

R U T C O R  
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ON EXACT BLOCKERS AND  
ANTI-BLOCKERS,  $\Delta$ -CONJECTURE, AND  
RELATED PROBLEMS <sup>a</sup>

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## RUTCOR RESEARCH REPORT

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# ON EXACT BLOCKERS AND ANTI-BLOCKERS, $\Delta$ -CONJECTURE, AND RELATED PROBLEMS <sup>1</sup>

Vladimir Gurvich

**Abstract.** Let us consider two binary systems of inequalities (i)  $Cx \geq e$  and (ii)  $Cx \leq e$ , where  $C \in \{0,1\}^{m \times n}$  is an  $m \times n$   $(0,1)$ -matrix,  $x \in \{0,1\}^n$ , and  $e$  is the vector of  $m$  ones. The set of all support-minimal (respectively, support-maximal) solutions  $x$  to (i) (respectively, to (ii)) is called the *blocker* (respectively, *anti-blocker*). A blocker  $\mathcal{B}$  (respectively, anti-blocker  $\mathcal{A}$ ) is called *exact* if  $Cx = e$  for every  $x \in \mathcal{B}$  (respectively,  $x \in \mathcal{A}$ ). Exact blockers can be completely characterized. There is a one-to-one correspondence between them and  $P_4$ -free graphs (along with a well known one-to-one correspondence between the latter and the so-called read-once Boolean functions). However, the class of exact anti-blockers is wider and more sophisticated. We demonstrate that it is closely related to the so-called CIS graphs, more general  $\ell$ -CIS  $d$ -graphs, and  $\Delta$ -conjecture.

**Key words:** blocker, anti-blocker, exact blocker, exact anti-blocker, read-once Boolean function, CIS graph, CIS  $d$ -graph,  $\Delta$ -conjecture, box-partition, solid box-partition.

# 1 Graphs and hypergraphs; basic definitions

A hypergraph  $\mathcal{H} \subseteq 2^V$  on the vertex-set  $V = V(\mathcal{H}) = \{v_1, \dots, v_n\}$  is a non-empty family of non-empty subsets  $H \subseteq V$  called its *edges*, that is,  $H \in \mathcal{H}$ . For convenience, we will assume from now on that every vertex belongs to an edge, or in other words, that  $V = \cup_{H \in \mathcal{H}} H$ .

A subset  $A \subseteq V$  is called an *independent (or stable) set* of  $\mathcal{H}$  if  $A$  contains no edge, that is, if  $H \subseteq A$  for no  $H \in \mathcal{H}$ . An independent set  $A$  is called *maximal* if none of its proper supersets  $A' \supset A$  is independent, that is, if  $A' \supseteq H$  for some  $H \in \mathcal{H}$ .

A subset  $B \subseteq V$  is called *transversal* to  $\mathcal{H}$  if  $B$  meets all edges of  $\mathcal{H}$ , that is, if  $B \cap H \neq \emptyset$  for every  $H \in \mathcal{H}$ . A transversal  $B$  is called *minimal* if none of its proper subsets  $B' \subset B$  is a transversal, that is, if  $B' \cap H = \emptyset$  for some  $H \in \mathcal{H}$ . Obviously, the complement to a (minimal) transversal is a (maximal) independent set and vice versa.

A hypergraph  $\mathcal{H} \subseteq 2^V$  is called a *graph* if each of its edges  $H \in \mathcal{H}$  consists of precisely two vertices; such vertices are called *adjacent*. Standardly, we denote a graph by  $G$  (rather than by  $\mathcal{H}$ ) and the set of its edges by  $E = E(G)$ . The complementary graph  $\overline{G}$  of  $G$  is defined by the same vertex-set,  $V(\overline{G}) = V(G)$ , and the complementary edge-set,  $(v', v'') \in E(\overline{G})$  if and only if  $(v', v'') \notin E(G)$  for any two distinct  $v', v'' \in V(G)$ .

A set of pairwise adjacent (respectively, non-adjacent) vertices of a graph  $G$  is called a *clique* (respectively, an independent or stable set) of  $G$ . Obviously, a (maximal) independent set of  $G$  is a (maximal) clique in  $\overline{G}$  and vice versa.

To each hypergraph  $\mathcal{H} \subseteq 2^V$  let us assign its *co-occurrence graph*  $G = G(\mathcal{H})$  on the same vertex-set  $V = V(G) = V(\mathcal{H})$  and such that two vertices  $v', v'' \in V$  are adjacent in  $G$  if and only if they are distinct,  $v' \neq v''$ , and adjacent in  $\mathcal{H}$ , that is,  $v', v'' \in H$  for an edge  $H \in \mathcal{H}$ .

**Example 1** *Distinct hypergraphs can have the same co-occurrence graph. Let us consider the following three examples:*

$$\mathcal{H}_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\} \text{ and } \mathcal{H}_2 = \{(v_1, v_2, v_3)\}$$

*both correspond to the complete graph on the ground set  $V = \{v_1, v_2, v_3\}$ ;*

$$\mathcal{H}_3 = \{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_1)\} \text{ and } \mathcal{H}_4 = \mathcal{H}_3 \cup \{(v_1, v_3, v_5)\}$$

*both generate the same graph, so-called sun or 3-anti-comb; finally*

$$\mathcal{H}_5 = \{(v_1, v_3, v_6), (v_1, v_4, v_5), (v_1, v_4, v_6), (v_2, v_3, v_5), (v_2, v_3, v_6), (v_2, v_4, v_5)\} \text{ and}$$

$$\mathcal{H}_6 = \mathcal{H}_5 \cup \{(v_1, v_3, v_5) \cup \{(v_2, v_4, v_6)\}$$

*both generate the same complete 3-partite graph of size  $2 \times 2 \times 2$ .*

Conversely, with a graph  $G$  let us associate its *clique-hypergraph*  $\mathcal{H}_C = \mathcal{H}_C(G)$  and its *stable-set-hypergraph*  $\mathcal{H}_S = \mathcal{H}_S(G)$  as follows:

both have the same vertex-set,  $V = V(G) = V(\mathcal{H}_C) = V(\mathcal{H}_S)$ , while the edges are all maximal cliques and all maximal stable sets of  $G$ , respectively.

A hypergraph  $\mathcal{H}$  will be called *completely clique-maximal* if it is the clique-hypergraph of its own co-occurrence graph, that is,  $\mathcal{H} = \mathcal{H}_C(G(\mathcal{H}))$ . Let us remark that completely clique-maximal hypergraphs are also called *normal* in the literature; see, for example, [19, 17, 16].

Above,  $\mathcal{H}_2$ ,  $\mathcal{H}_4$ , and  $\mathcal{H}_6$  are completely clique-maximal, while  $\mathcal{H}_1$ ,  $\mathcal{H}_3$ , and  $\mathcal{H}_5$  are not.

For any hypergraph  $\mathcal{H}'$  there is a unique completely clique-maximal hypergraph  $\mathcal{H}$  with the same co-occurrence graph,  $G = G(\mathcal{H}) = G(\mathcal{H}')$ ; obviously,  $\mathcal{H} = \mathcal{H}_C(G(\mathcal{H}'))$ .

Furthermore, a hypergraph  $\mathcal{H}$  will be called *clique-maximal* if  $\mathcal{H} \subseteq \mathcal{H}_C(G(\mathcal{H}))$ , or in other words, if each edge of  $\mathcal{H}$  is a *maximal* clique in  $G(\mathcal{H})$ . Yet, some maximal cliques of  $G(\mathcal{H})$  might be missing in  $\mathcal{H}$ . In Example 1 all hypergraphs are clique-maximal, except  $\mathcal{H}_1$ .

Finally, let us recall that  $\mathcal{H}$  is called a *Sperner* hypergraph if none of its edges contains another one, that is,  $H' \subseteq H''$  for no distinct  $H', H'' \in \mathcal{H}$ . Obviously, all six hypergraphs of Example 1 are Sperner ones.

It is easily seen that, in general, the above three families, of

(i) completely clique-maximal, (ii) clique-maximal, and (iii) Sperner hypergraphs, are nested,  $(i) \subset (ii) \subset (iii)$ . Example 1 shows that both containments are strict.

