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Optimal Portfolio Selection Based on Multiple Value at Risk Constraints

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Abstract. A variant of Kataoka's portfolio selection model is formulated in which lower bounds are imposed on several VaR values, where the bounds are taken from a reference probability distribution. Under mild assumptions, the problems are formulated as convex nonlinear programming problems, so that the global optimal solution can be found with a nonlinear programming solver. The numerical solution technique will be discussed and numerical examples will be presented.

1 Introduction

The portfolio selection problem in finance has been studied extensively since the well-known model by Markowitz ([2],[3]) was proposed. Suppose we hold n assets with fractions of investments $x_1, \dots, x_n$ and returns on investments $R_1, \dots, R_n$, respectively. Note that each R_i is a random variable. Let us denote

$$R = (R_1, \dots, R_n)^T, \quad x = (x_1, \dots, x_n)^T.$$

Let us denote μ the mean of R and C the covariance matrix of R:

$$\begin{aligned}
&\mu = (\mu_1, \dots, \mu_n)^T, \quad \mu_i = E[R_i] \text{ for } i \in \{1, \dots, n\},\\
&C = [c_{ik}] = E[(R-\mu)(R-\mu)^T], \quad c_{ik} = E[(R_i - \mu_i)(R_k - \mu_k)],\\
&(R-\mu)(R-\mu)^T = \begin{pmatrix} (R_1-\mu_1)(R_1-\mu_1) & \cdots & (R_1-\mu_1)(R_n-\mu_n) \\ \vdots & \ddots & \vdots \\ (R_n-\mu_n)(R_1-\mu_1) & \cdots & (R_n-\mu_n)(R_n-\mu_n) \end{pmatrix}.
\end{aligned}$$

Markowitz's model for portfolio selection is formulated in three different ways. The first model considers the maximization of the expected return given an upper bound on the variance of return. The second model considers the minimization of the variance of return given a lower bound on the expected return. The third model is the combination of the two, which is formulated as follows:

$$\begin{aligned}
&\text{Markowitz's model}\\
&\text{maximize} \quad \mu^T x - \beta x^T C x && (1)\\
&\text{subject to} \quad \sum_{i=1}^{n} x_i = 1, \quad x \geq 0. && (2)
\end{aligned}$$

$\beta > 0$ is a constant. The model considers the maximization of an estimated lower side profit, which is the expected return minus the amount affected by the variance of return.

Kataoka's model for portfolio selection ([1]), which is similar to the Markowitz's third model, is formulated as follows:

$$\begin{aligned}
&\text{Kataoka's model}\\
&(P_{\mathrm{K}}) \text{ maximize} \quad d && (3)\\
&\qquad \text{subject to} \quad \Pr\left(R^T x \geq d\right) \geq p && (4)\\
&\qquad \qquad \sum_{i=1}^{n} x_i = 1, \quad x \geq 0. && (5)
\end{aligned}$$

$p \approx 1$ is a fixed probability chosen by ourselves, e.g. $p = 0.95$. The model considers the maximization of the lower bound of the return which occurs with a high probability.

Let $\varphi(x)$ and $\Phi(x)$ denote the probability density function and the cumulative distribution function of the standard normal distribution, respectively:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right), \quad \Phi(x) = \int_{-\infty}^{x} \varphi(z)dz.$$

Suppose $R_1, \dots, R_n$ have joint normal distribution. We have the following theorem.

Theorem 1. *The set of x vectors satisfying*

$$\Pr\left(R^T x \geq d\right) \geq p \quad \text{where } 0 < p < 1$$

is the same as that satisfying

$$\mu^T x + \Phi^{-1}(1-p)\sqrt{x^T C x} \geq d.$$

Hence a new form of Kataoka's problem is as follows:

$$\begin{aligned}
(P'_{\mathrm{K}}) \quad & \text{maximize} \quad d && (6)\\
& \text{subject to} \quad \mu^T x + \Phi^{-1}(1-p)\sqrt{x^T C x} \geq d && (7)\\
& \sum_{i=1}^{n} x_i = 1 && (8)\\
& x \geq 0. && (9)
\end{aligned}$$

This can be further formulated as follows:

$$\begin{aligned}
(P''_{\mathrm{K}}) \quad & \text{maximize} \quad \mu^T x + \Phi^{-1}(1-p)\sqrt{x^T C x} && (10)\\
& \text{subject to} \quad \sum_{i=1}^{n} x_i = 1 && (11)\\
& x \geq 0. && (12)
\end{aligned}$$

