation network in which the conductance  $\lambda_e = \mu_e^{-1}$  is the capacity of a pipe or road  $e$ . Then,  $\lambda_{a,b} = \mu_{a,b}^{-1}$  is the capacity of the whole two-pole network  $(G, a, b)$  with terminals  $a$  and  $b$ . (Standardly, the capacity is defined as the amount of material that can be transported through  $e$ , or between  $a$  and  $b$ , per unit time.) In this case,  $\lambda_{a,b} = \lambda_{e'} + \lambda_{e''}$  for the parallel connection and  $\lambda_{a,b} = \min(\lambda_{e'}, \lambda_{e''})$  for the series connection. Hence,  $-s(t) \equiv -1$  and  $s(t)/r(t) \rightarrow \infty$ , that is,  $s(t) \equiv 1$  and  $r(t) \rightarrow 0$ , as in Figure 1, or more generally,  $s(t) \rightarrow 1$ , while  $r(t) \rightarrow 0$ , as  $t \rightarrow \infty$ .

### 4.3 Interpretation of values $s$ and $s/r$

As mentioned in Introduction, the above four examples can be viewed as transportation problems in which parameters  $s$  and  $s/r$  are interpreted as follows. Indeed, by the parallel and series connection of  $m$  edges, as in Figure 3 for  $m = 2$ , we obtain the convolutions (18), where  $t = -s$  and  $t = s/r$ , respectively.

Thus, parameter  $s$  can be viewed as a measure of divisibility of the transported material. Case  $s = 1$  corresponds to a fully divisible cargo, like gas, liquid, or electric charge in Examples 1 and 4, while  $s(t) \rightarrow \infty$  corresponds to an absolutely indivisible cargo, like an individual, ship, or car in Examples 2 and 3.

Respectively, the ratio  $s/r$  can be viewed as a measure of subadditivity of the transportation cost; so  $s/r = 1$  in examples 1 and 2, since in these cases the cost of transportation along a path is additive, i.e., is the sum of the costs for the edges that form this path; in contrast,  $s(t)/r(t) \rightarrow \infty$  for Examples 3 and 4, because in these cases only edges of the maximum cost ("the width of the bottleneck") matters.

Other values of  $s$  and  $s/r$ , between 1 and  $\infty$ , correspond to an intermediate divisibility of the transported material and subadditivity of the transportation cost, respectively.

### 4.4 Main result and conjecture

**Theorem 2** *In all four examples, the limits  $\mu_{a,b} = \lim_{t \rightarrow \infty} \mu_{a,b}(t)$  exist and equal the corresponding distances for all  $a, b \in V$ . In all four cases these distances form metric and the last two ultrametric spaces.*

**Proof** (sketch). The statement is obvious for Example 1 and it is also clear for the series-parallel circuits. Moreover, it holds in general too. Indeed, it is not difficult to demonstrate that for Examples 2 and 4 all currents tend to concentrate in, respectively, the shortest and bottleneck paths between  $a$  and  $b$ , as  $t \rightarrow \infty$ , that is,  $y_e^*(t) \rightarrow 0$  for every edge  $e$  that does not belong to such a path. Similarly, in Example 3, the limit currents  $y_e^*(t)$  tend to form a maximal flow between  $a$  and  $b$ , as  $t \rightarrow \infty$ . These arguments imply that limits  $\mu_{a,b} = \lim_{t \rightarrow \infty} \mu_{a,b}(t)$  exist and represent the corresponding distances.  $\square$

In general, given functions  $s(t)$  and  $r(t)$  such that the limits

$$s = \lim_{t \rightarrow \infty} s(t) \in [0, \infty] \quad \text{and} \quad p = \lim_{t \rightarrow \infty} s(t)/r(t) \in [0, \infty]$$

exist, we conjecture that limits  $\lim_{t \rightarrow \infty} \mu_{a,b}(t) = \mu_{a,b}$  also exist for all  $a, b \in V$ ; moreover, they depend only on  $s$  and  $p$ . In other words, given  $s(t), r(t)$  and  $s'(t), r'(t)$ , such that all four limits  $s, s', p, p'$  exist and  $p = p', s = s'$ , then the limits  $\mu_{a,b}$  and  $\mu'_{a,b}$  also exist and they are equal.

This conjecture is obvious for the parallel-series circuits and also in case when  $s$  and  $p$  take finite positive values,  $0 < s < \infty$  and  $0 < p < \infty$ , as in Example 1. By the previous Theorem, it holds for the Examples 2,3, and 4, as well.

## 5 $k$ -pole circuits with $r = s = 1$

By Proposition 1, in a two-pole circuit  $(G, a, b)$ , the total current  $y_{a,b}^*$  and voltage  $y_{a,b} = x_a - x_b$  are related by a (uniquely defined) conductivity function  $f_{a,b}$  with the same parameters  $r$  and  $s$  as in the functions  $f_e$  for each  $e \in E$ . In other words, every two-pole circuit  $(G, a, b)$  with parameters  $r$  and  $s$  can be effectively replaced by a single edge  $(a, b)$  with the same parameters.

**Remark 10** *In [15], Minty proved that the last claim holds not only for monomial but for arbitrary monotone conductivity laws, as well. More precisely, if  $f_e$  is non-decreasing for each edge  $e \in E$  then there is a (unique) non-decreasing conductivity function  $f_{a,b}$  such that the whole two-pole circuit  $(G, a, b)$  can be effectively replaced by the single edge  $(a, b)$ .*

In the case of standard electric resistances,  $r = s = 1$ , the above "effective replacement statement" can be extended from the two-pole circuits to the  $k$ -pole ones.

