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REPORT

More about Scarf and Sperner OIKs

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RRR 15-2010, October 2010

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Abstract. The Lemke-Howson exchange algorithm for finding a Nash equilibrium in bimatrix games, as well as the classical algorithm for finding the properly colored facet in Sperner's Lemma generalize and abstract to pure combinatorics.

In particular, the idea of Lemke pivoting is extended to an arbitrary family of oiks (*Euler complexes*). Given a room-partition, the corresponding algorithm finds another (distinct) room-partition by traversing an exchange graph.

In this paper we show that each family of k oiks $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ can be reduced to a pair of oiks $\mathcal{O}' = \{\mathcal{O}_1 + \ldots + \mathcal{O}_k, \mathcal{O}_0\}$ (one of which, $\mathcal{O}_0$, is a Sperner oik) such that the exchange graphs for $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic. Numerous application of Sperner's Lemma in combinatorial topology are well known.

We also formulate the famous Scarf Lemma in terms of oiks. This Lemma has two fundamental applications in game and graph theories. In 1967, Scarf derived from it core-solvability of balanced cooperative games. In 1996, it was shown that kernel-solvability of perfect graphs also results from this Lemma.

We show that Scarf's combinatorially defined oiks are in fact realized by polytopes. We also demonstrate that the pivoting path between room-partitions can be exponentially long in d already for two equal d-dimensional Scarf oiks on $2d$ vertices. A similar example is constructed for a pair of d-dimensional Scarf and Sperner oiks.

Keywords: Euler complex (oik), room, wall, manifold, cyclic polytope, matroid; exchange algorithm, pivot; bimatrix game, Nash equilibrium, Lemke-Howson; Sperner Lemma, Brouwer Fixed Point Theorem, KKM-Theorem; core, core-solvability, Scarf Lemma, balanced games; kernel, kernel-solvability, perfect graph

Acknowledgements: This research was partially supported by INRIA and University of Pierre and Marie Curie, Paris 6. The third author gratefully acknowledges also partial support by DIMACS, Center for Discrete Mathematics and Theoretical Computer Science, Rutgers University. The fourth author was partially supported by the U.S. Department of Homeland Security under Grant Award Number 2008-ST-104-000016. The views and conclusions contained in this document are those of the authors and should not be interpreted as necessarily representing the official policies, either expressed or implied, of the U.S. Department of Homeland Security.

1 Introduction

1.1 Oiks: definition and examples

The concept of an *oik* (short for Euler complex) was recently introduced in [12] as follows. Given two integers n and d such that $n > d > 1$, a *d-dimensional complex* $\mathcal{O} = (V, \mathcal{R})$ is a uniform hypergraph of edge-size d on the ground set V of cardinality n. Standardly, the elements $v \in V$ are called *vertices*, while the edges $R \in \mathcal{R}$ will be called *rooms*; each room consists of d vertices. Furthermore, given a room R and a vertex $v \in R$, the difference $W = R \setminus \{v\}$ (of cardinality $d-1$) is called a *wall*. A complex is called an *oik* if each wall W is contained in a **positive even** number $k(W)$ of rooms.

Two rooms $R, R' \in \mathcal{R}$ are called *adjacent* if their intersection is a wall, or in other words, if their symmetric difference $R \Delta R'$ is a pair of vertices $v \in R$ and $v' \in R'$.

An oik $\mathcal{O}$ will be called 2-*adjacent* if $k(W) \equiv 2$ for every wall W, that is, if each wall is contained in exactly two (adjacent) rooms.

The next four examples of 2-adjacent oiks are borrowed from [12].

Example 1 : Pseudo-manifolds. *A $(d-1)$-dimensional simplicial pseudo-manifold is a d-dimensional oik in which each $d-1$ vertices are contained in exactly zero or two rooms; in other words, each wall is in exactly two rooms.*

An important special case is a triangulation of a compact manifold M, oriented or not. In particular, if M is a d-dimensional sphere, the corresponding oik $\mathcal{O}(M)$ is realized by a d-dimensional polytope whose every facet is a simplex with d vertices.

The oiks generated by pseudo-manifolds, manifolds, and polytopes will be called PM-, M-, and P-oiks, respectively. The latter will be also called *polytopal*.

Example 2 : Polytopal oiks. *Let $Ax = b, x \geq 0$ be a tableau, as in the simplex method, that is, A is a $m \times n$ matrix that contains an $m \times m$ identity submatrix and all coordinates of $b \in \mathbb{R}^m$ are strictly positive. Let us also assume that the solution set is bounded and all basic feasible solutions are non-degenerate.*

Let V be the column set of A. By definition, its subset $R \subseteq V$ is a room if and only if $V \setminus R$ is a basis of the tableau. The hypergraph $\mathcal{O} = (V, \mathcal{R})$ of the rooms defines an oik of dimension $d = n - m$. This results from the following exchange property of the bases. Given a basic set of columns in A (the complement to a room), let us add to it an arbitrary "entering" column (thus getting the complement to a wall). Then there exists a unique "leaving" column such that all coefficients of the right-hand-side remain positive. Combinatorially the above oik is defined by the boundary of an $(n-m)$-dimensional simplicial polytope.

Remark 1 *The boundary (surface) of a simplicial polytope of dimension d is a manifold of dimension $d-1$. Thus, the corresponding oik can be called either d- or $(d-1)$-dimensional. Respectively, there are two options: to call an oik d-dimensional when its rooms are of cardinality d or $d+1$. Here we chose the first option, while the second one is chosen in [12].*

Let us consider two examples of special polytopal oiks.

Example 3 : Gale oiks. *Let us consider Gale's cyclic polytope $P = P(d,n) \subseteq \mathbb{R}^d$ with n vertices. In [15], David Gale proved that the rooms of the corresponding oiks are defined by the cyclic binary n-vectors $x \in \{0,1\}^n$ with d ones such that the following* Gale evenness condition *holds: If d is even then all sequences of successive ones in x are even. (Let us remark that the first and the last such sequences in x make one sequence s_0, since x is cyclic.) If d is odd then all above sequences are still even, except s_0, which must be odd.*

For more details we refer the reader to [17, 18].

Example 4 : Sperner oiks. *Let n elements of a set V be colored by d colors, where $d < n$. A subset $R \subset V$ is a room if and only if $V \setminus R$ contains exactly one vertex of each color.*

The defined hypergraph $\mathcal{O} = (V, \mathcal{R})$ is an oik of dimension $n-d$. Indeed, the complement to a wall, which is colored $\{1, 2, \ldots, d, j\}$, contains exactly two complements to rooms, which are colored $\{1, 2, \ldots, d\}$. This oik is polytopal. In particular, when V consists of $2d$ vertices and each color appears twice, $\{1, 1, 2, 2, \ldots, d, d\}$, the corresponding polytope is polar to the d-dimensional cube. We leave the proofs to the reader.

