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ON NASH EQUILIBRIA AND IMPROVEMENT CYCLES IN PURE POSITIONAL STRATEGIES FOR CHESS-LIKE AND BACKGAMMON-LIKE n-PERSON GAMES [a]

Endre Boros[b] Vladimir Gurvich[c]
Khaled Elbassioni[d] Kazuhisa Makino[e]

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RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

[a]The authors are thankful to DIMACS, Center for Discrete Mathematics and Theoretical Computer Science for partial support

[b]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ 08854-8003; (boros@rutcor.rutgers.edu)

[c]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ 08854-8003; (gurvich@rutcor.rutgers.edu)

[d]Max-Planck-Institute for Informatics; Stuhlsatzenhausweg 85, 66123, Saarbruecken, Germany (elbassio@mpi-sb.mpg.de)

[e]Graduate School of Information Science and Technology, University of Tokyo, Tokyo, 113-8656, Japan; (makino@mist.i.u-tokyo.ac.jp)

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On Nash equilibria and improvement cycles in pure positional strategies for Chess-like and Backgammon-like n-person games [1]

Endre Boros, Khaled Elbassioni, Vladimir Gurvich, Kazuhisa Makino

Abstract. We consider n-person positional games with perfect information modeled by finite directed graphs that may have directed cycles, assuming that all infinite plays form a single outcome a_∞, in addition to the standard outcomes $a_1, \ldots, a_m$ formed by the terminal positions. (For example, in case of Chess or Backgammon $n = 2$ and a_∞ is a draw.) These $m+1$ outcomes are ranked arbitrarily by n players. We study existence of (subgame perfect) Nash equilibria and improvement cycles in pure positional strategies and provide a systematic case analysis in terms of the following conditions:

(i) there are no directed cycles;

(ii) there are no random positions;

(iii) the "infinite outcome" c is ranked as the worst one by all n players;

(iv) $n = 2$;

(v) $n = 2$ and the payoff is zero-sum.

Key words: Back Gammon, Nash equilibrium, subgame perfect, improvement cycle, best reply, stochastic game, perfect information, position, move, random move

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1 Extended abstract

In this article we study Nash-solvability (NS) and improvement acyclicity (AC) in pure positional strategies of n-person finite positional games with perfect information.

Such a game is modeled by a finite directed graph (digraph) $G = (V, E)$ whose vertex-set V is partitioned into $n+2$ subsets $V = V_1 \cup \ldots \cup V_n \cup V_R \cup V_T$, where V_i are the positions of a player $i \in I = \{1, \ldots, n\}$, while V_R and $V_T = \{a_1, \ldots, a_m\}$ are the random and terminal positions. A game is called Chess-like if $V_R = \emptyset$ and Backgammon-like in general.

Digraph G may have directed cycles (di-cycles), yet, we assume that all infinite plays form a single outcome a_∞ (or c), in addition to the terminal outcomes V_T.

The standard Chess and Backgammon are two-person, $n = 2$, zero-sum games, in which a_∞ is defined as a draw. Yet, in a Chess-like game, n players $I = \{1, \ldots, n\}$ may rank $m+1$ outcomes $A = \{a_1, \ldots, a_m, a_\infty\}$ arbitrarily. Furthermore, an arbitrary real-valued *utility* (called also payoff) function $u : I \times A \to \mathbb{R}$ is defined for a Backgammon-like game.

Remark 1 *If $n = 2$ and $m = 1$ then the zero-sum Backgammon-like games turn into the so-called* simple stochastic games*, which were introduced in [12].*

All players are restricted to their pure positional strategies; see Section 2.8.

We shall consider two concepts of solution: the classical Nash equilibrium and also a stronger one, the absence of the (best response) improvement cycles.

The concept of *solvability* is defined as the existence of a solution for *any* utility function u, from a given family. Respectively, we arrive to the notions of *Nash-solvability (NS)* and *improvement acyclicity (AC)*. Both concepts will be considered in two cases: *initialized (I)*, that is, with respect to a fixed initial position $v_0 \in V \setminus V_T$ and *uniform (U)*, that is, for all possible initial positions simultaneously. In the literature, the latter case is frequently referred to as *ergodic* or *subgame perfect*.

Thus, we obtain four concepts: INS, UNS, IAC, UAC with the following implications:

UAC $\Rightarrow$ IAC $\Rightarrow$ INS, UAC $\Rightarrow$ UNS $\Rightarrow$ INS.

Let us remark that UAC and IAC are special cases of the so-called *restricted acyclicity*, which was suggested for trees in [31] and then extended to arbitrary digraphs in [1].

Our main results are given in the Table 1; they summarize criteria of solvability in terms of the following conditions:

(i) there are no directed cycles;

(ii) there are no random positions;

(iii) the "infinite outcome" a_∞ is ranked as the worst one by all n players;

(iv) $n = 2$;

(v) $n = 2$ and the payoff is zero-sum.

INS	IAC	UNS	UAC	games	INS	IAC	UNS	UAC
?	−	−	−	n-person	−	−	−	−
?	−	−	−	n-person and c is the worst	−	−	−	−
+	+	−	−	2-person	−	−	−	−
+	+	−	−	2-person and c is the worst	−	−	−	−
+	+	+	+	2-person zero-sum	+	−	+	−
Chess-like no random moves				↑ ↑ dicycles are possible but all form one outcome	Backgammon-like random moves possible			
+	+	+	+	← no dicycles →	+	+	+	+

Table 1: Main diagram. Notation: NS = Nash-solvability, AC = Acyclicity;
I = Initialized, given initial position, U = Uniform, any initial position.

First, UAC (together with the remaining three properties) hold for the acyclic digraphs. In this case, subgame perfect Nash equilibria exist (UNS) and all of them can be obtained by the standard backward induction [14, 29, 30]; furthermore, UAC (and IAC) follow from Theorem 2 of [1]; see also Section 4.

Then, IAC (and INS) hold for the two-person Chess-like games. In other words, (iv), even without (iii), implies IAC [1] and, in particular, INS [7]. However, these results do not extend neither the non-initialized nor Backgammon-like two-person (but non-zero-sum) games. More precisely, INS (and IAC) (respectively, UNS (and UAC)) may fail, even under assumption (iii), for the two-person initialized Backgammon-like (respectively, non-initialized Chess-like) games; see Section 3.

Remark 2 *Interestingly, in Table 1 lines 1 and 2, as well as lines 3 and 4, are equal. Thus, it is not clear whether assumption (iii) ever matters. Surely, it helps a lot in proving Nash-solvability, since for all players cycling becomes a punishing strategy.*

A more general type of payoffs is considered in [7] and it is shown that in this case assumption (iii) is *essential; see Section 8.*

The classical backward induction algorithm [29, 30, 14] for n-person Backgammon-like games on acyclic digraphs can be modified and applied to two-person zero-sum Chess-like games; in both cases UNS follows [2, 8]. Moreover, UAC holds too; see Sections 4 and 6.

In presence of random positions, IAC (and UAC) fail [12], while UNS (and INS) still hold. The latter result is derived in [8] from the existence of uniformly optimal strategies for the classical two-person zero-sum stochastic games with perfect information and limit mean payoff [16, 33]; see Section 5.

Finally, the fundamental question "whether a game with a fixed initial position and without random moves is Nash-solvable?" (that is, INS for the Chess-like games) remains open. For the two-person case, $n \leq 2$, the answer is positive [7, 1]. Yet, in general, there is no counterexample and no proof, even under assumption (iii). Although, in the latter case some partial results are known:

(j) INS holds when $m = |V_T| \leq 3$, that is, for at most 4 outcomes [10];

(jj) INS also holds for the play-once games, in which each player is in control of at most one position, $|V_i| \leq 1 \ \forall i \in I$ [7].

In [7], INS was conjectured for the Chess-like games satisfying (iii). In fact, this conjecture was extended to a much larger class of games with additive costs; see [7] and Section 8 for the definitions and more details. It is shown in [7] that condition (iii) holds for these games whenever all local costs are not negative and that INS may fail otherwise; in other words, condition (iii) is essential for INS in the considered case unless the conjecture fails.

2 Introduction

2.1 Positional forms

Given a finite digraph $G = (V, E)$ in which loops and multiple arcs are allowed; a vertex $v \in V$ is a *position* and a directed edge (or arc) $e = (v, v') \in E$ is a *move* from v to v'.

A position of out-degree 0 (in other words, with no moves) is called *terminal*. Let $V_T = \{a_1, \ldots, a_m\}$ denote the set of all terminal positions.

Furthermore, let us introduce a set of n players $I = \{1, \ldots, n\}$ and a partition $D : V = V_1 \cup \ldots \cup V_n \cup V_R \cup V_T$, assuming that each player $i \in I$ is in control of all positions of V_i, while V_R is the set of random positions, called also positions of chance. For each $v \in V_R$ a probabilistic distribution over the set of outgoing arcs is fixed.

An *initial* position $v_0 \in V$ may be fixed. The triplet (G, D, v_0) or pair (G, D) is called a *positional game form*, initialized or non-initialized, respectively.

A game form is called Chess-like if there are no positions of chance, $V_R = \emptyset$, and it is called Backgammon-like, in general. Eight examples of non-initialized Chess-like and initialized Backgammon-like game forms are given in Figure 1.

2.2 Plays, outcomes, preferences, and payoffs

Given an initialized game form (G, D, v_0), a *play* is defined as a directed path that begins in v_0 and either ends in a terminal position $a \in V_T$ or infinite.

In this article we assume that All Infinite Plays Form One Outcome a_∞, in addition to the standard Terminal outcomes of V_T. (In [8], this condition was referred to as AIPFOOT.)

Remark 3 *In contrast, in [18, 19, 20, 9] it was assumed that all di-cycles form pairwise distinct outcomes. A criterion of NS was obtained for the two-person bidirected case, that is, when $n = 2$ and $e = (v, v')$ is an arc if and only if $e' = (v', v)$ is.*

A *utility (or payoff)* function is a mapping $u : I \times A \to \mathbb{R}$, whose value $u(i, a)$ is interpreted as a profit of player $i \in I = \{1, \ldots, n\}$ in case of outcome $a \in A = \{a_1, \ldots, a_m, a_\infty\}$.