Given a hypergraph  $\mathcal{H}$  with  $n$  vertices,  $V(\mathcal{H}) = \{v_1, \dots, v_n\}$ , and  $m$  edges,  $\mathcal{H} = \{H_1, \dots, H_m\}$ , its *incidence* matrix  $C = C(\mathcal{H})$  is defined as an  $m \times n$   $(0, 1)$ -matrix whose entry  $c(i, j)$  is 1 whenever  $v_i \in H_j$  and 0 otherwise.

We refer the reader to the monograph [4], by Claude Berge, for more concepts and details.

## 2 Blockers and anti-blockers

The hypergraph  $\mathcal{B} = \mathcal{B}(\mathcal{H})$  of all minimal transversals to  $\mathcal{H}$  is called the *blocker* of  $\mathcal{H}$ .

By definition,  $\mathcal{B}$  is a Sperner hypergraph and  $\cup_{B \in \mathcal{B}} B = V(\mathcal{B}) \subseteq V$ .

If  $\mathcal{H}$  is a Sperner hypergraph too then it is obvious and well-known that

(i)  $V(\mathcal{B}) = \cup_{B \in \mathcal{B}} B = \cup_{H \in \mathcal{H}} H = V(\mathcal{H})$  and (ii)  $\mathcal{H}$  is the blocker of  $\mathcal{B}$ .

In this case, hypergraphs  $\mathcal{H}$  and  $\mathcal{B} = \mathcal{B}(\mathcal{H})$  are called dual and notation  $\mathcal{B} = \mathcal{H}^d$  or, equivalently,  $\mathcal{B}^d = \mathcal{H}$  is used. In other words, mapping  $\mathcal{B}$  is an involution, that is,  $\mathcal{B}(\mathcal{B}(\mathcal{H})) \equiv \mathcal{H}$  for any Sperner hypergraph  $\mathcal{H}$ .

In general, an arbitrary (not necessarily Sperner) hypergraph  $\mathcal{H}$  can be reduced to a Sperner hypergraph  $\mathcal{H}'$  by successive elimination of every edge that contains another edge. It is clear that  $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}')$ .

Given a hypergraph  $\mathcal{H} \subseteq 2^V$ , a subset  $A \subseteq V$  is called *anti-blocking* if  $A$  meets each edge of  $\mathcal{H}$  in at most one vertex, that is, if  $|A \cap H| \leq 1$  for all  $H \in \mathcal{H}$ . Standardly, an anti-blocking set  $A$  is called *maximal* if none of its proper superset is anti-blocking, that is, if  $|A' \cap H| \geq 2$  for some  $H \in \mathcal{H}$  whenever  $A' \supset A$ .

The hypergraph  $\mathcal{A} = \mathcal{A}(\mathcal{H})$  of all maximal anti-blocking sets of  $\mathcal{H}$  is called the *anti-blocker* of  $\mathcal{H}$ . By definition,  $\mathcal{A}(\mathcal{H})$  is a Sperner hypergraph and  $\cup_{A \in \mathcal{A}} A = V(\mathcal{A}) = V$ . It is also not difficult to verify that  $\mathcal{A}(\mathcal{H}_C(G)) = \mathcal{H}_S(G)$ . More generally,  $\mathcal{A}(\mathcal{H}) = \mathcal{H}_S(G(\mathcal{H}))$  for each hypergraph  $\mathcal{H}$ , Sperner or not.

In particular, the anti-blocker  $\mathcal{A}(\mathcal{H})$  depends only on the co-occurrence graph of  $\mathcal{H}$ . In other words, all hypergraphs with the same co-occurrence graph have the same anti-blocker. Of course, by symmetry,  $\mathcal{A}(\mathcal{H}_S(G)) = \mathcal{H}_C(G)$  for any graph  $G$ . In general,  $\mathcal{A}(\mathcal{A}(\mathcal{H})) = \mathcal{H}$  if

and only if  $\mathcal{H}$  is completely clique-maximal, or in other words, if  $\mathcal{H} = \mathcal{H}_C(G)$  (or equivalently,  $\mathcal{H} = \mathcal{H}_S(G)$ ) for a graph  $G$ .

Even more generally (but still obviously)  $\mathcal{A}(\mathcal{A}(\mathcal{H})) = H_C(G(\mathcal{H}))$  for all  $\mathcal{H}$ . In general, for an arbitrary (not necessarily clique-maximal or Sperner) hypergraph  $\mathcal{H}'$  let us consider the corresponding completely clique-maximal hypergraph  $\mathcal{H} = \mathcal{H}_C(G(\mathcal{H}'))$ . It is clear that  $G(\mathcal{H}) = G(\mathcal{H}')$  and  $\mathcal{A}(\mathcal{H}) = \mathcal{A}(\mathcal{H}')$ .

It is easily seen that blocker  $\mathcal{B}(\mathcal{H})$  (respectively, anti-blocker  $\mathcal{A}(\mathcal{H})$ ) can be equivalently redefined as the set of all support-minimal (respectively, support-maximal) binary solutions  $x \in \{0, 1\}^n$  of the binary system  $Cx \geq e$  (respectively,  $Cx \leq e$ ), where  $C = C(\mathcal{H})$  is the  $m \times n$  incidence matrix of  $\mathcal{H}$  and  $e$  is the vector of  $m$  ones.

For applications of blockers and anti-blockers, we refer, for example, to [15].

### 3 Exact blockers, exact anti-blockers, and $P_4$ -free graphs

A blocker  $\mathcal{B} = \mathcal{B}(\mathcal{H})$  (respectively, anti-blocker  $\mathcal{A} = \mathcal{A}(\mathcal{H})$ ) is called *exact* if every minimal transversal  $B \in \mathcal{B}$  (respectively, maximal anti-blocking set  $A \in \mathcal{A}$ ) and each edge of  $H \in \mathcal{H}$  have exactly one vertex in common, that is, if  $|B \cap H| = 1$  (respectively,  $|A \cap H| = 1$ ) for all  $H \in \mathcal{H}$ . Equivalently, in terms of the incidence matrix  $C = C(\mathcal{H})$ , a blocker (respectively, anti-blocker) is exact if and only if  $Cx \equiv e$  whenever  $x$  is a support-minimal (respectively, support-maximal) solution to  $Cx \geq e$  (respectively,  $Cx \leq e$ ).

Graph  $P_4$  is defined on four vertices by three edges  $P_4 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$ .

It is self-complementary, that is, the complementary graph  $\bar{P}_4 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$  is isomorphic to  $P_4$ . Standardly, a graph  $G$  is called  $P_4$ -free if it contains no induced  $P_4$ .

A hypergraph  $\mathcal{H}$  will be called *B-exact* (respectively, *A-exact*) if its blocker  $\mathcal{B}(\mathcal{H})$  (respectively, anti-blocker  $\mathcal{A}(\mathcal{H})$ ) is exact.

The B-exact hypergraphs are completely characterized by the following statement.