Remark 2 *Let us note that in the latter case the complement to a room is also a room (also see Remark 3 below). However, this is not the case in general for the above four examples.*

In Sections 1.3 and 3, we introduce one more family of 2-adjacent oiks based on the Scarf Lemma [33] and, in Section 4, we prove that these oiks are polytopal.

Now, we borrow from [12] another four examples of oiks related to Euler graphs and binary matroids. Let us note that these oiks might not necessarily be 2-adjacent.

Example 5 *An Euler graph, that is, a connected graph $G = (V, E)$ in which each vertex has an even degree is an oik, where E and V are the sets of rooms (edges) and walls (vertices), respectively. Let us remark that a disjoint union of Euler graphs is an oik, too.*

Example 6 *An Euler graph $G = (V, E)$ can be also interpreted as an oik $\mathcal{O} = (E, \mathcal{R})$ in a different way: E is the vertex-set of $\mathcal{O}$, while its rooms are the spanning trees of G. The dimension of this oik is $d = |V| - 1$.*

Example 7 *A connected bipartite graph $G = (V, E)$ defines a $|E| - |V| + 1$-dimensional oik $\mathcal{O} = (E, \mathcal{R})$ whose vertices are the edges of G and rooms are the complements to the spanning trees of G.*

Remark 3 *In particular, a bipartite Euler graph G defines two oiks whose rooms are complementary.*

The following example generalizes the previous two. A binary matroid M is the set of columns of a binary matrix A, $\mod 2$. The bases of M are the linearly independent sets of columns. The co-cycles are the supports of the row vectors generated by the rows of A. The co-circuits are the minimal co-cycles. Matroid M is called *Euler* when each row of A has an even number of ones. See, for example, [25, 34] for more details.

Example 8 *Let M be an Euler binary matroid of rank r in which each co-circuit (or equivalently, each co-cycle) is even. Then M defines an r-dimensional oik whose vertices are the elements and rooms are the bases of M.*

Remark 4 *As we recently learned, a similar "oik-like" concept was considered as early as in 1972 by Michael Todd in his thesis; see [40, 41, 42]. In these works, pairs of oiks were introduced under the names semi-primoid and semi-duoid.*

Interestingly, [42] refers to the first author of the present paper (private communications).

1.2 Finding another room-partition or room-selection of fixed odd degrees by the exchange algorithm

An *oik-family* is a set of k oiks $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ (of dimensions $d_1, \ldots, d_k$) defined on the same vertex-set V. Some of these oiks may be isomorphic or even identical.

Given an oik-family $\mathcal{O}$, a *room-selection* is a hypergraph $\mathcal{R} = \{R_1, \ldots, R_k\}$ in which R_i is a room of oik $\mathcal{O}_i$ for all $i \in [k] = \{1, \ldots, k\}$. Standardly, $\delta_{\mathcal{R}}(v)$ denote the degree of a vertex $v \in V$ in $\mathcal{R}$, that is, the number of rooms of $\mathcal{R}$ that contain v. A room-selection $\mathcal{R}$ is called *a room-partition* if $\delta_{\mathcal{R}}(v) \equiv 1$ for each $v \in V$.

It was shown in [12] that every oik-family has an even number of room-partitions.

Remark 5 *Let us note, however, that this number may be* 0*. Moreover, it might be NP-hard to verify the existence of a room-partition.*

Furthermore, given a room-partition, an *exchange algorithm* to get another one is suggested in [12]. This algorithm is based on constructing and, then, traversing the *exchange graph.* Given a family of oiks $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ on the common vertex-set ($\mathcal{O}_i = (V, \mathcal{R}_i)$, $i \in [k] = \{1, \ldots, k\}$), let us fix a special vertex $w \in V$ and define the exchange graph $\mathcal{G} = \mathcal{G}(\mathcal{O}, w)$ as follows.

A room-selection $\mathcal{R}$ is called a *skew room-partition* or a *butterfly* if $\delta_{\mathcal{R}}(w) = 0$, $\delta_{\mathcal{R}}(u) = 2$ for a unique vertex $u \in V$, and $\delta_{\mathcal{R}}(v) \equiv 1$ for all other vertices $v \in V \setminus \{u, w\}$. Let $\mathcal{V}$ and $\mathcal{V}^w$ denote the sets of all room-partitions and skew room-partitions, respectively.

Two room-selections $\mathcal{R} = \{R_1, \ldots, R_k\}$ and $\mathcal{R}' = \{R'_1, \ldots, R'_k\}$ are called *adjacent* if their symmetric difference $\mathcal{R} \Delta \mathcal{R}'$ is a pair of adjacent rooms (R_i, R'_i) from $\mathcal{O}_i$ for some $i \in [k]$. If also $\mathcal{R}, \mathcal{R}' \in \mathcal{V} \cup \mathcal{V}^w$ then let $(\mathcal{R}, \mathcal{R}') \in \mathcal{E}$. Thus, the exchange graph $\mathcal{G}(\mathcal{O}, w)$, with vertex set $\mathcal{V} \cup \mathcal{V}^w$ and edge set $\mathcal{E}$, is defined. (Let us recall that rooms R_i and R'_i are adjacent if their symmetric difference $R_i \Delta R'_i$ is a pair of vertices, or in other words, if their intersection $R_i \cap R'_i$ is a wall W_i in $\mathcal{O}_i$.)

It is easy to list all rooms adjacent to a given room R of a given oik $\mathcal{O}$. To do so, let us select a vertex $v \in R$ and enumerate all rooms of $\mathcal{O}$, except R, that contain the wall $W = R \setminus \{v\}$. By definition of an oik, there is an odd number $k(W) - 1$ of such rooms. We get all rooms of $\mathcal{O}$ adjacent to R by just repeating the above procedure for all $v \in R$.

Furthermore, by this procedure, it is also easy to obtain all room-selections adjacent to a given one $\mathcal{R} = \{R_1, \ldots, R_k\}$ in a given oik-family $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$.

The above definitions and observations immediately imply the following properties of the exchange graph $\mathcal{G}(\mathcal{O}, w)$.

Lemma 1 *Vertices of $\mathcal{V}$ (room-partitions) and $\mathcal{V}^w$ (skew room-partitions) have odd and even degrees in $\mathcal{G}$, respectively. No two vertices of $\mathcal{V}$ are adjacent.* □

Obviously, the number of room-partitions, $|\mathcal{V}|$, is even, since in any graph the number of odd degree vertices is even.

Furthermore, given a room-partition $\mathcal{R} \in \mathcal{V}$ as our starting point, let us traverse $\mathcal{G}$ arbitrarily, and passing no edge twice, until no possible move is left. In other words, we construct an Eulerian path beginning in an odd degree vertex (room-partition) $\mathcal{R} \in \mathcal{V}$. Obviously, any such path ends in another odd degree vertex (room-partition) $\mathcal{R}' \in \mathcal{V}$, distinct from $\mathcal{R}$. Indeed, $\mathcal{R}' \notin \mathcal{V}^w$, since all vertices of $\mathcal{V}^w$ have even degrees. Also $\mathcal{R}' \neq \mathcal{R}$, since vertex $\mathcal{R} \in \mathcal{V}$ is of odd degree. In particular, the following statement follows.