The Value at Risk, or VaR, is a widely used risk measure of the risk on a portfolio of assets. The VaR of confidence level $100p$ % of a random variable X (meaning revenue), denoted by $\mathrm{VaR}_p(X)$, is defined as the optimum value of the following optimization problem:

$$\begin{aligned}
& \text{maximize} \quad v && (13)\\
& \text{subject to} \quad \Pr(X \geq v) \geq p && (14)
\end{aligned}$$

If $X = R^T x$ is the total return of a portfolio where random returns R has joint normal distribution, then from Theorem 1 we can easily derive that

$$\mathrm{VaR}_p(R^T x) = \mu^T x + \Phi^{-1}(1-p)\sqrt{x^T C x}. \tag{15}$$

Note that this is the same as the objective function value of (P''_{K}). And it is also the $(1-p)$-quantile of $R^T x$.

2 Model I

In Kataoka's model (P'_K) we impose a condition (7) for the VaR. Now we handle two quantiles and optimize with respect to one of them while imposing a constraint for the other. Since we set $p \approx 1$, we want the p-quantile of $R^T x$ to be as large as possible and at the same time we want the $\mathrm{VaR}(R^T x)_p$ ($(1-p)$-quantile) to be not too small. So in connection with problem (P'_K) we consider a new problem as follows:

$$
\begin{aligned}
(P_2) \quad \text{maximize} \quad & \mathrm{VaR}_{1-p}(R^T x) = \mu^T x + \Phi^{-1}(p)\sqrt{x^T C x} && (16)\\
\text{subject to} \quad & \mathrm{VaR}_p(R^T x) = \mu^T x + \Phi^{-1}(1-p)\sqrt{x^T C x} \geq d && (17)\\
& \sum_{i=1}^{n} x_i = 1 && (18)\\
& x \geq 0. && (19)
\end{aligned}
$$

The value d is a lower bound on the VaR at confidence level p chosen by ourselves. C is supposed to be a positive definite matrix. Since $\sqrt{xCx}$ is convex, $\Phi^{-1}(p) > 0$, and $\mu^T x$ is convex, it follows that (16) is a convex function. Since $\Phi^{-1}(1-p) < 0$, we have that $\Phi^{-1}(1-p)\sqrt{x^T C x}$ is concave. So the left-hand side of (17) is a concave function, and the set satisfying (17) is convex. The sets satisfying (18) and (19) are both convex. Therefore the feasible set of problem (P_2) is convex. We have a convex feasible set and a convex objective function to be maximized. The optimal solution of (P_2) is attained on the boundary of the feasible set. In case of the optimal solution we have two cases with respect to the strict inequality or the equality of (17).

Suppose that (17) holds with strict inequality. Then the optimal solution is the same as that of the following problem:

$$
\begin{aligned}
\text{maximize} \quad & \mu^T x + \Phi^{-1}(p)\sqrt{x^T C x} && (20)\\
\text{subject to} \quad & \sum_{i=1}^{n} x_i = 1 && (21)\\
& x \geq 0. && (22)
\end{aligned}
$$

The optimal solution of this problem is attained at one of the unit vectors $e_1, \ldots, e_n$ and the optimum value is

$$
\max_{i \in \{1,\ldots,n\}} \left\{ \mu^T e_i + \Phi^{-1}(p)\sqrt{e_i^T C e_i} \right\} = \max_{i \in \{1,\ldots,n\}} \left\{ \mu_i + \Phi^{-1}(p) c_{ii} \right\}.
$$

Note that we have to check the strict inequality for (17) for the optimal solution. The optimal solution of this case is undesirable in practice and should be ignored.

Suppose that (17) holds with equality:

$$
\mu^T x + \Phi^{-1}(1-p)\sqrt{x^T C x} = d. \tag{23}
$$

$$
\Rightarrow \sqrt{x^T C x} = \frac{d - \mu^T x}{\Phi^{-1}(1-p)}. \tag{24}
$$

Plugging (24) into (16), we have the objective function as follows:

$$\mu^T x + \frac{\Phi^{-1}(p)}{\Phi^{-1}(1-p)}(d - \mu^T x) = 2\mu^T x - d \quad (\because \Phi^{-1}(1-p) = -\Phi^{-1}(p))$$

Under the assumption that (23) holds at optimality, we can reformulate problem (P_2) as

$$\begin{aligned} (P_2') \quad & \text{maximize} \quad && \mu^T x && (25)\\ & \text{subject to} \quad && \mu^T x + \Phi^{-1}(1-p)\sqrt{x^T C x} \geq d && (26)\\ & && \sum_{i=1}^{n} x_i = 1 && (27)\\ & && x \geq 0. && (28) \end{aligned}$$

The objective function is linear and the feasible set is convex. Thus we can find the optimum value of this problem using nonlinear programming. Note that we have to check the equality (23) for the optimal solution of (P_2').