**Theorem 1** ([19] and [23, 16, 21]). *The next properties of a hypergraph  $\mathcal{H}$  are equivalent:*

- (i) *the blocker  $\mathcal{B} = \mathcal{B}(\mathcal{H})$  to  $\mathcal{H}$  is exact, that is,  $\mathcal{H}$  is B-exact;*
- (ii)  *$\mathcal{H}$  is completely clique-maximal and its co-occurrence graph  $G(\mathcal{H})$  is  $P_4$ -free;*
- (iii)  *$|B \cap H| = 1$  for all  $B \in \mathcal{B}(\mathcal{H})$  and  $H \in \mathcal{H}$ ;*
- (iv) *the co-occurrence graphs  $G(\mathcal{H})$  and  $G(\mathcal{B}(\mathcal{H}))$  are edge-disjoint;*
- (v) *the co-occurrence graphs  $G(\mathcal{H})$  and  $G(\mathcal{B}(\mathcal{H}))$  are complementary.* □

**Remark 1** *It is also shown in [19, 20, 23, 21, 16] that  $P_4$ -free graphs are in one-to-one correspondence with the so-called read-once Boolean functions.*

*A simple recognition algorithm for the latter was suggested in [17], see also [16]. Given a DNF  $f$  of  $n$  variables, this algorithm can verify whether  $f$  is read-once and produces a (unique) read-once formula, when the answer is positive, in time  $O(n|f|)$ .*

**Remark 2** *To show that complete clique-maximality is essential in (ii) let us consider the hypergraphs  $\mathcal{H}_1$  and  $\mathcal{H}_5$  from Example 1. They both are clique-maximal, but not completely, and none of them is B-exact, although their co-occurrence graphs are  $P_4$ -free:  $G(\mathcal{H}_1) = K_3$  and  $G(\mathcal{H}_5)$  is the complete 3-partite  $2 \times 2 \times 2$  graph. Hypergraph  $\mathcal{H}_4$  is completely clique-maximal but not B-exact, since  $G(\mathcal{H}_4)$  contains a  $P_4$ .*

Moreover, the clique hypergraphs  $\mathcal{H}_C(G)$  of a  $P_4$ -free graph  $G$  is not only B-exact but also A-exact. Indeed, as we already know, if  $\mathcal{H} = \mathcal{H}_C(G)$  then  $\mathcal{A} = \mathcal{H}_S(G)$  is the anti-blocker of  $\mathcal{H}$ . Furthermore, if  $G$  is a  $P_4$ -free graph then this anti-blocker is exact, by (iii). Thus, both the anti-blocker  $\mathcal{A}(\mathcal{H})$  and blocker  $\mathcal{B}(\mathcal{H})$  are exact whenever  $\mathcal{H}$  satisfies (ii). By Theorem 1, (ii) is also *necessary* for B-exactness. Yet, not for A-exactness. In the next Section, we will show that each A-exact hypergraph is clique-maximal but it might be not completely clique-maximal and its co-occurrence graph might contain an induced  $P_4$ .

Some necessary and some sufficient conditions for A-exactness are given below.

## 4 On A-exact and clique-maximal hypergraphs

Clique-maximality is a necessary condition for A-exactness.

**Proposition 1** *A hypergraph  $\mathcal{H}$  is clique-maximal whenever it is A-exact.*

**Proof** . Let us assume indirectly that  $\mathcal{H}$  is not clique-maximal; in other words, it has an edge  $H_0 \in \mathcal{H}$  and its co-occurrence graph  $G = G(\mathcal{H})$  has a (maximal) clique  $C_0$  such that  $H_0 \subset C_0$  and containment is strict, i.e., there is a vertex  $v \in C_0 \setminus H_0$ . Let  $S_0$  be a maximal stable set in  $G$  that contains  $v$ . Then obviously,  $S_0$  is anti-blocking ( $|S_0 \cap H| \leq 1$  for any  $H \in \mathcal{H}$ ) and  $S_0 \cap H_0 = \emptyset$ . Thus,  $\mathcal{H}$  is *not* A-exact.  $\square$

We call  $G$  a *CIS graph* (or say that  $G$  has the CIS property) if  $C \cap S \neq \emptyset$  for every maximal clique  $C$  and every maximal stable set  $S$  in  $G$ . Each  $P_4$ -free graph is a CIS graph, yet, there are many others; see Section 5 and also [1] for more details. The following condition is sufficient for A-exactness.

**Proposition 2** *A hypergraph  $\mathcal{H}$  is A-exact whenever it is clique-maximal and its co-occurrence graph  $G(\mathcal{H})$  is a CIS graph.*

**Proof** . As we already know, a maximal anti-blocking set  $S$  of  $\mathcal{H}$  is a maximal stable set of  $G(\mathcal{H})$ . Hence,  $H \cap S \neq \emptyset$ , since  $H$  is clique-maximal and  $G(\mathcal{H})$  is a CIS graph.  $\square$

However,  $\mathcal{H}$  might be A-exact when  $G(\mathcal{H})$  is not a CIS graph.

**Example 2** *Recall hypergraph  $\mathcal{H}_3 = \{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_1)\}$  from Example 1.*

*It is easy to verify that the co-occurrence graph  $G(\mathcal{H}_3)$  is not a CIS graph, since  $C \cap S = \emptyset$  for  $C = \{(v_1, v_3, v_5)\}$  and  $S = \{(v_2, v_4, v_6)\}$ .*

*Yet, the anti-blocker  $\mathcal{A}(\mathcal{H}_3) = \{(v_1, v_4), (v_2, v_5), (v_3, v_6), (v_2, v_4, v_6)\}$  is exact.*

Let us notice that the hypergraph  $\mathcal{A}(\mathcal{H}_3)$  is not  $A$ -exact, although it is the exact anti-blocker to  $\mathcal{H}_3$ . Indeed,  $S = (v_2, v_4, v_6) \in \mathcal{A}(\mathcal{H}_3)$ , while it is easy to check that  $C = (v_1, v_3, v_5) \in \mathcal{A}(\mathcal{A}(\mathcal{H}_3))$ . Furthermore, for the same reason, the completely clique-maximal hypergraph  $\mathcal{H}_4 = \mathcal{H}_3 \cup \{(v_1, v_3, v_5)\}$  is not  $A$ -exact, either.

## 5 Main properties of CIS graphs

By definition, CIS graphs are closed under complementation.

It is also not difficult to show that they are *exactly closed* under substitution [1]. In other words, let notation  $G = G'(v \rightarrow G'')$  mean that graph  $G$  is obtained from graph  $G'$  by substituting graph  $G''$ , as a module, for a fixed vertex  $v$  in  $G'$ ; then,  $G$  is a CIS graph if and only if  $G'$  and  $G''$  are CIS graphs.

However, the family of CIS graphs is not hereditary. Let us consider the bull (or  $A$ -graph)  $G = (V, E)$  defined by  $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_2, v_5), (v_3, v_5)\}$ . It is easy to verify that  $G$  is a CIS graph, yet, it contains an induced  $P_4$ , which is not a CIS graph.

For this reason, CIS graphs cannot be characterized in terms of forbidden subgraphs. In fact, every graph  $G'$  is an induced subgraph of a CIS graph  $G$ . Given  $G'$ , to get  $G$  it is sufficient to add a simplicial vertex to each maximal clique of  $G'$  (which does not have one already in  $G'$ ). Let us note, however, that  $G$  might be exponential in the size of  $G'$ . See [1] for more details. Perhaps, for the same reason, no efficient characterization or recognition algorithm for CIS graphs is known. Yet, some necessary but not sufficient and sufficient but not necessary conditions are known.

For an integer  $k \geq 2$ , we define a  $k$ -comb  $G_k$  as a graph on  $2k$  vertices  $\{v_1, \dots, v_k; v'_1, \dots, v'_k\}$  with  $k(k+1)/2$  edges which form the clique on  $\{v_1, \dots, v_k\}$  and perfect matching  $(v_i, v'_i)$ ,  $i \in [k] = \{1, \dots, k\}$ . The complementary graph  $\overline{G}_k$  is called a  $k$ -anti-comb. Obviously, 2-comb, 2-anti-comb, and  $P_4$  are three isomorphic graphs. It is easy to see that a  $k$ -comb contains  $k$  induced  $(k-1)$ -combs for each  $k \geq 3$ . It is also clear that  $k$ -combs  $G_k$  and  $k$ -anti-combs  $\overline{G}_k$  are not CIS graphs. Indeed, two disjoint sets  $\{v_1, \dots, v_k\}$  and  $\{v'_1, \dots, v'_k\}$  induce a maximal clique and maximal stable set in  $G_k$ , and vice versa in  $\overline{G}_k$ .

In the 1980s, Claude Berge noticed that in a CIS graph  $G$  every induced  $P_4$  must be contained in an induced bull-graph; see [28]. More generally, for each  $k \geq 2$ , in a CIS graph  $G$ , every induced  $k$ -comb  $G_k$  (respectively,  $k$ -anti-comb  $\overline{G}_k$ ) must be settled, that is,  $G$  must contain a vertex  $v_0$  adjacent to every  $v_i$  and not adjacent to every  $v'_i$  for all  $i \in [k] = \{1, \dots, k\}$  (respectively, vice versa) [1]. Berge's necessary conditions correspond to the case  $k = 2$ . However, even for all  $k \geq 2$ , the above conditions do not imply the CIS property. The corresponding example was constructed by Ron Holzman in 1994; see [1].