Theorem 1 *Every oik-family $\mathcal{O}$ has an even number of room-partitions. Given a vertex $w \in V$ and a room-partition $\mathcal{R}$, we get another room-partition $\mathcal{R}'$ distinct from $\mathcal{R}$ by traversing the exchange graph $\mathcal{G}(\mathcal{O}, w)$ starting in $\mathcal{R}$ and passing no edge twice.* □

If $\mathcal{O}$ is a family of 2-adjacent oiks then obviously vertices of $\mathcal{V}$ and $\mathcal{V}^w$ have degrees 1 and 2 respectively. In this case the exchange graph has a very simple structure: it is a disjoint union of simple paths whose ends form $\mathcal{V}$ and simple cycles whose vertices form the rest of $\mathcal{V}^w$. These paths uniquely define the traversing procedure, as well as a matching on the set $\mathcal{V}$ of room-partitions.

The above results can be generalized in many ways; for example, as follows.

Let $\delta : V \to \mathbb{Z}_+$ be a mapping of V into set $\mathbb{Z}_+$ of the non-negative integers. A room-selection $\mathcal{R}$ is called a δ-selection if $\delta_{\mathcal{R}}(v) = \delta(v)$ for each $v \in V$. Given $\mathcal{O}$ and δ, let us define $\mathcal{V}$ as the set of all δ-selections and $\mathcal{V}^w$ as follows. Let us fix a vertex $w \in W$. A *skew* $(\delta \pm 1)$-*selection* (or a *dragonfly*) is defined as a room-selection $\mathcal{R}'$ such that $\delta_{\mathcal{R}'}(w) = \delta(w) - 1$, there is a vertex $u \in V$ such that $\delta_{\mathcal{R}'}(u) = \delta(u) + 1$, and $\delta_{\mathcal{R}'}(v) = \delta(v)$ for all other vertices $v \in V \setminus \{u, w\}$.

Given w and δ, let $\mathcal{V}^w$ be the set of all skew $(\delta \pm 1)$-selections. Adjacency relation $\mathcal{E}$ on the vertex-set $\mathcal{V} \cup \mathcal{V}^w$ and the exchange graph $\mathcal{G} = \mathcal{G}(\mathcal{O}, w)$ are defined exactly as before.

It is not difficult to verify that all claims of Lemma 1 and Theorem 1 still hold if $\delta(v)$ is *odd* for all $v \in V$.

Lemma 2 *If $\delta(v)$ is odd for each $v \in V$, then vertices of $\mathcal{V}$ (δ-selections) and $\mathcal{V}^w$ (skew $(\delta \pm 1)$-selections) have odd and even degrees in $\mathcal{G}$, respectively. Furthermore, no two vertices of $\mathcal{V}$ are adjacent in $\mathcal{G}$.* □

$$\left[\begin{array}{ccc|cccc} 0 & M & M & 1 & 2 & 3 & 4 \\ M & 0 & M & 3 & 1 & 2 & 4 \\ M & M & 0 & 4 & 3 & 2 & 1 \end{array}\right]$$

Figure 1: Example of a Scarf matrix with $m = 3$ and $n = 4$.

Theorem 2 *If $\delta(v)$ is odd for each $v \in V$ then every oik-family $\mathcal{O}$ has an even number of δ-selections. In particular, for a given vertex $w \in V$ and a δ-selection $\mathcal{R}$, we get another δ-selection $\mathcal{R}'$ distinct from $\mathcal{R}$ by traversing the exchange graph $\mathcal{G}(\mathcal{O}, w)$ starting at $\mathcal{R}$ and passing no edge twice.* □

1.3 Scarf's oiks; main results

Here we recall one more example of 2-adjacent oiks introduced by Herbert Scarf in [33]; see also [31, 32, 20]. An $m \times (m+n)$ real non-negative matrix A is called a *Scarf* matrix (see Figure 1), where $a(i,j)$ is the ijth entry of A, if

(i) $m \geq 2$ and $n \geq 1$;

(ii) $a(i,j) > a(i,m+k) > a(i,i) \geq 0$ for all $i,j \in [m] = \{1,\dots,m\}$, where $i \neq j$, and $k \in [n] = \{1,\dots,n\}$;

(iii) $a(i,m+k) \neq a(i,m+\ell)$ for all $i \in [m]$ and distinct $k,\ell \in [n]$.

Furthermore, a $m \times (m+n)$ Scarf matrix will be called *canonical* if

(iv) $a(i,i) = 0$ for any $i \in [m]$ and $a(i,j) = M > n$ for any distinct $i,j \in [m]$;

(v) in each row, the last n entries form a permutation of $[n] = \{1,\dots,n\}$.

Obviously, given n, m, and a constant $M > n$, there are $(n!)^m$ canonical Scarf matrices.

Let $V = [m+n] = \{1,\dots,m+n\}$ be the set of columns of a Scarf (not necessarily canonical) matrix A. A subset $J \subseteq V$ is called *dominating* if for each column $k \in [m+n]$ there is a row $i \in [m]$ such that $a(i,k) \leq a(i,j)$ for each $j \in J$.

Remark 6 *Names "subordinating" [2] or "primitive" set [20] also appear in the literature. We choose name "dominating," following [1, 23].*

Clearly, whenever J is dominating, each subset $J' \subseteq J$ is also dominating. Hence *domination* is a hereditary property defined on V. By definition, each column $j \in J$ is dominated by J, so there must exist at least one row $i^* \in [m]$ such that $a(i^*,j) = \min_{k \in J} a(i^*,k)$. Indeed, otherwise column $j \in J$ is not dominated by J. Clearly, each row contains at least one column, say j_i, such that $a(i, j_i) = \min_{j \in J} a(i, j)$ for each $i \in [m]$. Furthermore, we notice that $|\arg\min_{j \in J} a(i, j)| > 1$ if and only if $a(i, j_i) = M$, which can only happen if and only if $J \subset [m]$. Since the dimension of the submatrix of A restricted to the columns $j \in J$, call it A^J, is $m \times |J|$, these observations imply $|J| \leq m$.

So our properties for a dominating set $J \subseteq [m + n]$ are rewritten as

(vi) Each subset $J' \subseteq J$ is dominating.

(vii) Each column $j \in J$ contains a *row minimum* i^*, i.e. $a(i^*, j) = \min_{k \in J} a(i^*, k)$ for some $i^* \in [m]$.

($viii$) $|J| \leq m$.

(ix) If $|J| = m$, then for each row $i \in [m]$ of A^J there is a unique column j_i such that $a(i, j_i) = \min_{j \in J} a(i, j)$. Furthermore, since A^J is square, the sequence $(j_1, \ldots, j_m)$ forms a permutation of J.

The following simple observation will play an important role.