Given a weighted graph  $G = (V, E, \mu)$ , let us fix  $k \geq 2$  distinct poles  $A = \{a_1, \dots, a_k\} \subseteq V$  and add to equations (4), (6), (7) the first Kirchhoff law for all non-poles:

$$x_v^* = 0 \text{ for } v \in V \setminus A, \quad (19)$$

while in the  $k$  poles let us fix the potentials:

$$x_a = x_a^0 \text{ for } a \in A. \quad (20)$$

The above two equations in the two-pole case turn into (9) and (8), respectively.

**Lemma 6** *The obtained system of equations (4), (6), (7), (19), (20) has a unique solution.*

As in the two-pole case, we shall say that the corresponding (unique) potential vector  $x = x(G, A)$  solves the  $k$ -pole circuit  $(G, A)$  for  $x_a = x_a^0, a \in A$ .

**Proof** of the lemma is fully similar to the proof of Lemma 2. □

Two  $k$ -poles circuits  $(G; a_1, \dots, a_k)$  and  $(G'; a'_1, \dots, a'_k)$  are called *equivalent* if in them the corresponding fluxes are equal whenever the corresponding potentials are equal, or more accurately, if  $x_{a_i}^* = x_{a'_i}^*$  for all  $i \in [k] = \{1, \dots, k\}$  whenever  $x_{a_i} = x_{a'_i}$  for all  $i \in [k]$ .

**Proposition 2** *For every  $k$ -pole circuit with  $n$  nodes (where  $n \geq k$ ) there is an equivalent  $k$ -pole circuit with  $k$  nodes.*

**Proof** To show this, we shall explicitly reduce every  $k$ -pole circuit with  $n + 1$  nodes to a  $k$ -pole circuit with  $n$  nodes, whenever  $n \geq k$ . To do so, let us label the nodes of the former circuit  $G$  by  $0, 1, \dots, n$  and denote by  $\lambda_{i,j}$  the conductance of edge  $(i, j)$ . (If there is no such edge then  $\lambda_{i,j} = 0$ .) Let us construct a circuit  $G'$  whose  $n$  nodes are labeled by  $1, \dots, n$  and conductances are given by formula

$$\lambda'_{i,j} = \lambda_{i,j} + \frac{\lambda_{0,i}\lambda_{0,j}}{\sum_{m=1}^n \lambda_{0,m}}. \quad (21)$$

**Lemma 7** *The obtained two  $k$ -pole circuits  $(G, A)$  and  $(G', A)$  are equivalent.*

**Proof** (sketch). Since  $r = 1$  the conductance of a pair of parallel edgers is the sum of their conductances, we can assume, without any loss of generality, that  $G'$  is a star the with center at 0, that is,  $G'$  consists of  $n$  edges:  $(0, 1), \dots, (0, n)$ .

Due to linearity, it is sufficient to consider the  $n$  basic potential vectors  $x^i = (x_1^i, \dots, x_n^i)$  such that  $x_m^i = \delta_m^i$ , that is,  $x_i^i = 1$  and  $x_m^i = 0$  whenever  $m \neq i$ . For each such vector  $x_i$ , by the first Kirchhoff law at node 0, we obtain that

$$x_0^i = \frac{\lambda_{0,i}}{\sum_{m=1}^n \lambda_{0,m}}, \quad (22)$$

In its turn, this formula easily implies (21). □

Finally, we derive Proposition 2 applying Lemma 7 successively  $n - k$  times. □

**Remark 11** *Regarding the above proof, we should notice that:*

- $\lambda'_{i,j}$  gets the same value for vectors  $x^i$  and  $x^j$ ;
- Let  $G'$  be an  $n$ -star, that is,  $\lambda_{i,j} = 0$  for all distinct  $i$  and  $j$ . Then, we obtain a mapping that assigns a weighted  $n$ -clique  $K_n$  to each weighted  $n$ -star  $S_n$ . Obviously, this mapping is a bijection. In particular, for  $n = 3$ , the obtained one-to-one correspondence between the weighted claws and triangles is known as the  $Y$ - $\Delta$  transformation.

As a corollary, we obtain an alternative proof of the triangle inequality (2) in the linear case. Indeed, every three-pole network can be reduced to an equivalent triangle. In its turn, the triangle is equivalent to a claw and for the latter, the triangle inequality is obvious.

For the two-pole case, we can also obtain an important corollary, namely, an explicit formula for the effective conductance  $\lambda_{a,b}$ . To get it, let us consider the Kirchhoff  $n \times n$  conductivity matrix  $K$  defined as follows:  $K_{i,j} = \lambda_{i,j}$  when  $i \neq j$  and  $K(i, i) = -\sum_{j|j \neq i} \lambda_{i,j}$ .

Applying the reduction of Proposition 2 successively  $n - 2$  times we represent the effective conductance  $\lambda_{a,b}$  as the ratio of two determinants:

$$\lambda_{a,b} = \left| \frac{\det(K'_{a,b})}{\det(K''_{a,b})} \right|, \quad (23)$$

where  $K'$  and  $K''$  are two submatrices of  $K$  obtained by eliminating (i) row  $a$  and column  $b$  and, respectively, (ii) two rows  $a, b$  and two columns  $a, b$ .

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