By Theorem 1,  $G$  is a CIS graph whenever it is  $P_4$ -free. In this case,  $G$  contains no induced combs and anti-combs. In fact, the following relaxation still implies the CIS property.

**Theorem 2** *Graph  $G$  is a CIS graph whenever it contains no induced 3-combs and 3-anti-combs and every induced 2-comb is settled in  $G$ .*

This statement was conjectured in the early 1990s by Vasek Chvatal. His RUTCOR student Wenan Zang published first partial results in 1995 [28]. Finally, Theorem 2 was proved by Deng, Li, and Zang [11, 12], and independently in [1].

Graph  $G$  is called an *almost CIS* graph if every its maximal clique  $C$  and maximal stable set  $S$  intersect, except a unique pair. In contrast to CIS graphs, the family of almost CIS graphs admits a simple (although non-trivial) characterization.

**Theorem 3** *Graph  $G$  is an almost CIS graph if and only if  $G$  is a split graph with a unique split-partition.*

This statement was conjectured in [1]. First partial results were obtained in [6]. Recently, Theorem 3 was proved by Wu, Zang and Zhang [27].

## 6 On completely $\ell$ -clique-maximal hypergraphs and $(\ell, \ell')$ -CIS graphs

A Sperner hypergraph  $\mathcal{H} \subseteq 2^V$  will be called *completely  $\ell$ -clique-maximal* if every clique of cardinality at most  $\ell$  its co-occurrence graph  $G(\mathcal{H})$  is contained in an edge  $H \in \mathcal{H}$ .

First, without any loss of generality, we can assume that  $2 \leq \ell \leq \omega$ , where  $\omega = \omega(G)$  is the *clique number* of graph  $G$ , that is, the number of vertices of a maximum clique of  $G$ .

Indeed, every hypergraph is completely 2-clique-maximal, just by definition of the co-occurrence graph. Furthermore, if  $\mathcal{H}$  is a completely  $\omega$ -clique-maximal hypergraph then it is also completely  $\ell$ -clique-maximal for any  $\ell$ . In fact,  $\mathcal{H}$  is completely  $\omega$ -clique-maximal if and only if it is completely clique-maximal.

**Remark 3** *Yet, let us notice that a completely  $\ell$ -clique-maximal hypergraph might be not clique-maximal when  $\ell < \omega$ . In general, if a hypergraph is completely  $\ell$ -clique-maximal then obviously it is completely  $\ell'$ -clique-maximal whenever  $\ell \geq \ell'$ . Let us note also that  $\omega = \omega(G) \leq |V(G)| = n$  for every graph  $G$ .*

Given integer  $\ell$  and  $\ell'$ , a graph  $G = (V, E)$  will be called  $(\ell, \ell')$ -CIS graph if there exist completely  $\ell$ - and  $\ell'$ -clique-maximal hypergraphs  $\mathcal{H}$  and  $\mathcal{H}'$  whose co-occurrence graphs are  $G$  and  $\overline{G}$  respectively, that is,  $G(\mathcal{H}) = G$ ,  $G(\mathcal{H}') = \overline{G}$ , and whose edges pairwise intersect, that is,  $H \cap H' \neq \emptyset$  for all  $H \in \mathcal{H}$ ,  $H' \in \mathcal{H}'$ .

Again, without loss of generality, we assume that

$$2 \leq \ell \leq \omega(G) \text{ and } 2 \leq \ell' \leq \omega(\overline{G}) = \alpha(G),$$

where  $\alpha = \alpha(G)$  is the *stability number* of graph  $G$ . Moreover, the following statements hold.

**Proposition 3** *Hypergraphs  $\mathcal{H}$  and  $\mathcal{H}'$  are clique-maximal for every  $(\ell, \ell')$ -CIS graph  $G$ . If  $\ell' \geq \alpha$  then hypergraph  $\mathcal{H}'$  is completely clique-maximal, while hypergraph  $\mathcal{H}$  is A-exact.*



**Proof.** The first claim can be proved by copying the proof of Proposition 1, while the last two statements are straightforward.  $\square$

Let us note however that  $\mathcal{H}$  might be not completely clique-maximal and  $\mathcal{H}'$  not A-exact even when  $\ell' \geq \alpha$ ; see Example 2.

Finally, the above definitions easily result in the following characterization of A-exactness.

**Theorem 4** *Let  $\mathcal{H}$  be a hypergraph and  $G = G(\mathcal{H})$  be its co-occurrence graph. Then  $\mathcal{H}$  is not A-exact unless it is clique-maximal and  $G$  is a  $(2, \alpha(G))$ -CIS graph. When both conditions hold then  $\mathcal{H}$  is A-exact if and only if every its edge  $H \in \mathcal{H}$  and every maximal stable set  $S$  of  $G$  intersect,  $H \cap S \neq \emptyset$ .*  $\square$

By definition of the  $(\ell, \ell')$ -CIS property, such an A-exact hypergraph  $\mathcal{H}$  exists for every given  $(2, \alpha(G))$ -CIS graph  $G$ . Note that Theorem 4 strengthens Propositions 1 and 2.

In Section 9 we will extend the above  $(\ell, \ell')$ -CIS property from graphs to  $d$ -graphs.

Although such a generalization is not directly related to the exact anti-blockers, yet, it is of independent interest.

In the next Section, we extend the standard CIS property from graphs to  $d$ -graphs.

## 7 CIS $d$ -graphs

A  $d$ -graph  $\mathcal{G} = (V; E_1, \dots, E_d)$  is a complete graph on the vertex-set  $V = \{v_1, \dots, v_n\}$  whose  $\binom{n}{2}$  edges are partitioned into  $d$  subsets (colored by  $d$  colors) some of which might be empty.

We say that  $\mathcal{G}$  is  $\ell$ -colored if  $\ell$  is the number of its non-empty chromatic components  $E_i \neq \emptyset$  for  $i \in [d] = \{1, \dots, d\}$ . Obviously,  $\ell = 0$  if and only if  $\mathcal{G}$  consists of a unique vertex,  $|V| = 1$ . Such  $d$ -graph is called *trivial*.

In case  $d = 2$  a  $d$ -graph is just a graph, or more precisely, a pair that consists of a graph and its complement. Thus,  $d$ -graphs can be viewed as a generalization of graphs.

Given a  $d$ -graph  $\mathcal{G} = (V; E_1, \dots, E_d)$ , let  $G_i = (V, E_i)$  be its  $i$ th chromatic component, on the vertex-set  $V$  with the edge-set  $E_i$ ; furthermore, let  $S_i \subseteq V$  be a *maximal* stable set in  $G_i$ , where  $i \in [d]$ ; finally, let  $\mathcal{S} = \{S_i \mid i \in [d]\}$  be a collection of  $d$  such sets and let  $S = \bigcap_{i=1}^d S_i$ . Obviously,  $|S| \leq 1$  for every collection  $\mathcal{S}$ , since  $v, v' \in S$  implies that edge  $(v, v')$  has no color in  $\mathcal{G}$ . We call  $\mathcal{G}$  a *CIS  $d$ -graph*, or say that it has CIS  $d$ -property, if  $S \neq \emptyset$  for each collection  $\mathcal{S}$  defined above.

It is not difficult to verify that the family of CIS  $d$ -graphs is exactly closed with respect to substitution [1, 21]. More precisely, let  $\mathcal{G}'$  and  $\mathcal{G}''$  be two vertex-disjoint  $d$ -graphs and let  $\mathcal{G} = \mathcal{G}'(v \rightarrow \mathcal{G}'')$  denote the  $d$ -graph obtained by substituting  $\mathcal{G}''$  for a vertex  $v$  in  $\mathcal{G}'$ . Then,  $\mathcal{G}$  has the CIS property if and only if both  $\mathcal{G}'$  and  $\mathcal{G}''$  have it.