Lemma 3 *In an $m \times (m + n)$ Scarf matrix A, the first m columns, $J = [m]$, do not form a dominating set, while each proper subset $J \subset [m]$ is a dominating set.*

Proof: Each row of the $m \times m$ submatrix A^J contains 0. Hence, no column $i \notin J$ is dominated by J. In constrast, if $J \subseteq [m] \setminus \{i\}$ for an arbitrary $i \in [m]$, then $a(i, j) = M = \max_{j' \in [m+n]} a(i, j')$ for each column $j \in J$. Hence each column $j' \in [m + n]$ is dominated by J in row i. □

We refer to $J = [m]$ as *the special dominating set*, and shall henceforth include it when speaking of dominating sets.

It is straightforward to verify that properties (vi)-(ix) still hold after this extension. Moreover, the following key statement becomes true.

Theorem 3 *(**Scarf's Lemma**). Each dominating $(m-1)$-column set is contained in exactly two dominating m-columns sets; one of which may be the special dominating set $[m]$.*

Each dominating $(m - 1)$- and m-column set will be called a *wall* and *room*, respectively. By (vi) and Theorem 3, this structure defines a 2-adjacent oik of dimension $d = m$.

Remark 7 *In fact, Theorem 3 is stronger than the last statement, which would result from the following weaker claim: "each dominating set of cardinality $m - 1$ is contained either in two or in none of the dominating sets of cardinality m." Yet, by Theorem 3, the second option cannot hold.*

Remark 8 *Without convention on the special dominating set $[m]$, Theorem 3 and oik structure would fail. Indeed, by Lemma 3, for each $j \in [m]$ the column-set $J = [m] \setminus \{j\}$ is dominating in a Scarf matrix. By Theorem 3, J is contained in exactly two dominating sets one of which is $[m]$.*

Remark 9 *Theorem 3 would also fail for $n = 0$. Indeed, in this case every set of $m - 1$ columns is dominating but there is only one set of cardinality m. Thus, condition $n \geq 1$ is essential.*

Remark 10 *The definition and properties of the dominating sets are based only on the order of the entries $a(i, m+j)$ for $j \in [n]$ in each row $i \in [m]$. Given these m complete orders over $[n]$, the real values of the entries are irrelevant. Thus, without any loss of generality, we can fix some $M > n$ and restrict ourselves by $(n!)^m$ canonical Scarf matrices. In particular, it will be sufficient to prove Theorem 3 only for them.*

The following example shows that properties (ii) and (iii), which define the $m \times (m+n)$ Scarf matrices, can hardly be relaxed; even a slight modification of them might destroy the oik-structure.

Example 9 *Let us consider the following "almost" Scarf matrix.*

$$\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 2 & 0 & 1 \end{array}\right]$$

It is easy to verify that columns 1 and 3 form a dominating set, while 2 and 3 do not, since for them both row-minima are in column 2. Whether $\{1, 2\}$ is a special dominating set or it is not, still the oik properties fail. Indeed, set $\{3\} = \{1, 3\} \setminus \{1\}$ should be a wall, yet, it is contained in only one room.

The first proof of Theorem 3 given in [33] was then simplified in [20] and later in [2]. In Section 3, we present an even simpler proof.

Let us extend list (i)-(ix) by the following obvious observation:

(x) If $n = m$, then the last m columns of a Scarf oik form a room.

This room and the special room form a partition $([m], [2m] \setminus [m])$ of the set of all columns $V = [2m]$.

The rest of the paper is organized as follows. In Section 1.4, we discuss applications of oiks. In Section 2, we prove that an arbitrary oik-family can be reduced to a pair of oiks, one of which is a Sperner oik. In Section 3 we suggest a new proof of Scarf's Lemma (Theorem 3). In Section 4, we prove that the Scarf oiks are polytopal, that is, they can be realized by the construction given in Example 2. In Section 5, we construct two distinct sequences of $m \times 2m$ Scarf matrices, both of which exchange paths of length exponential in m for appropriately defined oik-pairs and their exchange graphs. In particular, for an oik-pair consisting of two Scarf oiks, there is an exchange path of length $\frac{3}{4}2^m - 1$, and for an oik-pair consisting of one Scarf oik and one Sperner oik, there is an exchange path of length $2^m - 1$. These constructions shows that the exchange algorithm can be exponential in dimension m already for $2m$ vertices. Finally, we close with a few comments and open problems.

1.4 Main applications of oiks

Several classical results can be explained in terms of oiks and exchange algorithms, which, given a room partition find another one. Let us consider the following three examples.

The Lemke-Howson algorithm [27] (of finding a Nash equilibrium in mixed strategies in a bimatrix game) can be interpreted as the exchange algorithm for two polytopal oiks. In [39], it was reformulated as an exchange algorithm for three oiks: two polytopal and one Sperner. More details can be found in [26, 28, 37].

The famous Sperner Lemma can be interpreted as Lemma 1 and Theorem 1 for an oik-family which consists of two oiks: a polytopal and Sperner one. In this case, given a multi-colored simplicial facet of a polytope, the exchange algorithm finds another one. This result has fundamental applications in combinatorial topology: Brouwer's Fixed Point Theorem [9] and KKM-Theorem [22]; see [3, 38] and also the next subsection for more details.

Similarly, the Scarf Lemma [33] can be interpreted as Lemma 1 and Theorem 1 for two oiks: polytopal and Scarf. In this case, the exchange algorithm begins with the origin and come to a dominating vertex of a given polytope; see Sections 3, 4, and 5 for more details.

This result has important applications in cooperative game theory. In [33], Scarf derived from his lemma existence of a non-empty core in every balanced game with non-transferrable utility (balanced NTU-game); see also [31, 32, 20, 35, 37, 10, 11, 21].

Interestingly, core-solvability of the NTU-games appears to be equivalent with kernel-solvability of perfect graphs [7, 2]; see also [1, 8, 23, 24]. The similarity between the Scarf and Sperner lemmas is discussed in [1, 23, 24].

Fractional versions of cores and kernels were considered in [2, 1, 23, 24]. In these papers, fractional core-solvability of all (not only balanced) NTU-games and fractional kernel-solvability of all (not only perfect) graphs were derived from Scarf's Lemma.

Remark 11 *Let us note that all applications mentioned above are related with partitioning of V in two rooms in a family which consists of two (distinct) oiks. It would be interesting to find an application of a partitioning (or δ-selection) in at least three rooms.*

2 Every oik-family can be reduced to a pair of oiks, one of which is a Sperner oik

Given a d-dimensional polytope (or, more generally, a $(d-1)$-dimensional manifold) P whose n vertices are colored by d colors $[d] = \{1, \dots, d\}$, we also assume that P is *simplicial*, that is, every facet of P contains only d vertices. A facet is called *multi-colored* if its d vertices are colored by d distinct colors. The classical Sperner's Lemma claims that the number of the multi-colored facets is even; moreover, given one of them, another one is uniquely determined by the exchange algorithm. This claim can be generalized in many ways [3, 38]. In particular, Lemma 1 and Theorem 1 generalize it to an arbitrary oik as follows.