3 Model II

Now we generalize the previous model further. We handle multiple quantiles and optimize with respect to one of them while imposing constraints for the others. Suppose we are given $r \geq 2$ reference quantiles $d_1, \ldots, d_r$ corresponding to the probabilities $p_1, \ldots, p_r$, respectively. We want to find x that satisfies the inequalities:

$$\Pr(R^T x \geq d_i) \geq p_i \quad \text{for } i \in \{1, \ldots, r\}. \tag{29}$$

By Theorem 1, (29) are equivalent to

$$\mathrm{VaR}_{p_i}(R^T x) = \mu^T x + \Phi^{-1}(1-p_i)\sqrt{x^T C x} \geq d_i \quad \text{for } i \in \{1, \ldots, r\}. \tag{30}$$

We select one index $h \in \{1, \ldots, r\}$ and consider a maximization problem with respect to the equation for h. Without loss of generality we may assume $h = 1$. We consider the following problem:

$$\begin{aligned} (P_3) \quad & \text{maximize} \quad && \mathrm{VaR}_{p_1}(R^T x) = \mu^T x + \Phi^{-1}(1-p_1)\sqrt{x^T C x} && (31)\\ & \text{subject to} \quad && \mathrm{VaR}_{p_i}(R^T x) = \mu^T x + \Phi^{-1}(1-p_i)\sqrt{x^T C x} \geq d_i \ \text{ for } i \in \{1, \ldots, r\} && (32)\\ & && \sum_{i=1}^{n} x_i = 1 && (33)\\ & && x \geq 0. && (34) \end{aligned}$$

If $p_i \geq 1/2$ for all $i \in \{1, \ldots, r\}$, then the objective function (31) is concave and the feasible set satisfying (32)-(34) is convex as in the argument of the previous section. We can find the optimum value of this problem using nonlinear programming.

Generally we may want to have any number of p_i's as $p_i \geq 1/2$ and any number of p_i's as $p_i \leq 1/2$. But in practice it is enough to have only one p_i where $p_i \leq 1/2$ and the rests are $p_i \geq 1/2$. Suppose p_i satisfy:

$$p_i \geq 1/2 \text{ for all } i \in \{1, \ldots, r-1\} \text{ but } p_r \leq 1/2. \tag{35}$$

Then the feasible set satisfying (32)-(34) is not convex anymore. However, in case of the optimal solution we have two cases.

(C_1) $\mu^T x + \Phi^{-1}(1-p_r)\sqrt{x^T C x} > d_r$.
In this case, we can remove the constraint for $i = r$ in (32) so that the problem reduces to the following problem (P_{3A}) where $p_i \geq 1/2$ for all $i \in \{1, \ldots, r-1\}$ and thus the feasible set is convex.

$$\begin{aligned}
(P_{3A}) \quad & \text{maximize} \quad && \mu^T x + \Phi^{-1}(1-p_1)\sqrt{x^T C x} && (36)\\
& \text{subject to} \quad && \mu^T x + \Phi^{-1}(1-p_i)\sqrt{x^T C x} \geq d_i \quad \text{for } i \in \{1, \ldots, r-1\} && (37)\\
& && \sum_{i=1}^{n} x_i = 1 && (38)\\
& && x \geq 0. && (39)
\end{aligned}$$

Note that we have to check the inequality (C_1) for the optimal solution of (P_{3A}).