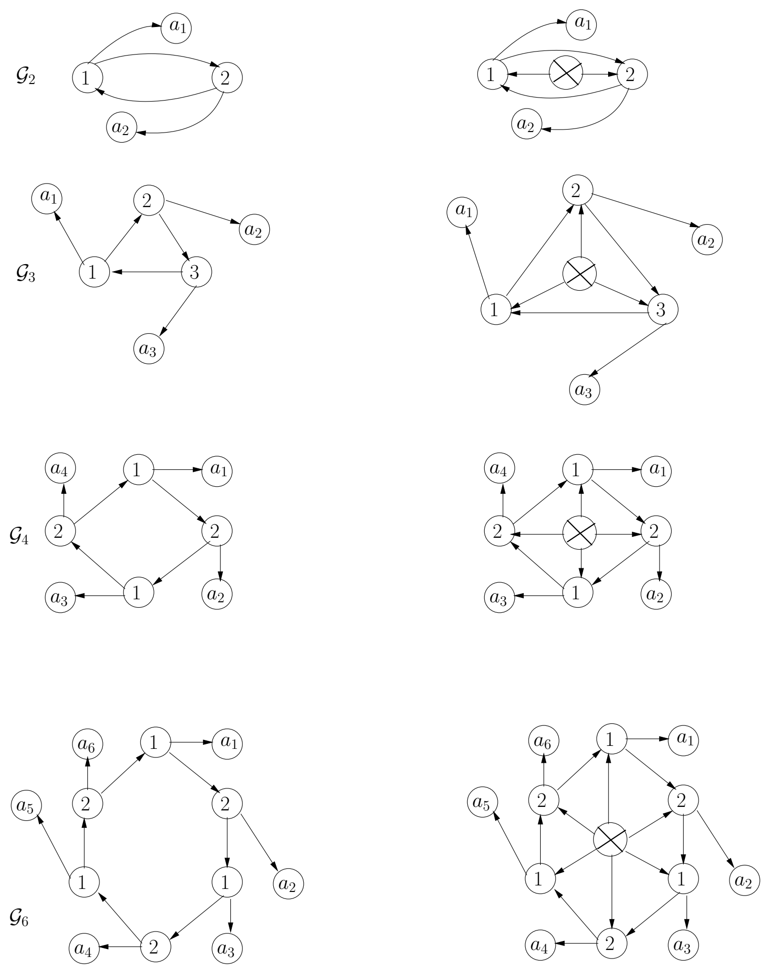


Figure 1: Non-initialized Chess-like and initialized Backgammon-like examples

A payoff is called *zero-sum* if $\sum_{i \in I} u(i, a) = 0$ for every $a \in A$. Two-person zero-sum games will play an important role. For example, standard Chess and Backgammon are two-person zero-sum games in which every infinite play is a draw, $u(1, a_\infty) = u(2, a_\infty)) = 0$.

Another important class of payoffs is defined by the condition $u(i, a_\infty) < u(i, a)$ for all $i \in I$ and $a \in V_T$; in other words, the infinite outcome a_∞ is the worst one for all players. Motivations of this assumption are discussed in [7, 8].

Quadruple (G, P, v_0, u) and triplet (G, P, u) will be called a Backgammon-like (or Chess-like, in case $V_R = \emptyset$) game, initialized and non-initialized, respectively.

For the Chess-like games only n pseudo-orders $o = \{o(u_i) \mid i \in I\}$ over A matter rather than payoffs $u(i, a)$ themselves. Moreover, in this case, ties can be excluded; in other words, we can assume that $o(u_i)$ is a *complete order* over A, called the *preference* of the player i. The set of all preferences $o = \{o(u_i) \mid i \in I\}$ is called a *preference profile*.

In contrast, for the Backgammon-like games payoffs $u_i = u(i, a)$ matter, since their probabilistic combinations are compared.

2.3 Strategies

Given a positional game form (G, D), a *strategy* x_i of a player $i \in I$ is defined as a mapping $x_i : V_i \to E_i$ which assigns to each position $v \in V_i$ a move (v, v') from this position.

Remark 4 *In this article, we restrict ourselves, or, more precisely, the players, to their* pure positional *strategies. In other words, the move (v, v') of a player $i \in I$ in a position $v \in V_i$ is deterministic (not random) and it depends only on the position v itself (not on preceding positions or moves). Every considered strategy is assumed to be pure and also positional unless it is explicitly said otherwise. The latter happens only in Section 2.8, where we consider the strategies with memory and show that all non-positional ones can be excluded.*

Let X_i be the set of all strategies of a player $i \in I$ and $X = \prod_{i \in I} X_i$ be the direct product of these sets. An element $x = (x_1, \ldots, x_n) \in X$ is called a *strategy profile or situation.*

2.4 Normal forms

A positional game form $\mathcal{G}$ can be represented in normal (or strategic) form. We will consider four cases: $\mathcal{G}$ is Chess- or Backgammon-like, initialized or not.

Given a *Chess-like initialized* positional game form $\mathcal{G} = (G, D, v_0)$ and a strategy profile $x \in X$, a play $p(x)$ is naturally and uniquely defined by the following rules: it begins in v_0 and in each position $v \in V_i$ proceeds with the arc (v, v') according to the strategy x_i. Obviously, $p(x)$ either ends in a terminal position $a \in V_T$, or $p(x)$ is infinite.

In the latter case $p(x)$ is a *lasso*, i.e., it consists of an initial part and a di-cycle repeated infinitely. This is because all players are restricted to their positional strategies.

Anyway, an outcome $a = a(x) \in A = \{a_1, \ldots, a_m, c\}$ is assigned to every strategy profile $x \in X$. Thus, a game form $g_{v_0} : X \to A$ is defined. It is called the *normal form* of $\mathcal{G}$.

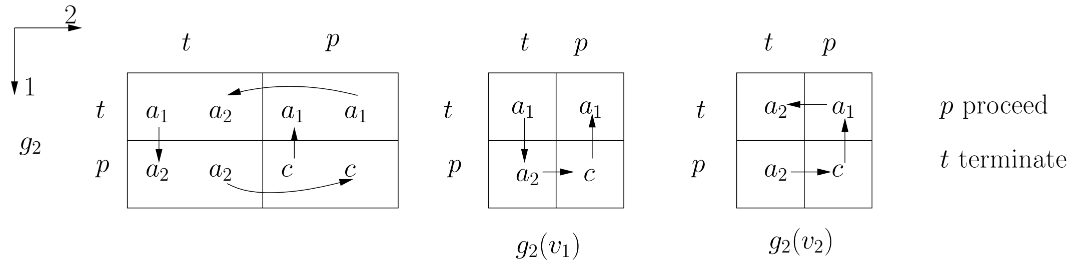
If game form $\mathcal{G} = (G, D)$ is not initialized then we repeat the above construction for every initial position $v_0 \in V \setminus V_T$ to obtain a play $p = p(x, v_0)$, outcome $a = a(x, v_0)$, and mapping $g : X \times (V \setminus V_T) \to A$ which is the *normal form* of $\mathcal{G}$ in this case.

For the Chess-like game forms in Figure 1 their normal forms are in Figures 2 and 3.

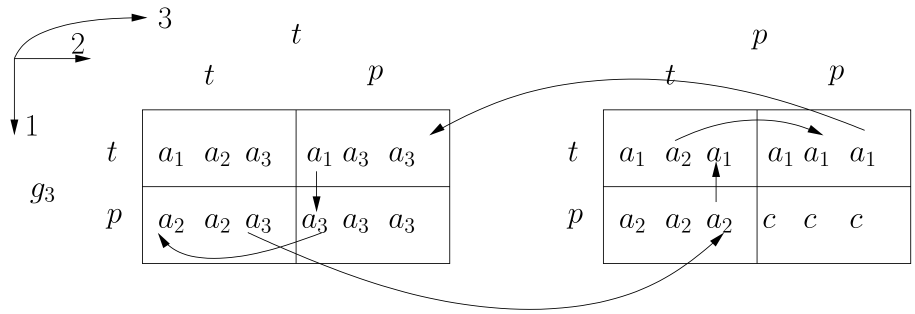
In a Backgammon-like non-initialized game form $\mathcal{G} = (G, D)$, each strategy profile $x \in X$ uniquely defines a Markov chain on V. Given also an initial position $v_0 \in V \setminus V_T$, this chain defines a unique limit probabilistic distribution $q = q(x, v_0)$ on $A = \{a_1, \ldots, a_m, a_\infty\}$, where $q(a) = q(x, v_0, a)$ is the probability to stop at $a \in V_T$, while $q(a_\infty) = q(x, v_0, a_\infty)$ is the probability of an infinite play. Let Q denote the set of all probabilistic distributions over A. The obtained mappings $g_{v_0} : X \to Q$ and $g : X \times (V \setminus V_T) \to Q$ are the *normal game forms* of the above Backgammon-like game forms, initialized and non-initialized, respectively.

Given also a payoff $u : I \times A \to \mathbb{R}$, the normal form games for the above two cases are defined by the pairs (g_{v_0}, u) and (g, u), respectively.

These games can be also represented by real-valued mappings $f_{v_0} : I \times X \to \mathbb{R}$ and $f : I \times X \times (V \setminus V_T) \to \mathbb{R}$. Indeed, let us fix $i \in I$, $x \in X$, and $v_0 \in V \setminus V_T$; having a probabilistic



$$u(1,c) > u(1,a_1) > u(1,a_2);\ \ u(2,a_1) > u(2,a_2) > u(2,c)$$



$$u(1,a_2) > u(1,a_1) > u(1,a_3) > u(1,c);$$
$$u(2,a_3) > u(2,a_2) > u(2,a_1) > u(2,c);$$
$$u(3,a_1) > u(3,a_3) > u(3,a_2) > u(3,c)$$

Figure 2: Normal forms of $\mathcal{G}_2$ and $\mathcal{G}_3$ from Figure 1.

distribution $q(x, v_0)$ over A and and real-valued payoff function $u_i : A \to \mathbb{R}$, where $u_i(a) = u(i, a)$, we compute the expected payoff by formula $f(i, x, v_0) = \sum_{a \in A} q(x, v_0, a) u(i, a)$.