Let $\mathcal{O}_1 = (V, \mathcal{R}_1)$ be an oik whose n vertices are colored by d colors, $c : V \to [d]$. A room $R_1 \in \mathcal{R}_1$ is *multi-colored* if $c(R_1) = [d]$. By Theorem 1, the number of multi-colored rooms is even; moreover, given one of them, another one can be obtained by the exchange algorithm.

To see that Theorem 1 is applicable let us add to the oik $\mathcal{O}_1$ a $(n-d)$-dimensional Sperner oik $\mathcal{O}_2 = (V, \mathcal{R}_2)$ defined on the same vertex-set V by the coloring c as follows. A set $R_2 \subseteq V$ is a room of oik $\mathcal{O}_2$ if and only if $|R_2| = n-d$ and the complementary set $V \setminus R_2$ of cardinality d is multi-colored; see Example 4. By this definition, a room $R_1 \in \mathcal{R}_1$ is multi-colored in oik $\mathcal{O}_1$ if and only if its complement $R_2 = V \setminus R_1$ is a room of $\mathcal{O}_2$, or in other words, sets R_1 and R_2 form a room-partition in the oik-pair $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)$.

Thus, Theorem 1 is applicable; in particular, it results in the standard "geometrical" Sperner Lemmas when $\mathcal{O}_1$ is a PM-, M-, or P-oik (see Example 1). In general, this approach is purely combinatorial and geometry is ignored. Moreover, the oik $\mathcal{O}_1$ might be not 2-adjacent. In this case, given a room-partition, another one, defined in Theorem 1, is not necessarily unique.

Now let $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)$ be an arbitrary oik-pair defined on a common vertex-set. Then, Theorem 1 results in

(i) the Sperner Lemma when $\mathcal{O}_1$ is a polytopal oik, while $\mathcal{O}_2$ is a Sperner oik,

(ii) the Scarf Theorem [33] when $\mathcal{O}_1$ is a polytopal oik, while $\mathcal{O}_2$ is a Scarf oik,

(iii) the Lemke-Howson exchange algorithm [27] when oiks $\mathcal{O}_1$ and $\mathcal{O}_2$ are polytopal.

Somewhat surprisingly, an arbitrary oik-family $\mathcal{O} = (\mathcal{O}_1, \ldots, \mathcal{O}_k)$ is equivalent with an oik-family or *oik-pair* $\mathcal{O}' = (\mathcal{O}_{k+1}, \mathcal{O}_0)$, where $\mathcal{O}_{k+1} = \mathcal{O}_1 + \ldots + \mathcal{O}_k$ is a sum, which will be defined below, and $\mathcal{O}_0$ is a Sperner oik, that is, the exchange graphs of $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic. Hence, one can execute the exchange algorithm for $\mathcal{O}'$ rather than for $\mathcal{O}$.

Remark 12 *In particular, due to this reduction, the Scarf theorem [33] can be derived from the Sperner Lemma as well as from the Scarf Lemma. The last observation is the main result of the recent paper by Kiraly and Pap [24].*

The reduction is simple. Let $\mathcal{O} = (\mathcal{O}_1, \ldots, \mathcal{O}_k)$ be an arbitrary oik-family in which $\mathcal{O}_i = (V, \mathcal{R}_i)$ is a d_i-dimensional oik for $i \in [k] = \{1, \ldots, k\}$ and $\sum_{i=1}^k d_i = n = |V|$. First, let us define the sum $\mathcal{O}_{k+1} = \sum_{i=1}^k \mathcal{O}_i$ as follows: $\mathcal{O}_{k+1} = (kV, \mathcal{R}_{k+1})$, where kV consists of k pairwise disjoint copies $V_1, \ldots, V_k$ of V and $R \in \mathcal{R}_{k+1}$ if and only if $R \cap V_i$ is a room of the oik $\mathcal{O}_i = (V_i, \mathcal{R}_i)$ (where V_i is a copy of V) for all $i \in [k] = \{1, \ldots, k\}$. In particular, $|kV| = kn$ and $d_{k+1} = \sum_{i=1}^k d_i = n$ are the size and dimension of the oik $\mathcal{O}_{k+1}$.

Let us color n vertices of V by n pairwise distinct colors and then copy this coloring in every V_i, $i \in [k]$, thus, coloring kn vertices of the set kV in n colors. This coloring standardly defines the Sperner oik $\mathcal{O}_0 = (kV, \mathcal{R}_0)$ in which $R \in \mathcal{R}_0$ if and only if $kV \setminus R$ is multi-colored. Thus, the oik-pair $\mathcal{O}' = (\mathcal{O}_{k+1}, \mathcal{O}_0)$ is defined. Let us choose two vertices: $w \in V$ and $w' \in kV$.

Theorem 4 *The two exchange graphs $\mathcal{G} = \mathcal{G}(\mathcal{O}, w)$ and $\mathcal{G}' = \mathcal{G}(\mathcal{O}', w')$ (defined above) are isomorphic whenever vertices w and w' are of the same color.*

Proof: . We will make use of the standard notation $V = \{v_1, \ldots, v_n\}$, $V_i = \{v_1^i, \ldots, v_n^i\}$ for all $i \in [k]$. Also let $\mathcal{V} \cup \mathcal{V}^w$ and $\mathcal{V}' \cup \mathcal{V}^{w'}$ be the vertex sets of $\mathcal{G}$ and $\mathcal{G}'$, respectively, and let $\mathcal{E}$ and $\mathcal{E}'$ be the corresponding edge sets, as in Section 1.2. Without any loss of generality, we can assume that vertices v_j and v_j^i are of color j for all $i \in [k]$, $j \in [n]$, and also that $w = v_1$ and $w' = v_1^1$.

Then $\mathcal{V}$ is the set of all room-partitions $\mathcal{R} = (R_1, \ldots, R_k)$ of V, that is, rooms R_i of oik $\mathcal{O}_i = (V, \mathcal{R}_i)$ for each $i \in [k]$ are pairwise disjoint and $\cup_{i=1}^k R_i = V$.

Respectively, $\mathcal{V}'$ consists of the set-families $\mathcal{R}' = (R_1', \ldots, R_k')$ such that R_i' is a room of $\mathcal{O}_i = (V, \mathcal{R}_i)$ for all $i \in [k]$ and the union $R' = \cup_{i=1}^k R_i' \subseteq kV$ is a multi-colored set of cardinality n. Indeed, by definition, in this and only in this case the complementary set $kV \setminus R'$ is a room of the Sperner oik $\mathcal{O}_0 = (kV, \mathcal{R}_0)$.

There is an obvious one-to-one correspondence f between the sets $\mathcal{V}$ and $\mathcal{V}'$.