(C_2) $\mu^T x + \Phi^{-1}(1-p_r)\sqrt{x^T C x} = d_r$.
In this case, we can plug $\sqrt{x^T C x} = (d_r - \mu^T x)/\Phi^{-1}(1-p_r)$ into (31)-(34) and we get the following problem:

$$\begin{aligned}
\text{maximize} \quad & \left(1 - \frac{\Phi^{-1}(1-p_1)}{\Phi^{-1}(1-p_r)}\right)\mu^T x + \frac{\Phi^{-1}(1-p_1)}{\Phi^{-1}(1-p_r)} d_r\\
\text{subject to} \quad & \mu^T x + \frac{\Phi^{-1}(1-p_i)}{\Phi^{-1}(1-p_r)}(d_r - \mu^T x) \geq d_i \quad \text{for } i \in \{1, \ldots, r-1\}\\
& \mu^T x + \Phi^{-1}(1-p_r)\sqrt{x^T C x} \leq d_r\\
& \sum_{i=1}^{n} x_i = 1\\
& x \geq 0.
\end{aligned}$$

This can be further reformulated as the following problem (P_{3B}). The objective function (40) is linear. The constraints (41),(43), and (44) are linear. Since $p_r < 1/2$ the constraint (42) is convex. Thus the feasible set is convex.

$$
\begin{aligned}
(P_{3B}) \quad & \text{maximize} \quad \mu^T x && (40)\\
& \text{subject to} \quad \mu^T x \geq \frac{d_i + K_i d_r}{1 + K_i} \quad \text{for } i \in \{1, \ldots, r-1\} && (41)\\
& \mu^T x + \Phi^{-1}(1 - p_r)\sqrt{x^T C x} \leq d_r && (42)\\
& \sum_{i=1}^{n} x_i = 1 && (43)\\
& x \geq 0. && (44)
\end{aligned}
$$

Here we have defined

$$K_i := -\Phi^{-1}(1 - p_i)/\Phi^{-1}(1 - p_r) \geq 0 \quad \text{for } i \in \{1, \ldots, r-1\}. \tag{45}$$

Note that we have to check the equality (C_2) for the optimal solution of (P_{3B}).

In order that the optimal solution of (P_{3B}) to be at the equality sign of (42), it is necessary that

$$
\begin{aligned}
& d_r - \Phi^{-1}(1 - p_r)\sqrt{x^T C x} \geq \frac{d_i + K_i d_r}{1 + K_i} \quad \text{for } i \in \{1, \ldots, r-1\}.\\
& \therefore d_r - \frac{d_i + K_i d_r}{1 + K_i} \geq \Phi^{-1}(1 - p_r)\sqrt{x^T C x} \geq 0. \quad \text{for } i \in \{1, \ldots, r-1\}\\
& \therefore d_r \geq d_i \quad \text{for } i \in \{1, \ldots, r-1\}. && (46)
\end{aligned}
$$

We can solve each of the two problems (P_{3A}) and (P_{3B}), and the solution that has larger objective value is the optimal solution.

4 Numerical Example

4.1 Example 1

In this example, we use the data in Tables 1 and 2 for μ and C taken by [6].

INSTRUMENT	S&P	Gov. Bond	Small Cap
MEAN RETURN	0.0101110	0.0043532	0.0137058

Table 1: Mean return μ for Example 1

The computational result is shown in Tables 3 and 4. In the solutions of Problem (P_4) and (P_5) at Table 4, the inequality (C_1) and equality (C_2) are both satisfied. The optimal solution of the original problem is the optimal solution of problem (P_5).

	S&P	Gov. Bond	Small Cap
S&P	0.00324625	0.00022983	0.00420395
Gov. Bond	0.00022983	0.00049937	0.00019247
Small Cap	0.00420395	0.00019247	0.00764097

Table 2: Covariance matrix C for Example 1

Optimum value ($2\mu^T x - d$)	Optimum sol (x)	d
0.0535089	0.659726 0.217319 0.122955	-0.04
0.0655171	0.531042 0.272793 0.196164	-0.05
0.0772562	0.419314 0.321588 0.259098	-0.06
0.0888812	0.315428 0.365837 0.318735	-0.07
0.100445	0.215317 0.408844 0.375838	-0.08

Table 3: Optimal solutions of Problem P_3 with $p = 0.95$ for Example 1

P_4 opt val	P_4 opt sol (x)	P_5 opt val	P_5 opt sol (x)	d_1 / p_1	d_2 / p_2	d_3 / p_3	d_4 / p_4
-0.0134292	0.872726 0.125375 0.001898	-0.00399	0.95206 0.04794 0	-0.03 0.8	-0.02 0.7	-0.01 0.6	0.01 0.3

Table 4: Optimal solutions of Problem P_4 and P_5 for Example 1

4.2 Example 2

In this example, we use the data in Tables 5 and 6 for μ and C taken by [6].