In case of the Chess-like games, Markov chains and the corresponding limit probabilistic distributions $q = q(x, v_0) \in Q$ over A are replaced by just plays and the corresponding outcomes $a = a(x, v_0) \in A$; respectively, $f(i, x, v_0) = u(i, a(x, v_0))$.

2.5 Nash equilibrium

The concept of Nash equilibrium (NE) is defined standardly for the normal form games.

First, let us consider the initialized case. Given $f_{v_0} : I \times X \to \mathbb{R}$, a strategy profile $x \in X$ is called a NE if $f_{v_0}(i, x) \geq f_{v_0}(i, x')$ for each player $i \in I$ and every strategy profile x' that can differ from x only in the ith component. In other words, no player $i \in I$ can profit by choosing a new strategy if all opponents keep their strategies unchanged.

In the non-initialized case, the same property must hold for every $v_0 \in V \setminus V_T$. Given $f : I \times X \times (V \setminus V_T) \to \mathbb{R}$ a strategy profile $x \in X$ is called a *uniform NE* if $f(i, x, v_0) \geq f(i, x', v_0)$ for each $i \in I$, for every x' defined as above, and for all $v_0 \in V \setminus V_T$, too.

g_4

	tt	tp	pt	pp
tt	$a_1\ a_2\ a_3\ a_4$	$a_1\ a_2\ a_3\ a_1$	NE $a_1\ a_3\ a_3\ a_4$	$a_1\ a_3\ a_3\ a_1$
tp	$a_2\ a_2\ a_3\ a_2$	$a_2\ a_2\ a_3\ a_4$	$a_3\ a_3\ a_3\ a_4$	$a_3\ a_3\ a_3\ a_3$
pt	$a_1\ a_2\ a_4\ a_4$	$a_1\ a_2\ a_1\ a_1$	$a_1\ a_4\ a_4\ a_4$	$a_1\ a_1\ a_1\ a_1$
pp	$a_2\ a_2\ a_4\ a_4$	$a_2\ a_2\ a_2\ a_2$ NE	$a_4\ a_4\ a_4\ a_4$	$c\ c\ c\ c$

$o_1:\ a_2 > a_1 > a_3 > a_4 > c, \qquad o_2:\ a_3 > a_2 > a_4 > a_1 > c.$

g_6

	ttt	ttp	tpt	tpp	ptt	ptp	ppt	ppp
ttt	$a_1a_2a_3a_4a_5a_6$	$a_1a_2a_3a_4a_5a_1$	$a_1a_2a_3a_5a_5a_6$	$a_1a_2a_3a_5a_5a_1$	○ $a_1a_3a_3a_4a_5a_6$	■ $a_1a_3a_3a_4a_5a_1$	$a_1a_3a_3a_5a_5a_6$	$a_1a_3a_3a_5a_5a_1$
ttp	$a_1a_2a_3a_4a_6a_6$	$a_1a_2a_3a_4a_1a_1$	$a_1a_2a_3a_6a_6a_6$	$a_1a_2a_3a_1a_1a_1$	■ $a_1a_3a_3a_4a_6a_6$	$a_1a_3a_3a_4a_1a_1$	○ $a_1a_3a_3a_6a_6a_6$	$a_1a_3a_3a_1a_1a_1$
tpt	○ $a_1a_2a_4a_4a_5a_6$	$a_1a_2a_4a_4a_5a_1$	$a_1a_2a_5a_5a_5a_6$	$a_1a_2a_5a_5a_5a_1$	$a_1a_4a_4a_4a_5a_6$	$a_1a_4a_4a_4a_5a_1$	$a_1a_5a_5a_5a_5a_6$	$a_1a_5a_5a_5a_5a_1$
tpp	$a_1a_2a_4a_4a_6a_6$	$a_1a_2a_4a_4a_1a_1$	○ $a_1a_2a_6a_6a_6a_6$	$a_1a_2a_1a_1a_1a_1$	$a_1a_4a_4a_4a_6a_6$	$a_1a_4a_4a_4a_1a_1$	$a_1a_6a_6a_6a_6a_6$	$a_1a_1a_1a_1a_1a_1$
ptt	$a_2a_2a_3a_4a_5a_6$	■ $a_2a_2a_3a_4a_5a_2$	$a_2a_2a_3a_5a_5a_6$	$a_2a_2a_3a_5a_5a_2$	$a_3a_3a_3a_4a_5a_6$	○ $a_3a_3a_3a_4a_5a_3$	$a_3a_3a_3a_5a_5a_6$	$a_3a_3a_3a_5a_5a_3$
ptp	■ $a_2a_2a_3a_4a_6a_6$	$a_2a_2a_3a_4a_2a_2$	$a_2a_2a_3a_6a_6a_6$	$a_2a_2a_3a_2a_2a_2$	$a_3a_3a_3a_4a_6a_6$	$a_3a_3a_3a_4a_3a_3$	$a_3a_3a_3a_6a_6a_6$	○ $a_3a_3a_3a_3a_3a_3$
ppt	$a_2a_2a_4a_4a_5a_6$	○ $a_2a_2a_4a_4a_5a_2$	$a_2a_2a_5a_5a_5a_6$	■ $a_2a_2a_5a_5a_5a_2$	$a_4a_4a_4a_4a_5a_6$	$a_4a_4a_4a_4a_5a_4$	$a_5a_5a_5a_5a_5a_6$	■ $a_5a_5a_5a_5a_5a_5$
ppp	$a_2a_2a_4a_4a_6a_6$	$a_2a_2a_4a_4a_2a_2$	■ $a_2a_2a_6a_6a_6a_6$	○ $a_2a_2a_2a_2a_2a_2$	$a_4a_4a_4a_4a_6a_6$	$a_4a_4a_4a_4a_4a_4$	■ $a_6a_6a_6a_6a_6a_6$	$c\ c\ c\ c\ c\ c$

$o_1:\ a_6 > a_5 > a_2 > a_1 > a_3 > a_4 > c, \qquad o_2:\ a_3 > a_2 > a_6 > a_4 > a_5 > c;\ \ a_6 > a_1 > c.$

Figure 3: Normal forms of $\mathcal{G}_4$ and $\mathcal{G}_6$ from Figure 1.

Remark 5 *In the literature, this concept is also called "a* subgame perfect *NE". This name is justified when the digraph $G = (V, E)$ is acyclic and each vertex $v \in V$ can be reached from v_0. Indeed, in this case (G, D, v, u) is a subgame of (G, D, v_0, u). However, if G has a di-cycle then then any two its vertices v' and v'' can be reached each from the other. Hence, (G, D, v', u) is a subgame of (G, D, v'', u) and vice versa. Thus, the name* a uniform (or ergodic) *NE is more accurate.*

Let us also notice that for a Chess-like game, the inequality $f_{v_0}(i, x) \geq f_{v_0}(i, x')$ is reduced to $u(i, a(v_0, x)) \geq u(i, a(v_0, x'))$, which can be verified whenever a pseudo-order o_i over A is known. It must be verified for all $i \in I$ and for a given (respectively, for each) $v_0 \in V \setminus T$ in the initialized (respectively, non-initialized) case.

2.6 On the UNS of Chess-like and INS of Backgammon-like games

From the above definitions, it follows immediately that UNS implies INS.

The non-initialized Chess-like game forms $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_6$ given in Figure 1 are not UNS. The corresponding payoffs u^2, u^3, u^4, u^6 will be constructed in Section 3. Moreover, u^3 and u^6 will satisfy condition (iii) of Abstract: $a_\infty = c$ is the worst outcome for all players. In contrast, games $(\mathcal{G}_2, u^2)$ and even $(\mathcal{G}_4, u^4)$ have uniform NE whenever u^2 and u^4 satisfy (iii). For this reason, a larger example $(\mathcal{G}_6, u^6)$ is needed to show that UNS may fail for two-person Chess-like games even under assumption (iii).

Let us also note that all obtained initialized game forms are INS for arbitrary initial positions. Moreover, it is an open problem, whether INS holds for all Chess-like games. This problem remains open even if (iii) holds. Yet, it is answered in the affirmative for the two-person games, even without assumption (iii) [7, 1]. Thus, we can fill the first four positions of the INS- and UNS-columns of the Chess-like (left-hand) side of Table 1.

Furthermore, the UNS of the Chess-like games can be reduced to the INS of some special Backgammon-like games as follows. Given a non-initialized Chess-like game $\mathcal{G} = (G, D, u)$, let us add to its graph $G = (V, E)$ an initial position of chance v_0 and a move (v_0, v) from v_0 to *each* position $v \in V \setminus V_T$ and define probabilities $q(v, v')$ so that $\sum_{v \in V \setminus V_T} q(v, v') = 1$; furthermore, denote the obtained probabilistic distribution and the initialized Backgammon-like game by $q(v_0)$ and $\mathcal{G}'$, respectively. By construction, games $\mathcal{G}$ and $\mathcal{G}'$ have the same set X of the strategy profiles.

Theorem 1 *(i) If x is a uniform NE in $\mathcal{G}$ then in $\mathcal{G}'$ it is a NE for every $q(v_0)$.*

(ii) If x is a NE in $\mathcal{G}'$ for a strictly positive $q(v_0)$ then x is a uniform NE in $\mathcal{G}$.

Part (i) is obvious; part (ii) will be proven in Section 7. The next corollary is immediate.

Corollary 1 *The following three claims are equivalent:*

(j) $\mathcal{G}$ is UNS;

(jj) $\mathcal{G}'$ is INS for some strictly positive $q(v_0)$;

(jj') $\mathcal{G}'$ is INS for each strictly positive $q(v_0)$.

This corollary allow us to fill the first four, negative, positions in the columns INS (and UNS) of the Backgammon (right-hand) side of Table 1.

The fifth row of the Table shows that NS holds for the two-person zero-sum games. This will be derived from the classical Gillette [16] theorem; see [8, 23], and also Section 6 below.