Furthermore, $\mathcal{V}^w$ is the set of all skew room-partitions (or butterflies) $\mathcal{R}^1 = (R_1, \ldots, R_k)$ in $\mathcal{O} = (\mathcal{O}_1, \ldots, \mathcal{O}_k)$, that is, $\cup_{i=1}^k R_i = V \setminus \{w\}$ and the rooms $R_i \in \mathcal{R}_i,\ i \in [k]$ are pairwise disjoint, except for a unique pair, which has a unique common vertex $u \in V$ distinct from w.

Respectively, $\mathcal{V}_1'$ consists of the set-families $\mathcal{R}^{1'} = (R_1', \ldots, R_k')$ in $\mathcal{O}' = (kV, \mathcal{R}')$ such that R_i' is a room of $\mathcal{O}_i = (V, \mathcal{R}_i)$ for all $i \in [k]$ and the union $R' = \cup_{i=1}^k R_i' \subseteq kV$ is an "almost" multi-colored set of cardinality n, that is, in its coloring 1 does not appear, while some other color appears twice.

Again, there is an obvious one-to-one correspondence f_1 between the sets $\mathcal{V}^w$ and $\mathcal{V}^{w'}$. Moreover, the obtained two mappings f and f_1 realize an isomorphism between the exchange graphs $\mathcal{G} = \mathcal{G}(\mathcal{O}, w)$ and $\mathcal{G}' = \mathcal{G}(\mathcal{O}', w')$. This is not difficult to derive just from the definitions of Section 1.2. □

It is obvious that size of $\mathcal{O}'$ is exponential in k but it is polynomial in size of $\mathcal{O}$. Hence, it is polynomial when k is a constant.

Thus, the room-partitions of an arbitrary oik-family $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ are in one-to-one correspondence with the multi-colored rooms of the sum $\mathcal{O}_1 + \ldots + \mathcal{O}_k$. In particular, instead of looking for another room-partition, one can search for another multi-colored room. This shows a sort of universality of the Sperner Lemma, cf. [24].

3 One more proof of the Scarf Lemma

By Remark 10, it is enough to prove Theorem 3 for the canonical Scarf matrices. Let A be an $m \times [m+n]$ canonical Scarf matrix.

Let $J \subseteq [m+n]$ be a dominating set of $m-1$ columns. In the corresponding $m \times (m-1)$ submatrix A^J, let us choose a minimum entry $a(i, j_i) = \min_{j \in J} a(i,j)$ in every row $i \in [m]$. By observation (vii) (see Section 1.3), each column $j \in J$ contains at least one such row minimum, since otherwise column j is not dominated by J.

Case I: $J \subseteq [m]$. Since $|J| = m-1$, $J = [m] \setminus \{i\}$ for some column $i \in [m]$. This implies that row i in matrix A^J consists of $m-1$ equal entries M. Let us show that in this case there are exactly two columns j^*, j^{**} which extend J to a dominating set of m columns: $J \cup \{j^*\}$ and $J \cup \{j^{**}\}$. Indeed, $j^{**} = i$ is one of them, since by convention, $J \cup \{i\} = [m]$ is a special dominating set.

Then, j^* is uniquely defined as follows: $a(i, j^*) = \max_{j \in [n]} a(i, m + j)$; in other words, $a(i, j^*)$ is the unique largest entry, distinct from M, in the ith row. Indeed, in this case $J_0 = J \cup \{j^*\}$ dominates every column $j \in [m + n]$:

- if $j > m$, then $a(i, j) \leq n \leq a(i, j')$ for each $j' \in J_0$, hence j is dominated by J_0;
- if $j \leq m$, then each column j contains 0, and since $a(j, k) \geq 0 \ \forall k \in J_0$, j is dominated by J_0.

Furthermore, one can substitute no other column i^* for j^*, because then column j^* is not dominated by $J \cup \{i^*\}$.

Finally, let us note that the above arguments are based on assumption (i) that $n \geq 1$, or in other words, $[n] \neq \emptyset$. Indeed, otherwise j^* fails to exist (see Remark 8).

Case II : $J \not\subseteq [m]$. It is easy to verify that in each row $i \in [m]$ of J, the minimum entry $a(i, j_i)$ is unique. Since there are m rows and $|J| = m-1$ columns, there is a (unique) column $i^* \in J$ that contains exactly two row minima, say (without loss of generality), $a(1, i^*)$ and $a(2, i^*)$ of rows 1 and 2, i.e. $j_1 = j_2 = i^*$.

Clearly, $a(1, i^*) = a(2, i^*) = 0$ cannot hold, since no column of A can contain two zeros.

Suppose $a(2, i^*) = 0$. Then all other entries of column i^* are equal to M. In particular, $a(1, i^*) = \min_{j \in J} a(1, j) = M$. Hence, $a(1, j) = M$ for all $j \in J$ and so $J = [m] \setminus \{1\} \subseteq [m]$, a contradiction.

Now let us consider the last case, $a(1, i^*) > 0$, $a(2, i^*) > 0$, and prove again that there are exactly two columns j^* and j^{**} which extend J to a dominating set of m columns.

For $k = 1, 2$, (i^* and $J \not\subseteq [m]$ both fixed) let us define

$$J_k = \{j \in [m + n] |\ a(k, i^*) > a(k, j) \text{ and } a(i, j_i) < a(i, j) \ \forall i \in [m] \setminus \{k\}\}, \tag{3.1}$$

where $j_i \in \arg\min_{j \in J} a(i, j)$ is in fact unique, since $J \not\subseteq [m]$. In other words, if we extend J by a column $j^k \in J_k$ to obtain a set $J_0 = J \cup \{j^k\}$, then in every row $i \in [m] \setminus \{k\}$ the (unique) row-minimum of J $a(i, j_i) \min_{j \in J_0} a(i, j)$ of J_0 remains the same as in J, while the row-minimum of $k = 1$ is updated from j^* to j^1 and for $k = 2$, j^{**} is updated to j^2, i.e. $a(k, j^k) = \min_{j \in J_0} a(k, j)$ for $k \in \{1, 2\}$.

Lemma 4 *The column-sets J_1, J_2, and J are non-empty and pairwise disjoint.*

Proof: First, $|J| = m - 1 > 0$, by definition. Hence $J \neq \emptyset$. Then also $J_k \neq \emptyset$, since in particular, $k \in J_k$ for $k \in \{1, 2\}$. Indeed, $a(k, k) = 0 = \min_{j \in [m+n]} a(k, j)$ and $a(i, k) = M = \max_{j \in [m+n]} a(i, j)$ whenever $i \neq k$. Furthermore, $J_k \cap J = \emptyset$ for $k \in \{1, 2\}$ by definition of J_k in (3.1). Finally, $J_1 \cap J_2 = \emptyset$ since $a(k, i^*) = a(k, j_k)$ for $k \in \{1, 2\}$. □

Also by (vii) of Section 1.3, containment $j_0 \in J_1 \cup J_2$ must hold for every dominating extension $J_0 = J \cup \{j_0\}$. Since $J_1 \cap J_2 = \emptyset$, let us consider two cases: $j_0 = j^1 \in J_1$

and $j_0 = j^2 \in J_2$. Furthermore, $j^k = \arg\max_{j \in J_k} a(k, j)$ must hold for $k = 1, 2$. Indeed, otherwise, for any other such $j_0 \notin J \cup \{j^k\}$, column j^k is not dominated by $J \cup \{j_0\}$.