0.77	0.82	0.8	0.62	0.6

Table 5: Mean return μ for Example 2

0.00421276	0.00004712	-0.00080459	-0.00022827	-0.00104229
0.00004712	0.00350952	-0.0011045	0.00172536	-0.00085248
-0.00080459	-0.0011045	0.00254941	-0.00001498	0.00105894
-0.00022827	0.00172536	-0.00001498	0.00173681	-0.00080253
-0.00104229	-0.00085248	0.00105894	-0.00080253	0.0010889

Table 6: Covariance matrix C for Example 2

The computational result is shown in Tables 7 and 8. In the solutions of Problem (P_4) and (P_5) at Table 8, the inequality (C_1) and equality (C_2) are both satisfied. The optimal solution of the original problem is the optimal solution of problem (P_4).

4.3 Example 3

In this example, we use the data in Tables 9 and 10 for μ and C. The data is computed by the annualized daily returns of foreign exchange rates of six currencies against US dollar (the value of one unit of a foreign currency in US dollar) from January 1 to June 30 in 2008. The meaning of the symbols in the table are the following: AUD=Australian Dollar, CAD=Canadian Dollar, CHF=Swiss Franc, EUR=European Euro, GBP=British Pound, and JPY=Japanese Yen.

The computational result is shown in Tables 11 and 12. In the solutions of Problem (P_4) and (P_5) at Table 12, the inequality (C_1) and equality (C_2) are both satisfied. The optimal solution of the original problem is the optimal solution of problem (P_4).

4.4 Code

The AMPL model file for the problem P_3 is the following.

```
set ASSET;
param MU {ASSET}; # mean
param C {ASSET,ASSET}; # covariance marix
param PHI_INV_P; # \Phi^{-1}(p)
param D;
```

Optimum value $(2\mu^T x - d)$	Optimum sol (x)	d
0.856958	0.041162	0.76
	0.485705	
	0.473133	
	0	
	0	
0.878339	0	0.75
	0.708468	
	0.291532	
	0	
	0	
0.907184	0	0.73
	0.92961	
	0.0703901	
	0	
	0	
0.92	0	0.72
	1	
	0	
	0	
	0	

Table 7: Optimal solutions of Problem P_3 with $p = 0.95$ for Example 2

P_4 opt val	P_4 opt sol (x)	P_5 opt val	P_5 opt sol (x)	d_1 / p_1	d_2 / p_2	d_3 / p_3	d_4 / p_4	d_5 / p_5
<u>0.77089</u>	0.08010	0.75779	0.20733	0.50	0.54	0.58	0.60	0.80
	0.45093		0.32272	0.9	0.8	0.7	0.6	0.3
	0.46897		0.40748					
	0		0					
	0		0.06247					

Table 8: Optimal solutions of Problem P_4 and P_5 for Example 2

AUD	0.189553
CAD	-0.054767
CHF	0.207133
EUR	0.143516
GBP	0.000331
JPY	0.118982

Table 9: Mean return μ for Example 3

	AUD	EUR	GBP	CAD	CHF	JPY
AUD	0.009719	0.004404	0.003313	0.004594	0.002253	0.000577
CAD	0.004404	0.007305	0.001093	0.002475	0.002467	-0.000936
CHF	0.003313	0.001093	0.012349	0.007801	0.002737	0.009873
EUR	0.004594	0.002475	0.007801	0.006503	0.002848	0.005011
GBP	0.002253	0.002467	0.002737	0.002848	0.004508	0.001213
JPY	0.000577	-0.000936	0.009873	0.005011	0.001213	0.011690

Table 10: Covariance matrix C for Example 3

Optimum value ($2\mu^T x - d$)	optimal solution (x)		d
0.4125	AUD	0.05013	0.030
	CAD	0	
	CHF	0.94987	
	EUR	0	
	GBP	0	
	JPY	0	
0.39427	AUD	0	0.020
	CAD	0	
	CHF	1	
	EUR	0	
	GBP	0	
	JPY	0	