Finally, the sixth row shows that UNS holds for the acyclic digraphs. In this case all uniform NE can be obtained by backward induction; see [14, 29, 30] and Section 4 below.

2.7 Uniform and lazy best responses

Again, let us start with the initialized case. Given the normal form $f_{v_0} : I \times X \to \mathbb{R}$ of an initialized Backgammon-like game, a player $i \in I$, and a pair of strategy profiles x, x' such that x' may differ from x only in the ith component, we say that x' *improves* x (for the player i) if $f_{v_0}(i, x) < f_{v_0}(i, x')$. Let us underline that the inequality is strict.

By this definition, situation x is a NE if and only if it can be improved for no $i \in I$.

Let us fix a player $i \in I$ and a situation $x = \{x_i, i \in I\}$. A strategy $x_i^* \in X_i$ is called a best response (BR) of i in x, if $f_{v_0}(i, x^*) \geq f_{v_0}(i, x')$ for any x', where x^* and x' are both obtained from x by replacement of its ith component x_i by x_i^* and x_i', respectively.

A BR x_i^* is not necessarily unique but the corresponding best achievable value $f_{v_0}(i, x^*)$ is, of course, unique. Moreover, somewhat surprisingly, such best values can be achieved by a BR x_i^* simultaneously for all initial positions $v_0 \in V \setminus V_T$.

Theorem 2 *Let us fix a player $i \in I$ and situation $x \in X$ in the normal form $f : I \times X \times (V \setminus V_T) \to \mathbb{R}$ of a non-initialized Backgammon-like game. Then there is a (pure positional) strategy $x_i^* \in X_i$ that is a BR of i in x for* all *initial positions $v_0 \in V \setminus T$ simultaneously.*

We will give a proof in Section 5, by viewing a Backgammon-like game as a stochastic game with perfect information with a (very special) limit mean payoff. Then, search for a BR in a given situation $x \in X$ is reduced to a controlled Markov chain problem. It is well-known that in this case a uniformly optimal (pure) strategy exists and can be found by linear programming; see, for example, [4, 34].

For the Chess-like games, Theorem 2 can be proven by the following, much simpler, arguments. Given a non-initialized Chess-like game $\mathcal{G} = (G, D, u)$, a player $i \in I$, and a strategy profile $x \in X$. First, in every position $v \in V \setminus (V_i \cup V_T)$ let us fix a move (v, v') in accordance with x and delete all other moves.

Then, let us order A according to the preference $u_i = u(i, *)$. Let $a^1 \in A$ be a best outcome. (Note that there might be several such outcomes and also that $a^1 = c$ might hold.) Let V^1 be the set of positions from which player i can reach a^1; in particular, $a^1 \in V^1$. Let us fix corresponding moves in $V^1 \cap V_i$. By definition, there are no moves to a^1 from $V \setminus V^1$. Moreover, if $a^1 = c$ then player i can reach no di-cycle beginning from $V \setminus V^1$; in particular, the induced digraph $G_1 = G[V \setminus V^1]$ contains no di-cycles. Then, let us consider an outcome a^2 that is the best for i in A, except maybe a^1, and repeat the same arguments as above for G_1 and a^2, etc. This procedure will result in a uniformly best response x_i^* of i in x. Indeed, obviously, the chosen moves of player i are optimal independently of v_0. □

Remark 6 *Let us mention that a uniformly BR exists and it can be found by the same procedure when the di-cycles form not one but several distinct outcomes, like in [18, 19, 20, 9].*

Let x_i^* be a BR of the player i in a situation $x = (x_1, \ldots, x_i, \ldots, x_n)$. It may happen that x_i and x_i^* recommend two distinct moves, $v, v')$ and (v, v''), in a position $v \in V_i$, although the original strategy x_i is already optimal with respect to v, that is, $a(i, x, v)$ is the (unique) optimal outcome for the initial position v; in other words, the considered BR changes an optimal move in v, which is not necessary.

A BR is called *lazy* if this happens for no $v \in V_i$.

A similar definition was introduced in [1] for the initialized Chess-like games.

In general, for the Backgammon-like games, a BR will be called *lazy* if the subset V_i in which x_i and x_i^* recommend the same move cannot be strictly increased.

Remark 7 *In this paper, we will not discuss how to verify the above condition. Yet, we will need it only for the Backgammon-like games with acyclic digraph, in which case the problem becomes simple. It is also simple for the Chess-like game on arbitrary digraphs.*

2.8 Best response and Nash equilibria in strategies with memory

In this paper, all players are restricted to their positional strategies. To justify this restriction, in this subsection we eliminate it and show that the result of the game will not change.

Let (G, D, v_0) be an initialized game form. A sequence of positions $d = \{v_0, v_1, \ldots, v_k\}$ such that $(v_{j-1}, v_j) \in E$ is a move for $j = 1, \ldots, k$ is called a *debut*. Let us notice that the same position may occur in d several times.

A *general strategy* (or strategy with memory) x_i of a player $i \in I$ is a mapping that assigns a move (v, v') to every debut $d = \{v_0, v_1, \ldots, v_k\}$ in which $v_k = v \in V_i$.

Let us recall that strategy x_i is *positional* if in each position $v \in V_i$ the chosen move depends only on v but not on preceding positions of the debut. Let X_i and X'_i denote the sets of all positional and general strategies of a player $i \in I$; by definition, $X_i \subseteq X'_i$. Furthermore, let $X = \prod_{i \in I} X_i$ and $X' = \prod_{i \in I} X'_i$ be the sets of general and positional strategy profiles; clearly, $X \subseteq X'$.

Now, we can strengthen Theorem 2 by simply noting that the positional strategy x^*_i which is a uniform BR of i in $x \in X$ is a uniform BR of i in $x \in X$ with respect to X' as well. The proof remains exactly the same for both Chess- and Backgammon-like games.

The new version of Theorem 2 immediately implies that each (uniform) NE in positional strategies is a general (uniform) NE; in other words, if $x^* \in X$ is a (uniform) NE with respect to X then it is a (uniform) NE with respect to X', as well. The claim holds for both initialized and non-initialized cases and shows that non-positional strategies can be ignored.

2.9 Improvement cycles and acyclicity

Let $f_{v_0} : I \times X \to \mathbb{R}$ (respectively, $f : I \times X \times (V \setminus V_T) \to \mathbb{R}$) be the normal form of an initialized (respectively, non-initialized) Backgammon-like game. A sequence $x^1, \ldots, x^k \in X$ of strategy profiles is called a *best response improvement cycle*, or an *im-cycle*, for short, if x^{j+1} is a uniform lazy BR improvement of x^j corresponding to the given initial position v_0 (respectively, to $v_0 = v_0(j) \in V$) by the player $i = i(j) \in I$, for $j = 1, \ldots, k$. Standardly the indices are taken modulo k, that is, $k + 1 = 1$ is assumed.

Let us underline that in both, initialized and non-initialized, cases we restrict ourselves by the *uniform and lazy BR* improvements.

Remark 8 *Kukushkin was the first who considered im-cycles in the initialized positional games [31]. He restricted himself to the trees and demonstrated that im-cycles may exist already in very simple two-person zero-sum games (with only four outcomes). To reduce the set of im-cycles he suggested the concept of the so-called* restricted acyclicity*, by which players are allowed to change their moves only in positions within the actual play. Making use of this restriction, he obtained several interesting criteria of restricted im-acyclicity for* *trees. In [1], the concept and results were extended to arbitrary finite digraphs and IAC was proven for the two-person and acyclic cases; see the last two lines of Table 1.*

In this article, we restrict ourselves, and the players, to the uniform lazy best response *improvements, in both non-initialized and initialized cases. The obtained IAC and UAC concepts look like a lucky alternative to Kukushkin's restricted acyclicity. In many cases, the latter is automatically implied by our "uniform and lazy BR" condition.*

A game is called *im-acyclic* (AC) if it has no im-cycles. Thus, the concepts of IAC and UAC appear for the initialized and non-initialized games. Obviously, UAC implies IAC. Indeed, an initialized im-cycle is also a non-initialized one, but not vice versa. The inverse holds only if all k improvements correspond to one initial position v_0, while in a non-initialized im-cycle they may correspond to different $v_0 = v_0(j) \in V$; see g_2 Figure 2.

The implications (IAC $\Rightarrow$ INS) and (UAC $\Rightarrow$ UNS) also follow immediately from the above definitions.

Examples of im-cycles will be given in Section 3 for non-initialized Chess-like games; see Figure 1. These examples allow us to fill with minuses the first four positions of the UAC-column in the left-hand side of Table 1. Furthermore, we refer to [1] to fill the IAC-column. Indeed, Theorems 2 and 3 of [1] imply that IAC holds for the acyclic and two-person cases; see also Section 4 and 6, respectively.

Now let us consider the concepts of IAC and UAC for the Backgammon-like games, that is, the right-hand-side of Table 1. UAC (and IAC) hold in case of the acyclic digraphs; see [1] and also Section 4. Yet, in presence of di-cycles, IAC (and UAC) fail in all five cases. For the first four, it follows from Theorem 1 and examples of Section 3, while for the two-person zero-sum games the corresponding example was constructed in 1993 by Anne Condon in [12].

2.10 On Nash-solvability of bidirected cyclic two-person games

For completeness, let us mention one more result on Nash-solvability obtained in [9].

Let payoff $u : I \times (C \cup V_T) \to \mathbb{R}$ be an arbitrary function, where $C = C(G)$ denotes the set of dicycles of G. In this model, every dicycle $c \in C$, as well as each terminal $a \in V_T$, is a *separate* outcome, in contrast to the AIPFOOT case.

A digraph G is called *bidirected* if (v, v') is its edge whenever (v', v) is.

Necessary and sufficient conditions of Nash-solvability were announced in [18, 19, 20] for the bidirected *bipartite* cyclic two-person game forms. Recently, it was shown that the bipartiteness is in fact irrelevant. Necessary and sufficient conditions for Nash-solvability of bidirected cyclic two-person game forms were shown in [9].