Let us also recall that the CIS-property is not hereditary already for  $d = 2$ .

## 8 On remarkable properties of $d$ -graphs $\Pi$ and $\Delta$

### 8.1 Definition

Two  $d$ -graphs  $\Pi$  and  $\Delta$  given in Figure 1 play an important role:

$\Pi$  is defined for any  $d \geq 2$  by  $V = \{v_1, v_2, v_3, v_4\}$ ;  
 $E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$ ,  $E_2 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$ , and  $E_i = \emptyset$  whenever  $i > 2$ ;  
 $\Delta$  is defined for any  $d \geq 3$  by  $V = \{v_1, v_2, v_3\}$ ,  
 $E_1 = \{(v_1, v_2)\}$ ,  $E_2 = \{(v_2, v_3)\}$ ,  $E_3 = \{(v_3, v_1)\}$ , and  $E_i = \emptyset$  whenever  $i > 3$ .

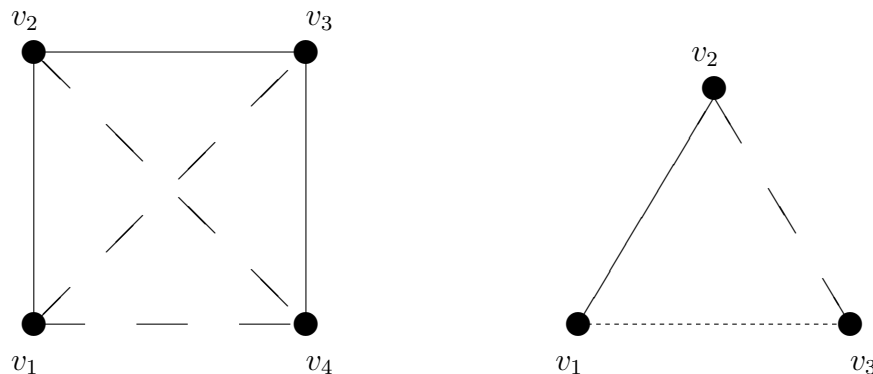


Figure 1: 2- and 3-colored  $d$ -graphs  $\Pi$  and  $\Delta$ .

Clearly,  $\Pi$  and  $\Delta$  are respectively 2- and 3-colored  $d$ -graphs; both non-empty chromatic components of  $\Pi$  are isomorphic to  $P_4$  and  $\Delta$  is a three-colored triangle.

Both  $d$ -graphs  $\Pi$  and  $\Delta$  were introduced in 1967 by Gallai in his seminal paper [14];  $\Delta$ -free  $d$ -graphs are frequently referred to as Gallai's graphs; we will call them Gallai's  $d$ -graphs, which is more accurate.

It is easy to verify that the class of Gallai's  $d$ -graphs is exactly closed under substitution [1, 21] and hereditary, just by definition. (Recall that CIS  $d$ -graphs have only the former but not the latter property.)

### 8.2 Minimal and locally minimal complementary connected $d$ -graphs

A  $d$ -graph  $\mathcal{G} = (V; E_1, \dots, E_d)$  is called *complementary connected* (CC) if the complement  $G(V, \overline{E_i})$  to its  $i$ th chromatic component  $G(V, E_i)$  is connected on  $V$  for all  $i \in [d]$ .

Obviously,  $\Pi$  and  $\Delta$  are minimal CC  $d$ -graphs, that is, they are CC, while all their proper sub- $d$ -graphs are not. (By convention, the trivial, single-vertex,  $d$ -graph is not CC.)

Moreover, except for  $\Pi$  and  $\Delta$ , there are no other minimal CC  $d$ -graphs.

**Theorem 5** *Each CC  $d$ -graph contains  $\Pi$  or  $\Delta$ . Moreover, for every  $\Pi$ - and  $\Delta$ -free  $d$ -graph  $\mathcal{G}$  there is a unique  $i \in [d]$  such that the graph  $\overline{G_i} = G(V, \overline{E_i})$  is not connected on  $V$ .*

This result was proven in [20]; see also [5, 21]. Let us split the graph  $\overline{G}_i$  into connected components and partition  $V$  accordingly. Since the corresponding sub- $d$ -graphs are still  $\Pi$ - and  $\Delta$ -free, we can proceed with such partitioning until obtain finally a unique canonical decomposition of  $\mathcal{G}$  [20, 21].

In case  $d = 2$ , this is the well-known modular decomposition of the  $P_4$ -free graphs.

As a corollary, we obtain a one-to-one correspondence between the  $\Pi$ - and  $\Delta$ -free  $d$ -graphs and extensive  $d$ -person game forms; see [20, 21] for more details. In [5], Theorem 5 was extended as follows:

**Theorem 6**  *$\Pi$  and  $\Delta$  are the only locally minimal CC  $d$ -graphs, that is, every other CC  $d$ -graph  $\mathcal{G}$  contains a vertex  $v \in V$  such that the sub- $d$ -graph  $\mathcal{G}[V \setminus \{v\}]$  is still CC.*

### 8.3 Minimal and locally minimal non-CIS $d$ -graphs

It is also easily seen that  $\Pi$  and  $\Delta$  are minimal non-CIS  $d$ -graphs, that is, the CIS property does not hold for  $\Pi$  and  $\Delta$  but it holds for all their proper sub- $d$ -graphs. Moreover, except for  $\Pi$  and  $\Delta$ , there are no other minimal non-CIS  $d$ -graphs. Let us notice that the trivial, single-vertex,  $d$ -graph has the CIS property.

**Theorem 7** *Every non-CIS  $d$ -graph contains a  $\Pi$  or  $\Delta$ , or in other words, all  $\Pi$ - and  $\Delta$ -free  $d$ -graphs have the CIS  $d$ -property.*

In [20, 21], this result was derived from the above canonical decomposition of the  $\Pi$ - and  $\Delta$ -free  $d$ -graphs. We will give a shorter proof (of a stronger statement) in the next section.

In [2], Theorem 7 was also strengthened as follows:

**Theorem 8** *The only locally minimal non-CIS  $d$ -graphs are  $\Pi$  and  $\Delta$ , that is, every other non-CIS  $d$ -graph  $\mathcal{G}$  contains a vertex  $v \in V$  such that  $\mathcal{G}[V \setminus \{v\}]$  is still a non-CIS  $d$ -graph.*

**Remark 4** *Thus,  $\Pi$  and  $\Delta$  are the only minimal and locally minimal elements in both classes, CC and non-CIS  $d$ -graphs. It was shown in [2] that these two classes are in general position: not nested and not disjoint.*

### 8.4 Another generalization of Theorem 7 and its proof

Theorem 7 follows from Theorem 8 but the prove of the latter in [2] is pretty long.

Also, Theorem 7 can be derived from Theorem 5 and resulting canonical decomposition of  $\Pi$ - and  $\Delta$ -free  $d$ -graphs. Yet, this plan, realized in [20, 21], is complicated too.

Here we suggest one more generalization of Theorem 7 and a relatively short proof of it obtained recently by Endre Boros and the author.

A cycle of a  $d$ -graph is called in [3] *colorful* if all its edges have pairwise distinct colors.

**Lemma 1** *A Gallai  $d$ -graph has no colorful cycles, or in other words, a  $d$ -graph with a colorful cycle has a colorful triangle, that is,  $\Delta$ .*

The induction on the number of edges of the cycle is obvious.  $\square$

This claim is instrumental in [3].

By definition, a non-CIS  $d$ -graph  $\mathcal{G} = (V; E_1, \dots, E_d)$  has a collection of vertex-sets  $\mathcal{S} = \{S_i \subseteq V \mid i \in [d]\}$  whose intersection is empty,  $S = \bigcap_{i=1}^d S_i = \emptyset$ , where  $S_i$  is a maximal independent set of the  $i$ th chromatic component  $G_i = (V, E_i)$  for each  $i \in [d] = \{1, \dots, d\}$ .