Table 11: Optimal solutions of Problem P_3 with $p = 0.95$ for Example 3

P_4 opt val	P_4 opt sol (x)		P_5 opt val	P_5 opt sol (x)		d_1 p_1	d_2 p_2	d_3 p_3	d_4 p_4	d_5 p_5
0.08981	AUD	0.51004	0.08389	AUD	0.55928	0.010	0.012	0.014	0.016	0.23
	CAD	0		CAD	0	0.9	0.8	0.7	0.6	0.3
	CHF	0.48995		CHF	0.33037					
	EUR	0		EUR	0					
	GBP	0		GBP	0					
	JPY	0		JPY	0.11035					

Table 12: Optimal solutions of Problem P_4 and P_5 for Example 3

```
var x {ASSET} >= 0;
var mux;
var xCx >= 1d-20;

maximize Max_Portfolio: 2 * mux;

subject to MU_X:
  mux = sum {i in ASSET} MU[i] * x[i];

subject to X_C_X:
  xCx = sum {i in ASSET, j in ASSET} C[i,j]*x[i]*x[j];

subject to Low_Bound_Return:
  mux - PHI_INV_P * sqrt(xCx) >= D;

subject to Sum_Portfolio:
  sum {i in ASSET} x[i] = 1;
```

The AMPL model file for the problem P_4 is the following.

```
set ASSET;
param MU {ASSET}; # mean
param C {ASSET,ASSET}; # covariance marix
set REF_DIST ordered; # reference distribution
param RD_PHI_INV_P {REF_DIST}; # \Phi^{-1}(p_i)
param RD_D {REF_DIST};

# The following two params are not used in this model.
param PHI_INV_P;
param D;
```

```
var x {ASSET} >= 0;
var mux;
var xCx >= 1d-20;

maximize Max_One_Bound:
  mux - RD_PHI_INV_P[first(REF_DIST)]*sqrt(xCx);

subject to MU_X:
  mux = sum {i in ASSET} MU[i] * x[i];

subject to X_C_X:
  xCx = sum {i in ASSET, j in ASSET} C[i,j]*x[i]*x[j];

subject to Lower_Bound_of_Return {k in REF_DIST diff {last(REF_DIST)}}:
  mux - RD_PHI_INV_P[k] * sqrt(xCx) >= RD_D[k];

subject to Sum_Portfolio:
  sum {i in ASSET} x[i] = 1;
```

The AMPL model file for the problem P_5 is the following.

```
set ASSET;
param MU {ASSET}; # mean
param C {ASSET,ASSET}; # covariance marix
set REF_DIST ordered; # reference distribution
param RD_PHI_INV_P {REF_DIST}; # \Phi^{-1}(p_i)
param K {k in REF_DIST} = -RD_PHI_INV_P[k]/RD_PHI_INV_P[last(REF_DIST)];
param RD_D {REF_DIST};

# The following two params are not used in this model.
param PHI_INV_P;
param D;

var x {ASSET} >= 0;
var mux;
var xCx >= 1d-20;

maximize Max_One_Bound: mux;

subject to MU_X:
  mux = sum {i in ASSET} MU[i] * x[i];

subject to X_C_X:
```

```
  xCx = sum {i in ASSET, j in ASSET} C[i,j]*x[i]*x[j];

subject to Lower_Bound_of_Return {k in REF_DIST diff {last(REF_DIST)}}:
  mux >= (RD_D[k] + K[k]*RD_D[last(REF_DIST)])/(1+K[k]);

subject to Upper_Bound_of_Return:
  mux - RD_PHI_INV_P[last(REF_DIST)]*sqrt(xCx) <= RD_D[last(REF_DIST)];

subject to Sum_Portfolio:
  sum {i in ASSET} x[i] = 1;
```

The AMPL data file for Example 1 is the following.

```
# Data file for example 1

set ASSET := SandP GovBond SmallCap; # asset

param: MU := # mean
SandP    0.0101110
GovBond  0.0043532
SmallCap 0.0137058;

param C: SandP      GovBond    SmallCap := # covariance
SandP    0.00324625 0.00022983 0.00420395
GovBond  0.00022983 0.00049937 0.00019247
SmallCap 0.00420395 0.00019247 0.00764097;

param PHI_INV_P:=1.644854; # = \Phi^{-1}(p), p=0.95

param D:=-0.08;

set REF_DIST := 1 2 3 4; # reference distribution

param: RD_D   RD_PHI_INV_P :=
1      -0.03  0.841621     # p=0.8
2      -0.02  0.524401     # p=0.7
3      -0.01  0.253347     # p=0.6
4      0.01   -0.524401;   # p=0.3
```

References

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