3 NE-free games with a unique di-cycle

Let us consider four non-initialized Chess-like game forms $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_6$ given in Figure 1. Each digraph $G_j = (V_j, E_j)$ consists of a unique di-cycle C_j of length j and a matching connecting each vertex v_k^j of C_j to a terminal a_k^j, where $k = 1, \dots, j$ and $j = 2, 3, 4, 6$. Graphs

G_2, G_4 and G_6 are bipartite; in other words, $\mathcal{G}_2, \mathcal{G}_4$ and $\mathcal{G}_6$ are two-person game forms in which two players move in turn; $\mathcal{G}_3$ is a three-person game form; $\mathcal{G}_2$ and $\mathcal{G}_3$ are play-once, that is, each player controls a unique position. In each non-terminal position v_k^j there are only two moves; one of them immediately terminates in a_k^j, while the other proceeds along the di-cycle. Thus, in $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_6$ each player has $2, 2, 4, 8$ strategies, respectively. The corresponding normal non-initialized game forms g_2, g_3, g_4, g_6 of size $2 \times 2, 2 \times 2 \times 2, 4 \times 4$, and 8×8 are given in Figures 2 and 3.

Let us start with g_2 and consider the preference profile $o_1 : c > a_1 > a_2,\ o_2 : a_1 > a_2 > c$; in other words, player 1 likes and 2 dislikes c the most, while both prefer a_1 to a_2. It is easily seen that all four situations of g_2 form an im-cycle; hence, none of them is a uniform NE.

However, INS holds for this example; in other words, if an initial position, v_1 or v_2, is fixed then the obtained initialized game form has a NE for every payoff.

Both game forms, $g_2(v_1)$ and $g_2(v_2)$, are given in Figure 1. It is easy to see that the considered im-cycle is broken in each. Thus, the obtained non-initialized Chess-like game is not UNS, although, it is INS for each initial position.

In positional form, this example appeared in [1], Figure 10; see also Figure 1 of [8].

In addition to the above counterclockwise im-cycle, the clockwise one is defined by the preference profile $o_1 : a_2 > a_1 > c,\ o_2 : c > a_2 > a_1$. Obviously, there can be no more im-cycles in a 2×2 game form. In both cases, the preferences o_1 and o_2 are not opposite and outcome c is the best for one player and worst for the other.

Based on these simple observations, one could boldly conjecture that UNS holds for the two-person non-initialized Chess-like games whenever:

(A) they are zero-sum or (B) a di-cycle c is the worst outcome for both players.

Conjecture (A) holds, indeed; moreover, not only for the Chess-like but even for the Backgammon-like games; see Section 6. However, (B), unfortunately, fails. Although, $\mathcal{G}_2$ (and even $\mathcal{G}_4$) are too small to provide a counterexample, yet, $(\mathcal{G}_6, o)$ has no uniform NE whenever the preferences $o = (o_1, o_2)$ satisfy:

$$o_1 :\ a_6 > a_5 > a_2 > a_1 > a_3 > a_4 > c,\ \ o_2 :\ a_3 > a_2 > a_6 > a_4 > a_5 > c,\ a_6 > a_1 > c.$$

To verify this, let us consider the normal form g_2 in Figure 2. By Theorem 2, we determine the best responses of player 2 for all strategies of player 1 and vice versa. It is easy to check that the obtained two sets of the best responses (white circles and black squares in Figure 2) are disjoint. Hence, there is no uniform NE. Furthermore, the obtained 16 best responses induce an im-cycle of length 10 and two im-paths of lengths 2 and 4 that end in this im-cycle.

Remark 9 *Im-cycles exist already in g_4; one of them is shown in Figure 3 for the preferences* $o_1 :\ a_2 > a_1 > a_3 > a_4 > c,\ \ o_2 :\ a_3 > a_2 > a_4 > a_1 > c.$

However, a uniform NE also exists in this case. Moreover, by a tedious case analysis, one can show that $\mathcal{G}_4$ (and, of course, $\mathcal{G}_2$) are UNS, unlike $\mathcal{G}_6$.

For $n = 3$ a simpler counterexample to (B) is known [7]. Let us consider the game defined by $\mathcal{G}_3$ and the following preference profile $o = (o_1, o_2, o_3)$:

$o_1 : a_2 > a_1 > a_3 > c, \; o_2 : a_3 > a_2 > a_1 > c, \; o_3 : a_1 > a_3 > a_2 > c.$

In other words, for each player $i \in I = \{1, 2, 3\}$ to terminate is an average outcome; it is better (worse) when the next (previous) player terminates; finally, if nobody does then the di-cycle c appears, which is the worst outcome for all. The considered game has an im-cycle of length 6, which is shown in Figure 2. Indeed, let player 1 terminates at a_1, while 2 and 3 proceed. The corresponding situation (a_1, a_1, a_1) can be improved by 2 to (a_1, a_2, a_1), which in its turn can be improved by 1 to (a_2, a_2, a_2). Let us repeat the same procedure two times more to obtain the im-cycle of length six shown in Figure 2.

The remaining two situations, (a_1, a_2, a_3) and (c, c, c) appear when all three players terminate or proceed, respectively. They are not NE either. Moreover, each of them can be improved by every player. Thus, there is no uniform NE in the obtained game.

Finally, let us recall that, by Theorem 1, we can assign an initialized Backgammon-like not INS game to each non-initialized Chess-like not UNS game; see Figure 1.

The Backgammon-like not INS game corresponding to the last example appeared in [7].

4 Backward induction in case of acyclic digraphs

Let us show that UAC (together with the remaining three properties UNS, IAC, and INS) hold if digraph G has no di-cycles. The proof will be indirect. Let game $\mathcal{G} = (G, D, u)$ be a minimal counterexample, that is, $\mathcal{G}$ has an im-cycle $Y \subseteq X$, digraph $G = (V, E)$ has no di-cycle, and there is no similar example $\mathcal{G}'$ with a digraph G' smaller than G.

By minimality of G, for every move $e = (v, v') \in E$ of a player i (that is, $v \in V_i$) there are two strategy profiles $x, x' \in Y$ such that e is chosen by x_i and not chosen by x'_i. Indeed, otherwise we can eliminate e from G (if e is not chosen by any strategy of Y) or all other arcs from v (if e is chosen by all strategies of Y), and Y remains an im-cycle in the reduced game $\mathcal{G}'$, as well.

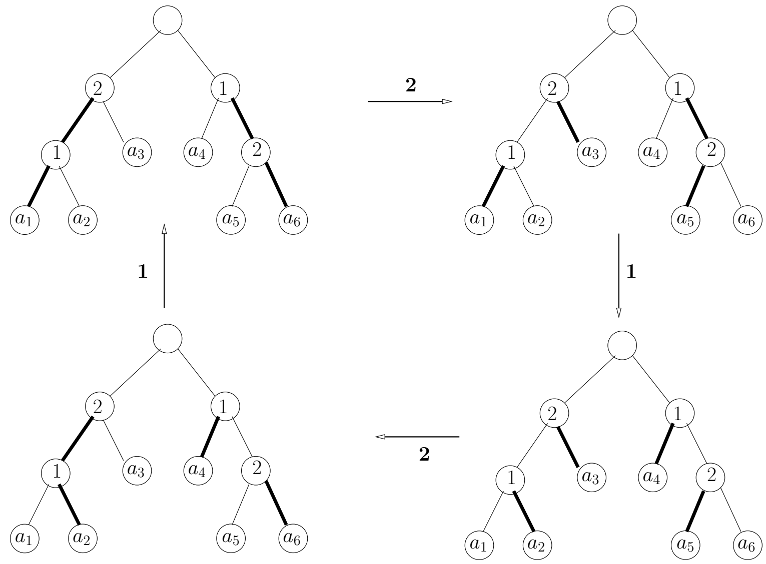
By acyclicity of G, there is a position $v \in V \setminus V_T$ such that each move (v, v') from it leads to a terminal position $v' \in V_T$. Let us choose such a position v, delete all arcs (v, v') from G, and denote the obtained reduced graph by G' and the corresponding game by $\mathcal{G}'$. By construction, v is a terminal position in G'. Let us consider two cases:

Case 1: $v \in V_R$ is a position of chance. Then, for each player $i \in I$, let us define the terminal payoff $u(i, v)$ as an average of $u(i, v')$ for all moves (v, v') from v in G.

Case 2: $v \in V_i$ is a private position of a player $i \in I$. Then, $u(i, v)$ will be defined as the maximum of $u(i, v')$ for all moves (v, v') from v. Let such a maximum be realized in v'_0. Then $u(i', v) = u(i', v'_0)$ for all $i' \in I$. Obviously, every strategy profile of an im-cycle will chose in v a move (v, v'_0). Such a move may be not unique, yet, by the laziness condition, it will remain unchanged in the considered im-cycle.

In both cases, games $\mathcal{G}$ and $\mathcal{G}'$ are obviously equivalent. In particular, the im-cycle in $\mathcal{G}$ remains an im-cycle in $\mathcal{G}'$, in contradiction with the minimality of G.

In fact, the above arguments result in a stronger and more constructive statement.



$u(1, a_1) = u(1, a_2);\ u(1, a_6) > u(1, a_4) > u(1, a_5)$

$u(2, a_5) = u(2, a_6);\ u(2, a_2) > u(2, a_3) > u(2, a_1)$

Figure 4: Laziness assumption is essential for UAC.

Let us choose successively the pre-terminal positions $v \in V \setminus V_T$ and make them terminal, as shown above. Furthermore, we will fix an optimal move, maximizing $u(i, *)$, whenever $v \in V_i$ is a private position. This recursive procedure is known as *backward induction*. It was suggested in the early fifties by Kuhn [29, 30] and Gale [14]. Clearly, in $k = |V \setminus V_T|$ steps it results in a strategy profile x^*, which is called a backward induction (or sophisticated) equilibrium (BIE). It is also clear that a BIE is a uniform NE and vice versa.

Furthermore, a Chess-like game has a unique BIE whenever payoffs $u(i, *)$ have no ties.

The above arguments also show that any sequence of lazy BR improvements (an im-path) cannot cycle; moreover, it terminates in a BIE whenever cannot be extended. In particular, UAC holds if G is an acyclic digraph.