Let us choose a vertex  $v_{i_1} \in V$ . It does not belong to a maximal independent set of  $\mathcal{S}$ , say, to  $S_{i_2}$ , since  $S = \emptyset$ . Then, there is a vertex  $v_{i_2} \in S_{i_2}$  such that  $(v_{i_1}, v_{i_2}) \in E_{i_2}$ , since otherwise set  $S_{i_2} \cup \{v_{i_1}\}$  would be independent in  $G_{i_2}$ , in contradiction with maximality of  $S_{i_2}$ . In its turn, vertex  $v_{i_2}$  does not belong to a maximal independent set, say, to  $S_{i_3}$ . Again by maximality, there is a  $v_{i_3} \in S_{i_3}$  such that  $(v_{i_2}, v_{i_3}) \in E_{i_3}$ , etc. Since  $d$ -graph  $\mathcal{G}$  is finite, this procedure will result in a cycle  $C$  that consists of  $k$  distinct vertices  $v_{i_j} \in S_{i_j}$  and  $k$  edges  $(v_{i_{j-1}}, v_{i_j}) \in E_{i_j}$ , where  $j \in [k] = \{1, \dots, k\}$  and standardly the indices are taken modulo  $k$ , that is,  $v_{i_0} = v_{i_k}$ . A cycle  $C$  obtained in such a way will be called a  $\Pi\Delta$ -cycle in  $\mathcal{G}$ .

Let us generalize this concept slightly and extend it to all, CIS or non-CIS,  $d$ -graphs. To do so, we just relax the above definition a bit assuming now that  $S_{i_j}$  is a (*not necessarily maximal*) independent set of  $G_{i_j} = (V, E_{i_j})$ , for  $j \in [k], i_j \in [d]$ . In other words, a  $\Pi\Delta$ -cycle  $C$  in  $\mathcal{G}$  is defined by the following condition:

if edges  $(v_{i_{r-1}}, v_{i_r})$  and  $(v_{i_{s-1}}, v_{i_s})$  of  $C$  are of the same color then  $(v_{i_r}, v_{i_s})$  must be colored with a different color:  $(v_{i_{r-1}}, v_{i_r}), (v_{i_{s-1}}, v_{i_s}) \in E_m \Rightarrow (v_{i_r}, v_{i_s}) \notin E_m \quad \forall r, s \in [k], m \in [d]$ .

**Remark 5** *The order of vertices in  $C$  cannot be reversed; in fact,  $C$  is a directed cycle.*

**Lemma 2** *The  $d$ -graphs  $\Pi$  and  $\Delta$  contain Hamiltonian  $\Pi\Delta$ -cycles.*

**Proof** Indeed, in  $\Pi$  such a cycle  $C$  is specified by the sequence of vertices  $\{v_1, v_2, v_4, v_3\}$ , in other words,  $i_1 = 1, i_2 = 2, i_3 = 4, i_4 = 3$ ; thus, colors in  $C$  alternate:  $v_1, v_4 \in S_1, v_2, v_3 \in S_2$ , while in  $\Delta$  all colors are distinct:  $S_{i_j} = j$  for  $j \in \{1, 2, 3\}$ ; see Figure 1.  $\square$

By Lemma 2 and Theorem 5, all non-CIS  $d$ -graphs contain  $\Pi\Delta$ -cycles.

**Remark 6** *As we know, for these special  $\Pi\Delta$ -cycles, it can be assumed that  $S_{i_j}$  is a maximal independent set of  $G_{i_j} = (V, E_{i_j})$  for each  $j \in [k]$ , while in general these independent sets may not be maximal.*

**Remark 7** *Let us also mention that CIS  $d$ -graphs can contain  $\Pi\Delta$ -cycles as well, already for  $d = 2$ . For example, 2-graph  $\Pi$ , which has a  $\Pi\Delta$ -cycle  $C$ , can be extended to an  $A$ -graph (also called bull-graph), which has the CIS property and still contains  $C$ .*

In contrast, the  $\Pi$ - and  $\Delta$ -free  $d$ -graphs are characterized by the absence of the  $\Pi\Delta$ -cycles.

**Theorem 9** *A  $d$ -graph contains a  $\Pi$  or  $\Delta$  if and only if it contains a  $\Pi\Delta$ -cycle.*

This statement implies Theorem 7, since each non-CIS  $d$ -graph contains a  $\Pi\Delta$ -cycle.

**Proof** of the theorem. The "only if part" follows from Lemma 2. To prove the "if part", let us assume indirectly that a  $\Pi$ - and  $\Delta$ -free  $d$ -graph  $\mathcal{G} = (V; E_1, \dots, E_d)$  contains a  $\Pi\Delta$ -cycle  $C$ . Without loss of generality, we can also assume that  $C$  is a *shortest*  $\Pi\Delta$ -cycle in *all*  $\Pi$ - and  $\Delta$ -free  $d$ -graphs.

**Lemma 3** *An edge  $(v_{i_{j-1}}, v_{i_j})$  and diagonal  $(v_{i_r}, v_{i_j})$  in  $C$  are colored differently, while edge  $(v_{i_r}, v_{i_{j-1}})$  is colored with one of these two colors, that is,*

$$(v_{i_{j-1}}, v_{i_j}) \in E_m \text{ and } (v_{i_r}, v_{i_j}) \in E_\ell \Rightarrow m \neq \ell \text{ and } (v_{i_r}, v_{i_{j-1}}) \in E_m \cup E_\ell.$$

**Proof** Indeed, if  $m = \ell$  then a  $\Pi\Delta$ -cycle shorter than  $C$  can be constructed in  $\mathcal{G}$ .

Furthermore, if  $(v_{i_r}, v_{i_{j-1}}) \notin E_m \cup E_\ell$  then three vertices  $v_{i_{j-1}}, v_{i_j}$  and  $v_{i_r}$  form a  $\Delta$ .  $\square$

By Lemma 1, any cycle  $C$  whose edges are colored with distinct colors contains a  $\Delta$ .

Hence, we can assume that a  $\Pi\Delta$  cycle  $C$  contains two edges of the same color.

Let  $D$  be the distance between such a pair of edges.

We consider the cases  $D = 0, D = 1$ , and  $D \geq 2$  and in each case obtain a contradiction.

**Lemma 4** *Any two successive edges of a  $\Pi\Delta$ -cycle are colored with distinct colors.*

**Proof** It follows immediately from the definition if we set  $i_r = i_{s-1}$ .  $\square$

**Lemma 5** *Any two edges of  $C$  at distance 1 are colored with distinct colors too, that is,*

$$(v_{i_{j-1}}, v_{i_j}) \in E_m, (v_{i_{j+1}}, v_{i_{j+2}}) \in E_{m'} \Rightarrow m \neq m'.$$

**Proof** Assume indirectly that  $m = m'$  and let  $(v_{i_j}, v_{i_{j+1}}) \in E_\ell$ . As we already know,  $\ell \neq m$ . Then, by Lemma 3, we conclude that the considered four successive vertices form a  $\Pi$ , since

$$(v_{i_{j-1}}, v_{i_j}), (v_{i_j}, v_{i_{j+2}}), (v_{i_{j+2}}, v_{i_{j+1}}) \in E_m; (v_{i_j}, v_{i_{j+1}}), (v_{i_{j+1}}, v_{i_{j-1}}), (v_{i_{j-1}}, v_{i_{j+2}}) \in E_\ell. \quad \square$$

Finally, it remains to consider two remote edges of the same color.

**Lemma 6** *Let  $(v_{i_{r-1}}, v_{i_r}), (v_{i_{s-1}}, v_{i_s}) \in E_m$  be two edges (of the same color  $m$ ) in  $C$  at distance at least 2 then all four diagonals are colored with the same color  $\ell$ , which is distinct from  $m$ , that is,  $(v_{i_r}, v_{i_s}), (v_{i_r}, v_{i_{s-1}}), (v_{i_{r-1}}, v_{i_s}), (v_{i_{r-1}}, v_{i_{s-1}}) \in E_\ell$  for some  $\ell \neq m$ .*

Now, we can finish the proof as follows. Let us merge the above two pairs of vertices,  $v_{i_r}$  with  $v_{i_{r-1}}$  and  $v_{i_s}$  with  $v_{i_{s-1}}$ , assuming that the obtained two vertices  $v'_{i_r}$  and  $v'_{i_s}$  belong to  $S_{i_{r-1}}$  and  $S_{i_{s-1}}$ , respectively. By this operation, we also merge four edges listed in Lemma 6. Since all four are of the same color  $\ell$ , let us color the obtained edge by color  $\ell$  too, that is,  $(v'_{i_r}, v'_{i_s}) \in E_\ell$ . The above operations result in a reduced  $d$ -graph  $\mathcal{G}'$  and cycle  $C'$  in it which is shorter than  $C$  by two edges.