Finally, if both above conditions $j^k \in J_k$ and $a(k, j^k) = \max_{j \in J_k} a(k, j)$ hold, then both extensions $J \cup \{j^k\}$, $k = 1, 2$, are dominating sets, by choice of J_k in (3.1). □

4 Scarf's oiks are polytopal

Given an $m \times (m+n)$ Scarf matrix A (not necessarily in canonical form) defined by formulae (i)-(iii) of Section 1.3, introduce an $n \times (m + n)$ matrix B as follows:

(i) $b(i, m + j) = \delta_i^j$ for $i, j \in [n]$, where standardly $\delta_i^j = 1$ if $i = j$, and $\delta_i^j = 0$ if $i \neq j$,

(ii) $b(j, i) = 1 - a(i, m + j)^{-K}$ for $i \in [m]$, $j \in [n]$, where $K > 0$ is a large constant.

It is easy to see that the last n columns of matrix B form the $n \times n$ identity matrix and the ijth entry of first m columns of B is associated with jith entry of the last n columns of A. The transformation $f(a) = 1 - a^{-K}$ was suggested by Scarf in [31]. Obviously, we have the following properties of f:

(iii) f is a monotone increasing function for any fixed positive K,

(iv) $f(a) = 1 - 1/a^K > 0$ when $a > 1$, and $f(a) \leq 0$ when $0 < a \leq 1$.

Due to (iv), it will be convenient to assume that

(v) $a(i, m + j) > 1$ for all $i \in [m], j \in [n]$.

It is clear that (v) can be assumed without any loss of generality, since the Scarf oik $\mathcal{O}_A = (V, \mathcal{E})$, defined with respect to matrix A, remains the same if we add a constant to these entries of A. Furthermore, by construction, matrices A and B have the common column-set $V = [m + n]$.

Let us consider the system of n equations $Bx = e_n$ of $m + n$ real variables $x \in \mathbb{R}^{m+n}$, where $e_n \in \mathbb{R}^n$ is the vector of n ones. Standardly, a set of columns $J \subseteq V$ is called *basic* if $Bx = e_n$ for a non-negative x such that $x_j = 0$ whenever $j \notin J$. (Conversely, $x_j > 0$ for $j \in J$, because matrix B is not degenerate for any Scarf matrix A and sufficiently large K.) Obviously, the set $J_0 = \{m + j \mid j \in [n]\}$ of the last n columns in B is basic. Moreover, it is well known that each basic set J can be obtained from J_0 by a sequence of exchanges produced by the Simplex Method (see Example 2). In that example, we assigned the room $V \setminus J$ to each basic set J in B thus getting an m-dimensional oik $\mathcal{O}_B$. The next theorem shows that A and B generate the same oik, i.e. $\mathcal{O}_A = \mathcal{O}_B$. In other words, we now prove that the Scarf oik $\mathcal{O}_A$ is polytopal.

Theorem 5 *A column-set J is basic in B if and only if the complementary set $V \setminus J$ is dominating in A, provided $K > 0$ is sufficiently large.*

Proof: . It is easily seen that the original basis $J_0 \subseteq V$ in B is complementary to the special dominating set $[m]$ in A.

As an exercise, let us exchange one column in J_0, say, $m+1$ by m. We have to show that column-set $J = (J_0 \setminus \{m+1\}) \cup \{m\}$ is basic in B if and only if $V \setminus J = [m-1] \cup \{m+1\}$ is dominating in A. As we know, J is basic iff the following system of equations has a (strictly) positive solution:

$$\begin{aligned} b(1,m)x_m &= 1 \\ b(i,m)x_m + x_{m+i} &= 1 \quad i = 2, \ldots, n. \end{aligned} \tag{4.2}$$

Let us substitute $b(i,m) = 1 - a(m, m+i)^{-K}$ for $i \in [n]$ and obtain

$$\begin{aligned} x_m/a(m, m+1)^K &= x_m - 1, \\ x_m/a(m, m+i)^K - x_{m+i} &= x_m - 1 \quad i = 2, \ldots, n. \end{aligned} \tag{4.3}$$

It is easy to see that $x_m = (1 - a(m, m+1)^{-K})^{-1} > 1$, since $a(m, m+1) > 1$, by (v). This further implies that the right-hand side of system (4.3), $x_m - 1$, is positive. Furthermore, for $i = 2, \ldots, n$, $x_{m+i} > 0$ if and only if $a(m, m+1) > a(m, m+i)$. Thus the following proposition holds.

Proposition 1 *The following four statements are equivalent*

(vi) $a(m, m+1) > a(m, m+i)$ *for* $i = 2, \ldots n$*;*

(vii) column-set $V \setminus J = \{1, \ldots, m-1, m+1\}$ *is dominating in* A*;*

(viii the system (4.3) *has a (strictly) positive solution;*

(ix) column-set $J = \{m, m+2, \ldots, m+n\}$ *is basic in* B*.* □

Now let us consider the general case. Without any loss of generality, we can assume that J consists of the first ℓ and last $n-\ell$ columns of B, that is, $J = \{1, \ldots, \ell, m+\ell+1, \ldots, m+n\}$. The corresponding system of equations is

$$\begin{aligned} \sum_{k \in [\ell]} b(i,k)x_k &= 1 \quad i \in [\ell] = \{1, \ldots, \ell\}, \\ \sum_{k \in [\ell]} b(\ell+i,k)x_k + x_{m+\ell+i} &= 1 \quad i \in [n-\ell] = \{1, \ldots, n-\ell\}. \end{aligned} \tag{4.4}$$

Substituting $b(i,k) = 1 - a(k, m+i)^{-K}$ for each entry of B we get

$$\sum_{k\in[\ell]} x_k/a(k,m+i)^K = \sum_{k\in[\ell]} x_k - 1 \quad i \in [\ell],$$
$$\sum_{k\in[\ell]} x_k/a(k,m+\ell+i)^K - x_{m+\ell+i} = \sum_{k\in[\ell]} x_k - 1 \quad i \in [n-\ell]. \tag{4.5}$$

Let us notice that the right-hand side of system (4.5) is fixed at $\sum_{k\in[\ell]} x_k - 1$. We will show that this value is positive.

Let us recall that, by (v), $a(k, m+i) > 1$ for all entries $i \in [n]$, $k \in [m]$, of the Scarf matrix A. Also, in each row of A, the last n entries are pairwise distinct. Hence, for each variable x_k, $k \in [\ell]$ on the left-hand side of (4.5), the coefficient $a(k, m+i)^{-K}$ corresponding to the smallest entry $a(k, m+i)$, over $i \in [n]$, is much larger than all other row entries, provided $K > 0$ is very large. Let $i(k)$ be such a row with respect to column k, and let us scale variables $x_k, k \in [\ell]$ to make such largest coefficients equal. We shall show that $i(k) \neq i(k')$ for each $k \neq k' \in [\ell]$.