Remark 10 *It is a curious combinatorial problem to bound the length of a shortest (or longest) im-path from a situation x to a BIE x^*. Yet, both problems hardly have a practical value, since it is much easier to obtain a BIE x^* by backward induction (in exactly $|V \setminus V_T|$ steps) rather than by constructing an im-path from a situation x.*

Remark 11 *Let us note that the laziness assumption is essential. Indeed, otherwise, an im-cycle may appear, as the example in Figure 4 shows. In this example, payoffs have ties. However, even if the terminals make no ties, they still may appear in the backward induction procedure, in presence of random positions.*

Finally, let us note that backward induction (and even INS) may fail in presence of a di-cycle; see examples of Section 3. However, a modified backward induction works for the two-person zero-sum Chess-like games; see Section 6.

5 Mean payoff n-person games

Let us recall the model of Section 2 and introduce the following modifications. Given a non-initialized game form (G, D), where $G = (V, E)$ is a digraph and $D : V = V_1 \cup \ldots \cup V_n \cup V_R \cup V_T$ is a partition of its positions, we will assume that there are no terminals, $V_T = \emptyset$. Instead, we introduce a local reward function $r : I \times E \to \mathbb{R}$, whose value $r(i, e)$ is standardly interpreted as a profit obtained by the player $i \in I$ whenever the play passes the arc $e \in E$.

A strategy profile $x \in X$ uniquely defines a Markov chain on V. Whenever we fix also an initial position $v_0 \in V$, this chain defines a unique limit probabilistic distribution $q = q(x, v_0)$ on V, that is, $q(v) = q(x, v_0, v) = \lim_{t\to\infty} t(v)/t$, where $t(v)$ is the expected part of the total time t that the play spends in v. Respectively, for a move $e = (v, v')$ in a random position $v \in V_R$, we obtain the limit probability $q(e) = q(x, v_0, e) = q(v)q(v, v')$, where $q(v, v')$ is the probability of the considered move (v, v') in v.

Then, the effective limit payoff (or in other words, the game in normal form) is defined as a real-valued function $f : I \times X \times (V \setminus V_T) \to \mathbb{R}$, where $f(i, x, v_0) = \sum_{e\in E} q(x, v_0, e)r(i, e)$.

The deterministic case, $V_R = \emptyset$, is also of interest. In this case, given $x \in X$ and $v_0 \in V$, we obtain a play $p = p(x, v_0)$, which can be viewed as a deterministic Markov chain. Since all players are restricted to their positional strategies, $p(x)$ is a lasso, which consists of an initial part and a di-cycle $C = C(x, v_0)$ in G repeated infinitely. Then, $q(e) = q(x, v_0, e) = |C|^{-1}$ for $e \in C$ and $q(e) = 0$ for $e \notin C$.

In the literature on mean payoff games usually the two-person zero-sum are considered, since not much is known in other cases. The most important statement is as follows.

Theorem 3 *Two-person zero-sum mean payoff games are UNS; in other words, every such game has a saddle point (NE) in uniformly optimal strategies.* □

This is just a reformulation of the fundamental result of [16, 33]. For the two-person zero-sum case, the above model (first introduced in [24]) and the classical Gillette model [16] are equivalent [8, 6].

However, IAC (and UAC) may fail for the two-person zero-sum mean payoff games even in absence of random positions. Let us consider the complete bipartite 2 by 2 digraph G on four vertices $v_1, v_3 \in V_1$ and $v_2, v_4 \in V_2$; see Figure 5. Let the local rewards of player 1 be

$$r(1, (v_1, v_2)) = r(1, (v_2, v_1)) = r(1, (v_3, v_4)) = r(1, (v_4, v_3)) = 1,$$
$$r(1, (v_1, v_4)) = r(1, (v_4, v_1)) = r(1, (v_2, v_3)) = r(1, (v_3, v_2)) = -1,$$

while $r(2, e) + r(1, e) \equiv 0$ for all arcs $e \in E$, since the game is zero-sum.

The corresponding normal form is of size 4×4.

It is not difficult to verify that the strategy profile $x = (x_1, x_2)$ defined by the strategies $x_1 = \{(v_1, v_2), (v_3, v_4)\}$ and $x_2 = \{(v_2, v_3), (v_4, v_1)\}$
is a uniform saddle point, which results in the di-cycle (v_1, v_2, v_3, v_4); see Figure 5.

Yet, the following four strategy profiles define an im-cycle
$x^1 = (x'_1, x'_2),\ x^2 = (x'_1, x''_2),\ x^3 = (x''_1, x''_2),\ x^4 = (x''_1, x'_2)$, where

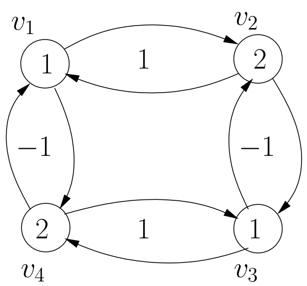


Figure 5: Im-cycle in a mean-payoff game.

$x_1' = \{(v_1, v_2), (v_3, v_2)\}, x_1'' = \{(v_1, v_4), (v_3, v_4)\}, x_2' = \{(v_2, v_1), (v_4, v_1)\}, x_2'' = \{(v_2, v_3), (v_4, v_3)\}$, which is invariant with respect to all possible initial positions.

Remark 12 *Moreover, it is not difficult to verify that the BR property holds but laziness condition fails for the above im-cycle.*

It is an open general question whether BR lazy im-cycles exist for mean-payoff games. If not then obviously every BR lazy im-path results in a saddle point.

It is also known that INS (and UNS) may fail for a two-person (but not zero-sum) mean payoff game, even in absence of random positions. In other words, the zero-sum assumption is essential in Theorem 3. The corresponding example was given in [21]; see also [24, 8].

Let G be the complete bipartite 3 by 3 digraph and the local rewards of players 1 and 2 on its 18 arcs be given by the following two 3×3 real matrices:

$$\begin{matrix} 0 & 0 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \end{matrix} \qquad\qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-\varepsilon & 0 & 1 \end{matrix}$$

where $r(i, (v, v')) = r(i, (v', v))$ for each player $i \in I = \{1, 2\}$ and every pair of oppositely directed arcs $(v, v'), (v', v)$, while ε is a small positive number, say, 0.1.

Let us fix an arbitrary initial position, construct the corresponding normal form of size $3^3 \times 3^3 = 27 \times 27$, and choose the BR of player 1 in every row and BR of player 2 in every column. It was shown in [21] that the obtained two sets of dicycles are disjoint. Hence, there is no NE. In [22], it was shown that the above example is minimal, since INS holds for the games on 2 by k digraphs.

Now let us demonstrate that the n-person Backgammon-like games form a very special subclass of the n-person mean payoff games [8]. The reduction is simple.

Let $\mathcal{G} = (G, D, u)$ be a non-initialized Backgammon-like n-person game.

First, without any loss of generality, we can assume that $u(i, a_\infty) = 0$ for all $i \in I$. Indeed, to enforce this condition it is sufficient to subtract $u(i, a_\infty)$ from the payoff $u(i, *)$ of each player $i \in I$. More precisely, given a payoff $u : I \times A \to \mathbb{R}$ let us define a payoff

$u' : I \times A \to \mathbb{R}$ by formula $u'(i,a) = u(i,a) - u(i,a_\infty)$ for all $i \in I,\ a \in V_T$, and denote the obtained game by $\mathcal{G}' = (G, D, u')$.

Then, let us get rid of V_T by adding a loop $e_v = (v,v)$ to every terminal position $v \in V_T$. Furthermore, let us set the local reward $r(i,e_v) = u'(i,v)$ for each such loop and $r(i,e) \equiv 0$ for all other arcs $e \in E$ and all players $i \in I$. Thus, we obtain a non-initialized mean payoff game $\mathcal{G}'' = G', D', r)$. The following statements are obvious:

- all three games $\mathcal{G}, \mathcal{G}'$, and $\mathcal{G}''$ are equivalent;
- game $\mathcal{G}$ is Chess-like iff $\mathcal{G}''$ (and $\mathcal{G}'$) are deterministic iff $V_R = \emptyset$;
- all rewards $r(i,e)$ are positive (non-negative) in $\mathcal{G}''$ if and only if a_∞ is the worst (maybe, not strictly) outcome for all players $i \in I$ of $\mathcal{G}$;
- games $\mathcal{G}, \mathcal{G}'$, and $\mathcal{G}''$ are two-person and/or zero-sum simultaneously.

6 Two-person (zero-sum) games

Let us recall that UNS (and UAC) may fail for the two-person Chess-like games, even when c is the worst outcome for all players; see examples g_2 and g_6 of Section 3. Respectively, by Theorem 1, INS may fail for the corresponding Backgammon-like games.

In contrast, IAC (and INS) hold for the Chess-like two-person games, not necessarily zero-sum. This follows from Theorem 2 of [1]; for INS see also [7, 8].

Remark 13 *It it is a fundamental open problem whether INS still holds for the n-person Chess-like games; see Section 8. It is known that UNS may fail already when $n = 3$ and c is the worst outcome for all three players; see example g_3 of Section 3. For IAC, a similar example is known [1] only for $n = 4$, while for $n = 3$ the question remains open.*

From now on, in this section, we assume that all considered game are two-person and zero-sum, without repeating this each time explicitly.

Let us recall that IAC (and UAC) may fail for the Backgammon-like games; an example was given in 1993 by Anne Condon [12].

Yet, UNS (and INS) hold and result from Theorem 3 and the reduction of Section 5.

Remark 14 *This proof looks like "a big gun for a small fly". Indeed, all known proofs of Theorem 3 are sophisticated and based on the Hardy-Littlewood Tauberian theorems [26] that summarize relations between the Abel and Cesaro averages. In 1957, in his proof Gillette [16] did not verify the corresponding conditions accurately and this flaw was repaired only in 1969 by Liggett and Lippman [33].*

We have no simpler proof for the UNS of the Backgammon-like games, although they are a *very* special case of the mean payoff games. In contrast, for the Chess-like games, UAC (and the remaining three properties) hold and can be proven much easier as follows.