It is not difficult to verify that, by construction,  $C'$  is a  $\Pi\Delta$ -cycle in  $\mathcal{G}'$ . Furthermore,  $\mathcal{G}'$  is a  $\Pi$ - and  $\Delta$ -free  $d$ -graph, since  $\mathcal{G}$  has this property, by assumption, and  $\mathcal{G}$  contains a  $\Pi$  or  $\Delta$  whenever  $\mathcal{G}'$  does. The last claim is obvious when  $\mathcal{G}'$  does not contain the new edge  $(v'_{i_r}, v'_{i_s})$  and also easy to verify when it does. Thus,  $C'$  is a  $\Pi\Delta$ -cycle shorter than  $C$  in a  $\Pi$ - and  $\Delta$ -free  $d$ -graph  $\mathcal{G}'$ , in contradiction to our assumption.  $\square$

## 8.5 $\Delta$ -conjecture

Although many CIS  $d$ -graphs contain a  $\Pi$ , yet, it seems that they cannot contain a  $\Delta$ .

**$\Delta$ -Conjecture** ([20], page 71; remark after Claim 17). **Each CIS  $d$ -graph is a Gallai  $d$ -graph; or in other words, no CIS  $d$ -graph contains a  $\Delta$ .**

Several partial results in this direction are obtained in [1]; in particular,  $\Delta$ -conjecture for an arbitrary  $d$  is reduced to the case  $d = 3$ .

It is also shown in [1] (Sections 1.6, 1.7, and 4) that, modulo  $\Delta$ -conjecture, the problem of characterizing the CIS  $d$ -graphs can be reduced to the case  $d = 2$ , that is, to characterization of the CIS graphs. Let us remark, however, that case  $d = 2$  is still very difficult [11, 12, 1].

The above reduction is based on the general concept of modular decomposition applied to the  $\Delta$ -free  $d$ -graphs; [1, 3, 2, 7, 8, 9, 13, 14, 21, 22, 24, 25].

## 9 On $\ell$ -CIS $d$ -graphs

Now, let us extend the concept of CIS  $d$ -graph as follows.

Let  $\ell = (\ell_1, \dots, \ell_d)$  be a positive integer vector. A  $d$ -graph  $\mathcal{G} = (V; E_1, \dots, E_d)$  will be called an  $\ell$ -CIS  $d$ -graph if for each  $i \in [d] = \{1, \dots, d\}$  there is an completely  $\ell_i$ -clique-maximal hypergraph  $\mathcal{H}_i$  whose co-occurrence graph is  $\overline{G}_i$  (hence, without loss of generality, we can assume that  $\ell_i \geq 2$ ) and such that  $\bigcap_{i=1}^d H_i \neq \emptyset$  for every edge-selection  $\{H_i \in \mathcal{H}_i \mid i \in [d]\}$ . Obviously, the  $\ell$ -CIS  $d$ -graphs turn into the standard CIS  $d$ -graphs when  $\ell = (n, \dots, n)$  and  $n = |V|$ . In this case, all  $\mathcal{H}_i$  are completely clique-maximal hypergraphs. In general, it is not difficult to demonstrate (just by copying the proof of Proposition 1) that all  $\mathcal{H}_i$  are clique-maximal hypergraphs whenever  $\mathcal{G}$  is an  $\ell$ -CIS  $d$ -graph. Furthermore, copying case analysis from [1], it is also easy to verify that

**$\ell$ -CIS  $d$ -graphs are exactly closed under substitution.**

Hence, the  $\Delta$ -free (Gallai)  $\ell$ -CIS  $d$ -graphs can be reduced to  $\ell$ -CIS 2-graphs (that is, graphs) by modular decomposition, in accordance with [1, 21]; see also [3, 7, 8, 9].

However,  $\Delta$ -conjecture does not extend to the case  $d = 3$  and  $\ell = (2, 2, 2)$  (or even  $\ell = (2, 2, 5)$ ). The next example was constructed by Andrey Gol'berg (1954 - 1985) in 1984.

**Example 3** *Let us consider the 3-graph  $\mathcal{G}$  on nine vertices  $V = \{v_0, v_1, \dots, v_8\}$  in Figure 2, where solid (dotted) lines are colored by color 3 (respectively, 2), and each edge between  $\{v_1, v_2, v_3, v_4\}$  and  $\{v_5, v_6, v_7, v_8\}$  is of color 1.*

*It is easy to verify that  $\mathcal{G}$  contains eight  $\Delta$ s induced by the vertex-sets*

$$(v_0, v_1, v_6), (v_0, v_1, v_7), (v_0, v_4, v_6), (v_0, v_4, v_7), (v_0, v_2, v_5), (v_0, v_2, v_8), (v_0, v_3, v_5), (v_0, v_3, v_8).$$

*Let us consider the following three hypergraphs:  $\mathcal{H}_1 = \{(v_0, v_1, v_2, v_3, v_4), (v_0, v_5, v_6, v_7, v_8)\}$ ;*

$$\mathcal{H}_2 = \{(v_0, v_2, v_3, v_6, v_7), (v_1, v_2, v_5, v_6), (v_1, v_2, v_7, v_8), (v_3, v_4, v_5, v_6), (v_3, v_4, v_7, v_8)\};$$

$$\mathcal{H}_3 = \{(v_0, v_1, v_4, v_5, v_8), (v_1, v_3, v_5, v_7), (v_1, v_3, v_6, v_8), (v_2, v_4, v_5, v_7), (v_2, v_4, v_6, v_8)\}.$$

*It is also easy to verify that:*

- (a) their co-occurrence graphs are  $\overline{G}_1, \overline{G}_2,$  and  $\overline{G}_3,$  respectively;
- (b)  $\mathcal{H}_1$  is completely clique-maximal, while  $\mathcal{H}_2$  and  $\mathcal{H}_3$  are not, more precisely, they are completely 2-clique-maximal but not completely 3-clique-maximal; indeed, set  $\{v_1, v_4, v_6, v_8\}$  is a 4-clique of  $\overline{G}_3,$  every its 2-subset is contained in an edge of  $\mathcal{H}_3,$  while the 3-subset  $\{v_1, v_4, v_6\}$  is already not;
- (c)  $H_1 \cap H_2 \cap H_3 \neq \emptyset$  (in fact,  $|H_1 \cap H_2 \cap H_3| = 1$ ) for every  $H_1 \in \mathcal{H}_1, H_2 \in \mathcal{H}_2, H_3 \in \mathcal{H}_3.$

The corresponding  $2 \times 5 \times 5$  intersection table is given below.