First, let us show that in a Chess-like game $\mathcal{G} = (G, D, u)$ a uniform NE (that is, a saddle point) $x^* = (x_1^*, x_2^*)$ exists and can be found by the following "modified backward induction" algorithm, which was recently and independently suggested in [2] and [8].

In every position $v \in V \setminus V_T = V_1 \cup V_2$ we shall select one or several arcs so that a strategy profile $x = (x_1, x_2)$ is a uniform saddle point whenever the strategy x_i chooses a selected arc in each $v \in V_i$ for $i \in I = \{1, 2\}$.

Remark 15 *In contrast with the n-person Backgammon like games on acyclic digraphs, in this case we obtain some, but not necessarily all, optimal strategies of the players.*

Without any loss of generality, we can make the following two assumptions:

(a) There are no parallel arcs in G, i.e., there is at most one arc $e = (v, v')$ from v to v'. Indeed, if there are several such arcs we can just identify them.

(b) Players 1 (respectively, player 2) prefers m terminal outcomes in the order $a_1, \ldots, a_m$ (respectively, in the reverse order). Moreover, we can assume that this order is strict, that is, there are no ties. Indeed, if several successive outcomes form a tie then we can just identify the corresponding terminal positions in G.

Then, the infinite outcome c should be placed somewhere between $a_1, \ldots, a_m$ and we will assume, for simplicity, that there are no ties, again.

Remark 16 *This time we say "for simplicity" rather than "without loss of generality", because some uniformly optimal strategies might be lost due to the last assumption. However, this may happen even if we allow a tie between c and some terminal a_j.*

In particular, c might be the best outcome for player 1 or 2, but obviously not for both unless $m = 0$ (due to the last assumption; otherwise it would be also possible that $m = 1$).

In case $m = 0$ there are no terminals at all. Hence, each situation $x \in X$ results in c and x is a uniform NE. Thus, in each position every move is optimal. Otherwise, by assumption (b), terminal a_1 is the best outcome for player 1. If a_1 is an isolated vertex, we eliminate it from G; otherwise, let us fix a move (v, a_1) in G and consider the following two cases.

If $v \in V_1$, let us select the move (v, a_1) in G and then reduce G by contracting (v, a_1) and eliminating all other moves from v.

Remark 17 *At this moment, some optimal moves in v might be lost.*

If $v \in V_2$ then let us eliminate only the arc (v, a_1) from G. It may happen that there are no other moves from v in G. Then v becomes a new terminal. In this case, let us identify it with a_1 and also select (v, a_1) in the original digraph G. In both cases, (v, a_1) may be a unique move to a_1. Then, let us eliminate the obtained isolated terminal from G.

In all cases the original digraph G is reduced. Let us denote the obtained digraph by G' and repeat the above procedure again, etc. Each time we choose the best terminal of player

1 or 2 until they are all eliminated and only infinite plays remain. Then, let us select all remaining arcs in the original digraph G.

Let us notice that arcs are eliminated in the current (reduced) digraphs, while they are selected in the original digraph G, the copy of which we keep all time.

By construction, in every position $v \in V_1 \cup V_2$ at least one arc is selected and, moreover, a strategy profile x is a uniform NE whenever x chooses a selected arc in each position $v \in V_1 \cup V_2$. The obtained procedure, as well as the standard backward induction, provide linear time algorithms searching for a uniform NE in the Chess-like and acyclic games, respectively; see, however, Remarks 15 and 17.

Remark 18 *In contrast, complexity of getting a uniform NE (saddle point) in a Backgammon-like game is a fundamental open problem, although its existence results from Theorem 3, by the reduction of Section 5. Unfortunately, the above algorithm cannot be extended to the Backgammon-like games, which, already in case $m = 1$, turn into the simple stochastic games [12]. The latter are known to be in the intersection of NP and co-NP and, yet, no polynomial or even pseudo-polynomial algorithm for their solution is known [12]. Moreover, it was recently shown that Condon's simple stochastic games and general Gillette's stochastic games with perfect information are polynomially equivalent [3], see also [25].*

To prove UAC of the Chess-like games we just combine the above arguments, which prove UNS, and the arguments Section 4, which prove UAC of the Backgammon-like n-person games on acyclic digraphs. Again, in a minimal counterexample $\mathcal{G} = (G, D, u)$ with an im-cycle Y, for every edge $e = (v, v') \in V_i,\ i \in I = \{1, 2\}$, there are two strategy profiles $x, x' \in Y$ such that e is chosen by x_i and not chosen by x'_i. Yet, the arc (v, a_1) (considered in the above proof of UNS) cannot satisfy this property. Indeed, if $v \in V_1$ then this move will be chosen by every strategy profile $x \in Y$, by the optimality of a_1 for player 1.

Remark 19 *It is important to recall at this moment that the players are restricted to their lazy BR improvements.*

In contrast, if $v \in V_2$ then move (v, a_1) will be chosen by no $x \in Y$ unless (v, a_1) is a unique (forced) move in v. In both cases, we get a contradiction with minimality of G.

Thus, no im-path can cycle and, hence, it either can be extended or terminates in a uniform saddle point. This implies UAC, which, in its turn, implies UNS, IAC, and INS.

7 Proof of Theorems 1 and 2

Clearly, Theorem 2 results from Theorem 3. Indeed, given an n-person mean payoff game in which all players but one have their strategies fixed, we obtain a mean payoff game of the remaining single player i, or in other words, a controlled Markov chain. It is well known that in this case a uniformly optimal strategy x_i^* exists; see, for example, [4, 34]. This and the above reduction imply the statement of Theorem 2 for the Backgammon-like games.

Furthermore, part (i) of Theorem 1 is obvious. In its turn, part (ii) can be derived from the Chess-like case of Theorem 2 (see Section 2.7) as follows.

Let us assume indirectly that a Chess-like game $\mathcal{G}$ has no uniform NE. Then, for each situation $x \in X$ of its normal form there is a BR improvement for a player $i \in I$, that is, a situation $y = y(i, x) \in X$ such that y may differ from x only in the ith coordinate, $u(i, a(x, v_0)) \leq u(i, a(y, v_0))$ for all $v_0 \in V \setminus T$, and the inequality is strict for at least one v_0.

By assumption, $q(v_0)$ is strictly positive, that is, $q(v_0, v) > 0$ for all $v \in V \setminus (V_T \cup \{v_0\})$. Hence, in $\mathcal{G}'$, situation y is still an improvement of x (although not necessarily BR) for the same player i. Since this holds for all $x \in X$, we conclude that there is no NE in $\mathcal{G}'$. □

Remark 20 *The strict positivity of $q(v_0)$ is essential in the above arguments. For example, if $q(v_0, v) = 0$ for all but one $v \in V \setminus (V_T \cup \{v_0\})$ then $\mathcal{G}'$ is a Chess-like game too and its INS (with respect to v_0) holds for all known examples, perhaps, even in general.*

8 Chess-like n-person games and INS-conjecture for additive payoffs

According to Table 1, INS of the n-person Chess-like games remains an open problem. There is no counterexample and no proof, even under assumption (iii): c is the worst outcome for all n players. Yet, we conjecture that under this assumption INS holds.

In this section we shall recall a much stronger conjecture suggested in [7].

Given an initialized Chess-like game form (G, D, v_0), let us introduce a local reward function $r : I \times E \to \mathbb{R}$. Standardly, $r(i, e)$ is interpreted as the profit obtained by player $i \in I$ when the play passes arc $e \in E$.

Let us recall that, in absence of random moves, each strategy profile $x \in X$ defines a unique play $p = p(x)$ that begins in the initial position v_0 and either (j) terminates at $a = a(x) \in V_T$ or (jj) results in a lasso that consists of an initial part and a simple di-cycle $c = c(x) \in C(G)$. The *additive effective payoff* $u : I \times X \to \mathbb{R}$ is defined in case (j) as the sum of all local rewards of the obtained play, $u(i, x) = \sum_{e \in p(x)} r(i, e)$, and in case (jj) $u(i, x) \equiv -\infty$, for all $i \in I$. In other words, in case (jj) all infinite plays are equivalent and ranked as the worst outcome by all players, in agreement with (iii). Yet, in case (j) payoffs may depend not only on the terminal position $a(x)$ but on the entire play $p(x)$.

The following two equivalent assumptions were considered in [7]:

(t) all local rewards are non-positive, $r(i, e) \leq 0$ for all $e \in E$, $i \in I$, and

(tt) all di-cycles are non-positive, $\sum_{e \in c} r(i, e) \leq 0$ for all $c \in C = C(G)$, $i \in I$.

Obviously, (t) implies (tt). Moreover, it was shown in 1958 by Gallai [15] that in fact these two assumptions are equivalent, since (t) can be enforced by a potential transformation whenever (tt) holds; see [15] and also [7] for the definitions and more details.

Remark 21 *In [7], players $i \in I$ minimize cost functions $-u(i,x)$ rather than maximize payoffs $u(i,x)$. Hence, (t) and (tt) turn into non-negativity conditions.*

It was proven in [7] that INS holds under conditions (t) and (tt) for the so-called play-once games defined by the following extra condition

(ttt) $|V_i \leq 1| \; \forall \, i \in I$; in other words, each player controls at most one position.

Also, it was conjectured in [7] that assumptions (t) and (tt) imply INS in general, even without (ttt), and this INS-conjecture was verified for several non-trivial examples.

It was also shown in [7] that conditions (t) and (tt) are essential, that is, (t), (tt), and INS may fail, even when (ttt) holds.

It is easily seen that each terminal payoff is a special case of an additive one [7]. To show this, let us just set $r(i,e) = u(i,a)$ whenever $e = (v,a)$ is a terminal move in G and $r(i,e) \equiv 0$ otherwise. Let us also notice that no terminal move can belong to a di-cycle. Hence, conditions (t) and (tt) hold for a terminal payoff automatically.

Obviously, the INS-conjecture, if true, answers in positive the second question of Table 1, yet, even in this case, the first one would remain open.