$v_4$	$v_4$	$v_2$	$v_2$	$v_2$	$v_6$	$v_8$	$v_6$	$v_6$	$v_8$
$v_4$	$v_4$	$v_2$	$v_2$	$v_2$	$v_5$	$v_7$	$v_7$	$v_5$	$v_7$
$v_4$	$v_4$	$v_0$	$v_1$	$v_1$	$v_5$	$v_8$	$v_0$	$v_5$	$v_8$
$v_3$	$v_3$	$v_3$	$v_1$	$v_1$	$v_6$	$v_8$	$v_6$	$v_6$	$v_8$
$v_3$	$v_3$	$v_3$	$v_1$	$v_1$	$v_5$	$v_7$	$v_7$	$v_5$	$v_7$

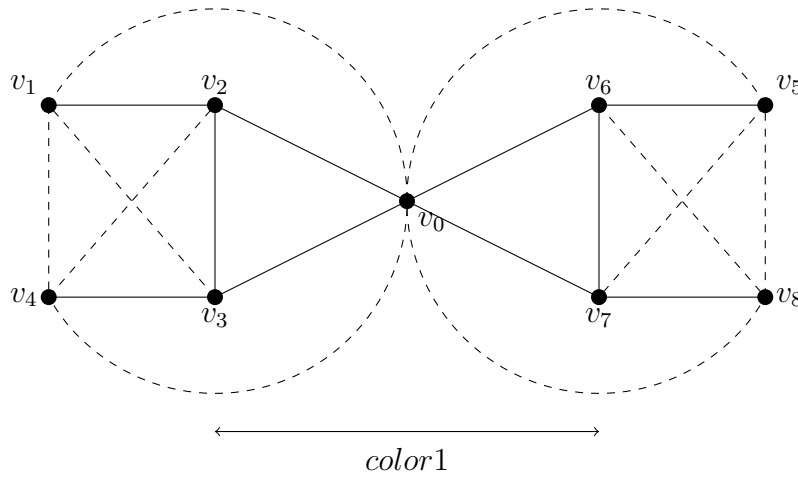


Figure 2: A  $(2, 2, 5)$ -CIS 3-graph that contains eight  $\Delta$ s

This table represents a 3-dimensional box-partition with many interesting properties [26].

As we just mentioned, the hypergraphs  $\mathcal{H}_2$  and  $\mathcal{H}_3$  are clique-maximal but not completely clique-maximal. Their completely clique-maximal extensions are

$$\mathcal{H}'_2 = \mathcal{H}_2 \cup \{(v_1, v_2, v_6, v_7), (v_3, v_4, v_6, v_7), (v_2, v_3, v_5, v_6), (v_2, v_3, v_7, v_8)\}$$

$$\mathcal{H}'_3 = \mathcal{H}_3 \cup \{(v_1, v_4, v_5, v_7), (v_1, v_4, v_6, v_8), (v_1, v_3, v_5, v_8), (v_2, v_4, v_5, v_8)\}.$$

However, for the triplet  $\mathcal{H}_1, \mathcal{H}'_2,$  and  $\mathcal{H}'_3$  the intersection property fails. For example,

$$\{v_0, v_5, v_6, v_7, v_8\} \cap \{v_1, v_2, v_6, v_7\} \cap \{v_1, v_3, v_5, v_8\} = \emptyset.$$

Hence, there is no contradiction with the "standard"  $(n, n, n)$ -CIS  $\Delta$ -conjecture.

This example shows, in particular, that a  $(2, 2, 5)$ -CIS 3-graph can contain a  $\Delta$ , while an  $(n, n, n)$ -one cannot, if  $\Delta$ -conjecture holds.

In general, one can ask for which  $\ell$ , if any, the  $\ell$ -CIS  $d$ -graphs contain no  $\Delta$ .

## 10 On Gallai $d$ -graphs and complete, normal, and solid $d$ -dimensional box-partitions

The obtained intersection table  $g : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \rightarrow \{v_0, v_1, \dots, v_8\}$  represents a box-partition of the total  $2 \times 5 \times 5$  box  $\mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$  into nine boxes  $\{v_0, v_1, \dots, v_8\}$ . Let us notice that the first five boxes in this box-partition are *solid*, that is, the corresponding edges got successive numbers in the given edge-enumeration of hypergraphs  $\mathcal{H}_1$ ,  $\mathcal{H}_2$ , and  $\mathcal{H}_3$ , while the last four boxes are *not* solid. It is easy to verify that there is no enumeration of edges in these three hypergraphs such that *all* boxes are solid.

In fact, each  $d$ -graph generated by a  $d$ -dimensional solid box partition is a *Gallai  $d$ -graphs*.

Let us give more details. A collection of  $d$  hypergraphs  $\mathcal{H} = \{\mathcal{H}_i \subseteq 2^V \mid i \in [d] = \{1, \dots, d\}\}$  defined on a common vertex-set  $V$  will be called a CIS collection (or we will say that it has the CIS property) if

$$\left| \bigcap_{i=1}^d H_i \right| = 1 \text{ for all edge-selections } \mathbf{H} = \{H_i \in \mathcal{H}_i \mid i \in [d]\}.$$

Without any loss of generality, we assume that each vertex  $v \in V$  is realized as the edge-intersection  $v = \bigcap_{i=1}^d H_i$  of such an edge-selection  $\mathbf{H}$ ; indeed, all other vertices can be just removed.

Let us consider a mapping  $g : \mathcal{H} \rightarrow V$  that assigns the intersection-vertex  $v = v(\mathbf{H}) = \bigcap_{i=1}^d H_i$  to every edge-selection  $\mathbf{H} = \{H_i \in \mathcal{H}_i \mid i \in [d]\}$  of a CIS collection  $\mathcal{H}$  of hypergraphs.

Alternatively, this mapping  $g$  can be interpreted as a *box-partition* in which every vertex  $v \in V$  is a box. To each such box-partition  $g$  we will assign a  $(d+1)$ -graph  $\mathcal{G} = \mathcal{G}(g) = (V, E_0, E_1, \dots, E_d)$  as follows. For every two distinct vertices  $v, v' \in V$ , let us define a subset  $s(v, v') \subseteq [d]$  by the condition:

$$i \notin s(v, v') \text{ if and only if } v, v' \in H_i \text{ for an edge } H_i \in \mathcal{H}_i.$$

Obviously,  $s(v, v') = \emptyset$  means that  $\mathcal{H}$  is not a CIS collection (and  $g(\mathcal{H})$  is not a box-partition), since boxes  $v$  and  $v'$  intersect. Further,  $|s(v, v')| = 1$ , say,  $s(v, v') = \{i\} \in [d]$  if and only if projections *of the interiors* of boxes  $v$  and  $v'$  in the direction  $i$  intersect. In this case let  $(v, v') \in E_i$  in  $\mathcal{G}$ . Finally,  $|s(v, v')| > 1$  if and only if projections *of the interiors* of  $v$  and  $v'$  intersect in no direction  $i \in [d]$ . In this case let  $(v, v') \in E_0$  in  $\mathcal{G}$ . By this rule, to each box-partition  $g : \mathcal{H} \rightarrow V$  a  $(d+1)$ -graph  $\mathcal{G}(g) = (V; E_0, E_1, \dots, E_d)$  is assigned. It is not difficult to verify that  $\mathcal{G}(g)$  contains a  $\Delta$  whenever  $E_0 \neq \emptyset$ .

Furthermore, a box-partition  $g = g(\mathcal{H})$  will be called:



- (i) *complete* if  $E_0 = \emptyset$ , or in other words, if for each  $v, v' \in V$  there is a direction  $i \in [d]$  such that projections of the interiors of boxes  $v$  and  $v'$  in this direction intersect;
- (ii) *normal* if  $g$  is complete and all  $d$  hypergraphs of  $\mathcal{H}$  are completely clique-maximal; or in other words, if for every direction  $i \in [d]$  the following Helly property holds: projections, in the direction  $i$ , of the interiors of a family of boxes intersect whenever they are pairwise intersect;
- (iii) *Gallai's* if  $g$  is complete and the corresponding  $d$ -graph  $\mathcal{G}(g)$  is  $\Delta$ -free;
- (iv) *solid* if there is an enumeration of the edges in each of the  $d$  hypergraphs of  $\mathcal{H}$  such that all boxes of the box-partition  $g$  are solid.

As we know, the box-partition  $g$  from Example 3 is complete but not normal, not solid, and not Gallai's. Obviously, the  $\Delta$ -conjecture can be reformulated as follows:

**any normal box-partition is Gallai's.**

Let us remark that every complete and solid box-partition is Gallai's, indeed.

**Theorem 10** *If a box-partition  $g$  is complete and solid then  $d$ -graph  $\mathcal{G}(g)$  contains no  $\Delta$ .  $\square$*

This statement was announced in [18], a proof first appeared in [26].

The result admits a natural geometric interpretation:

**no three solid boxes that induce a  $\Delta$   
can be extended to a complete solid box partition.**

However, Example 3 shows that the similar statement fails if boxes may be not solid.

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