Remark 22 *The concept of an additive payoff can be naturally extended to the n-person Back-gammon-like games. Yet, INS for these games may fail already for terminal payoffs; see Section 3.*

9 Why Chess and Backgammon can be solved in pure positional uniformly optimal strategies

Chess and Backgammon can be viewed as two-person zero-sum stochastic games with perfect information and limit mean effective payoff; see section 6 or, for example, [8]. Solvability of such games in pure positional uniformly optimal strategies was proven in 1957 by Gillette [16]. The proof is quite involved; it is based on the classical Hardy-Littlewood (1931) Tauberian theorems [26] that summarize relations between the Abel and Cesaro averages. Conditions of [26] were not accurately verified in [16] and this flaw was repaired only in 1969 by Liggett and Lippman [33]. Let us recall that the above reduction results in a very special local rewards for Chess and Backgammon (as well as for n-person Chess- and Backgammon-like games): $r(i,e) = 0$ unless $e = (v,v)$ is the loop corresponding to a terminal $v \in V_T$.

Nevertheless, in case of Backgammon, we are not aware of any simpler proof of solvability.

Also, in presence of random positions, no efficient solution algorithm is known. Moreover, Andersson and Miltersen [3] (see also [25]) recently demonstrated that general Gilette model (and, in particular, the two-person zero-sum Backgammon-like games with m terminals) can be polynomially reduced to the case $m = 1$, that is, to the simple stochastic games, introduced in 1993 by Condon [12]. Although the latter class is in the intersection of NP and co-NP [12], yet, no even pseudo-polynomial algorithm is currently known for it.

In contrast, the situation with the two-person zero-sum Chess-like games (in particular, with Chess) is much better. Proofs of solvability were given in the early works by Zermelo [41], König [28], and Kalmar [27]. Surveys of these results and, in particular, of their relations to the classical Chess defined by the FIDE rules can be found in [40, 37, 13]. Yet, somewhat surprisingly, an efficient algorithm of solution was given only in 2009, independently in [2] and [8]. This algorithm (a modified of backward induction) finds some, but not all, positional uniformly optimal strategies of both players in (almost) linear time; see [2] for more details.

Finally, let us mention that the two-person zero-sum Chess- and Backgammon-like games are naturally extended to the n-person case. The concept of equilibrium (NE) suggested in 1950 by Nash [35, 36] is a natural replacement of the saddle points. Very soon after this, Kuhn [29, 30] and Gale [14] suggested an algorithm (backward induction) which finds *all* uniform (subgame perfect) NE in positional strategies in case of acyclic digraphs.

In fact, both classical and modified backward inductions imply not only UNS but UAC as well; see Sections 4 and 6, respectively.

Several questions related to NS and AC of the n-person Chess- and Backgammon-like games are summarized by Table 1. All of them but two are resolved in this article.

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References

[1] D. Andersson, V. Gurvich, and T. Hansen, On acyclicity of games with cycles, RUTCOR Research Report RRR-18-2008 and DIMACS Technical Report DTR-2009-09, Rutgers University; Algorithmic Aspects in Information and Management (AAIM); Lecture Notes in Computer Science 5564 (2009) 15-28; Discrete Applied Math. 158 (10) (2010) 1049-1063.

[2] D. Andersson, K.A. Hansen, P.B. Miltersen, and T.B. Sorensen, Deterministic graphical games revisited, J. Logic and Computation (2010) in press.

[3] D. Andersson and P.B. Miltersen, The Complexity of Solving Stochastic Games on Graphs, Algorithms and Computation: 20th International Symposium, ISAAC, December 16-18, 2009, Hawaii, USA, Y. Dong, D.-Z. Du, and O. Ibarra, eds. Lecture Notes in Computer Science 5878 (2009) 112-121.

[4] D. Blackwell, Discrete dynamic programming, Annals of Mathematical Statistics 33 (1962) 719726.

[5] E. Boros, K. Elbassioni, Gurvich, and K. Makino, On effectity functions of game forms, RUTCOR Research Report, RRR-03-2009; Games and Economic Behaviour 68:2 (2010) 512–531.

[6] E. Boros, K. Elbassioni, Gurvich, and K. Makino, A Pumping Algorithm for Ergodic Stochastic Mean Payoff Games with Perfect Information; RUTCOR Research Report, RRR-19-2009, Rutgers University; 14th International Conference in Integer Programming and Combinatorial Optimization (IPCO) Lausanne, Switzerland, June 9-11, 2010, Proceedings, F. Eisenbrand and F.B. Shepherd eds., Lecture Notes in Computer Science LNCS 6080 (2010).

[7] E. Boros and V. Gurvich, On Nash-solvability in pure strategies of finite games with perfect information which may have cycles. Math. Social Sciences 46 (2003), 207-241.

[8] E. Boros and V. Gurvich, Why Chess and Backgammon can be solved in pure positional uniformly optimal strategies, RUTCOR Research Report, RRR-21-2009.

[9] E. Boros, V. Gurvich, K. Makino, and Wei Shao, Nash-solvable bidirected cyclic two-person game forms, DIMACS Technical Report, DTR-2008-13 and RUTCOR Research Report, RRR-30-2007, revised in RRR-20-2009, Rutgers University.

[10] E. Boros and R. Rand, Terminal games with three terminals have proper Nash equilibria, RUTCOR Research Report, RRR-22-2009, Rutgers University.

[11] A. Condon, The complexity of stochastic games, Information and Computation 96 (1992) 203–224.

[12] A. Condon, An algorithm for simple stochastic games, in Advances in computational complexity theory, volume 13 of DIMACS series in discrete mathematics and theoretical computer science, 1993.

[13] C. Ewerhart, Backward Induction and the Game-Theoretic Analysis of Chess, Games and Economic Behavior (39) 2 (2002) 206-214.

[14] D. Gale, A theory of N-person games with perfect information, Proc. Natl. Acad. Sci. 39 (1953) 496–501.

[15] T. Gallai, Maximum-minimum Sätze über Graphen, Acta Mathematica Academiae Scientiarum Hungaricae 9 (1958) 395–434.

[16] D. Gillette, Stochastic games with zero stop probabilities; in Contribution to the Theory of Games III, in Annals of Mathematics Studies 39 (1957) 179–187; M. Dresher, A.W. Tucker, and P. Wolfe eds, Princeton University Press.

[17] H. Gimbert, Jeux positionnels, PhD Thesis, Univ. Paris 7, 2006 (in French) available at $http://www.labri.fr/perso/gimbert/recherche/These_{G}imbert_{J}eux_{P}ositionnels.pdf$

[18] A.I. Gol'berg and V.A. Gurvich, Tight cyclic game forms, Russian Mathematical Surveys, 46:2 (1991) 241-243.

[19] A.I. Gol'berg and V.A. Gurvich, Some properties of tight cyclic game forms, Russian Acad. Sci. Dokl. Math., 43:3 (1991) 898-903.

[20] A.I. Gol'berg and V.A. Gurvich, A tightness criterion for reciprocal bipartite cyclic game forms, Russian Acad. Sci. Dokl. Math., 45 (2) (1992) 348-354.

[21] V. Gurvich, A stochastic game with complete information and without equilibrium situations in pure stationary strategies, Russian Math. Surveys 43:2 (1988) 171–172.

[22] V. Gurvich, A theorem on the existence of equilibrium situations in pure stationary strategies for ergodic extensions of $(2 \times k)$ bimatrix games, Russian Math. Surveys 45:4 (1990) 170–172.

[23] V. Gurvich, Saddle point in pure strategies, Russian Acad. of Sci. Dokl. Math. 42:2 (1990) 497–501.

[24] V. Gurvich, A. Karzanov, and L. Khachiyan, Cyclic games and an algorithm to find minimax cycle means in directed graphs, USSR Computational Mathematics and Mathematical Physics 28:5 (1988) 85-91.

[25] V. Gurvich and P. B. Miltersen, On the computational complexity of solving stochastic mean-payoff games, Manuscript (2008) CoRR abs/0812.0486 at arxiv.org/abs/0812.0486, DBLP, http://dblp.uni-trier.de

[26] G. H. Hardy and J. E. Littlewood, Notes on the theory of series (XVI): two Tauberian theorems, J. of London Mathematical Society 6 (1931) 281–286.

[27] L. Kalmar, Zur Theorie der abstrakten Spiele, Acta Sci. Math. Szeged 4 (1928-1929) 65-85.

[28] D. König, Über eine Schlussweise aus dem Endlichen ins Unendliche, Acta Sci. Math. Szeged. 3 (1927) 121-130.

[29] H. Kuhn, Extensive games, Proc. Natl. Acad. Sci. 36 (1950) 286–295.

[30] H. Kuhn, Extensive games and the problem of information, in Contributions to the theory of games, Volume 2, Princeton (1953) 193–216.

[31] N.S. Kukushkin, Perfect information for potential games, Games and Economic Behavior 38 (2002) 306–317.

[32] N.S. Kukushkin, Acyclicity of improvements in finite game forms, Int. J. Game Theory (2010) in press; Munich Personal RePEc Archive, Nov. 27 2008, online at $http://mpra.ub.uni-muenchen.de/11802/1/MPRA_paper_11802.pdf$;

[33] T.M. Liggett and S.A. Lippman, Stochastic Games with Perfect Information and Time Average Payoff, SIAM Review 11 (1969) 604607.

[34] H. Mine and S. Osaki, Markovian decision process, American Elsevier Publishing Co, New York, 1970.

[35] J. Nash, Equilibrium points in n-person games, Proceedings of the National Academy of Sciences 36 (1) (1950) 48-49.

[36] J. Nash, Non-Cooperative Games, Annals of Mathematics 54:2 (1951) 286–295.

[37] U. Schwalbe and P. Walker, Zermelo and the Early History of Game Theory, Games and Economic Behavior 34 (2001) 123-137.

[38] L. S. Shapley, Stochastic games, in Proc. Nat. Acad. Science, USA 39 (1953) 1095–1100.

[39] S. Vorobyov, Cyclic games and linear programming, Discrete Applied Mathematics 156:11 (2008) 2195–2231.

[40] A.R. Washburn, Deterministic graphical games, Journal of Mathematical Analysis and Applications 153 (1990) 8496.

[41] E. Zermelo, Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels; in: Proc. 5th Int. Cong. Math. Cambridge 1912, vol. II; Cambridge University Press, Cambridge (1913) 501-504.