If the first ℓ equations of system (4.5) have a positive solution for a sufficiently large K then each such equation contains (exactly) one of these largest coefficients. Equivalently, all columns $\{m+i |\ i \in [\ell]\}$ are dominated by $V \setminus J$ in A. Indeed, let $i, i' \in [\ell]$ be two equations of (4.5) such that i contains a largest coefficient, while i' does not. Then, obviously, the left-hand sides of i and i' cannot be equal, while the right-hand sides are equivalent, which is a contradiction.

Thus, without loss of generality, we can assume that $a = a(i, m+i) \leq a(j, m+s)$ for all $i, j, s \in [\ell]$ and equality holds only when $j = s$. In other words, the left-hand side matrix of equations $i \in [\ell]$, of (4.5), has a constant main diagonal $1/a$ which is the larger than all non-diagonal entries.

Let us substitute for each equation $i \in [\ell]$ of (4.5) the following approximating system

$$x_i/a^K = \sum_{k\in[\ell]} x_k - 1, \quad i \in [\ell]. \tag{4.6}$$

Obviously, (4.6) has a unique solution $x^*(K) \in \mathbb{R}^\ell$, where $x_i^* = x_i^*(K) = (\ell - a^{-K})^{-1} > 1/\ell$ for all $i \in [\ell]$. In particular, $\sum_{k\in[\ell]} x_k^* - 1 > 0$ for each $K > 0$ and $\lim_{K\to+\infty} x^*(K) = (1/\ell, \ldots, 1/\ell)$. It is also clear that

(x) for any sufficiently large positive K, the system of the first ℓ equation in (4.5) has a unique solution $x^* = x^*(K) = (x_1^*, \ldots, x_\ell^*)$, where $x_i^* > 1/\ell$ for all $i \in [\ell]$; in particular, (4.5) has a positive right-hand side $(x_1^* + \ldots + x_\ell^*) - 1$; moreover, $x^*(K)$ tends to the same constant vector $(1/\ell, \ldots, 1/\ell)$, as $K \to +\infty$.

Now let us further consider the remaining $n - \ell$ equations of (4.5).

Proposition 2 *Equation $\ell + i$ for $i \in [n-\ell]$ of (4.5) has a (strictly) positive solution if and only if the corresponding column $m + \ell + i$ of A is dominated by $V \setminus J$.*

Proof: If column $m+\ell+i$ is dominated then, by definition of B, the coefficient of x_j for some $j \in [\ell]$ (on the left-hand side of (4.5)) in row $\ell+i$ is much larger than any other coefficient of x_j in (4.5). Hence, by (x),

$$\sum_{k\in[\ell]} x_k/a(k,m+\ell+i)^K > \sum_{k\in[\ell]} x_k - 1 \tag{4.7}$$

and there is a (unique) positive $x_{m+\ell+i}$ which settles the equation $\ell+i$ for $i \in [n]$ of (4.5).

Conversely, if the considered column $m+\ell+i$ is not dominated by $V \setminus J$ in A then the left-hand side of each equation $j \in [\ell]$ of (4.5) is such that for some $x_{k(j)}$, the coefficient $1/a(k(j),m+\ell+j)^K$ is much larger than each such coefficient of equation $m+\ell+i$ of (4.5). Hence, $x_{m+\ell+i}$ must be negative and the set of columns J is not basic in B. □

This concludes the proof of Theorem 5. □

5 Exponential Length Exchange-Paths

The exchange path between two room-partitions of a d-dimensional oik $\mathcal{O} = (V,\mathcal{R})$ with $n = |V|$, may be exponential:

(i) in the number of vertices n, already for dimension $d = 3$, and

(ii) in dimension d already for $n = 2d$ vertices, or in other words, for only 2 rooms in a partition.

A construction for (i) was given in [13]. For each $k \geq 3$, there is a 3-dimensional oik $\mathcal{O}$ defined by $12(k-2)$ rooms (triangles) on $n = 3k$ vertices. (Hence, each room-partition consists of k rooms.) This oik has an exchange path of length $7 \times 2^{k-1} - 5$ between two room-partitions.

Here, for each $d \geq 2$ we construct two particular $d \times 2d$ canonical Scarf matrices, the first of which yields an exponential length path between two room-partitions where both oiks are Scarf, while the second also yields an exponential length path between two room-partitions, but both oiks are not identical; one is a Scarf oik and the other is a Sperner oik.

We show, by induction, that the first oik has an exchange path of length $\frac{3}{4}\times 2^d - 1$ between two room-partitions, while the second oik yields a path of length $2^d - 1$. We actually show a slightly stronger result than the first statement; the two room partitions of our path are the *only* room partitions of O_d, and, furthermore, the exchange graph is comprised *only* of this path.

The first oik-pair we consider, $\mathcal{O}_d^{ff} = (\mathcal{O}_d, \mathcal{O}_d)$ where $\mathcal{O}_d$ is a d-dimensional Scarf oik defined over a particular $d \times 2d$ Scarf matrix A_d, is referred to as *Scarf-Scarf*. The second oik-pair, $\mathcal{O}_d^{fr} = (\mathcal{O}'_d, \mathcal{O}''_d)$ where $\mathcal{O}'_d$ and $\mathcal{O}''_d$ are d-dimensional Scarf and Sperner oiks, respectively, defined over a particular $d \times 2d$ Scarf matrix A'_d, is called *Scarf-Sperner*.

Before we establish the results, we record an observation made in the proof of Scarf's Lemma (see section 3), which will be useful in both of the following subsections.

Lemma 5 *Let $W \subseteq V = [2d]$ be a wall of oik $\mathcal{O}$ such that $W \nsubseteq [d]$, where $\mathcal{O} = (V, \mathcal{R})$ is a d-dimensional Scarf oik with V being the set of columns of $d \times 2d$ Scarf matrix A.*

(i) There exists a unique column $j_0 \in W$ which has exactly two rows, say, i_1 and i_2, in which column j_0 is dominated. Furthermore, the rows i_1 and i_2 of column j_0 are such that $a(i_k, j_0) < a(i_k, j)$ for each $j \in W \setminus \{j_0\}$ and $k \in \{i_1, i_2\}$, i.e. they are unique row minima.

(ii) The set $W \cup \{j^k\}$ is a room of oik $\mathcal{O}_d$ if $j^k = \arg\max_{j \in J_k} a_d(k, j)$ for $k \in \{i_1, i_2\}$, where

$$J_k = \{j \in [2d] |\ a_d(k, j) < a_d(k, j_0), a_d(i, j) > a_d(i, j_i),\ \forall i \in [d] \setminus \{k\}\}, \tag{5.8}$$

and standardly for a fixed W, $j_i = \arg\min_{j \in W} a_d(i, j)$ for each row $i \in [d]$. □

5.1 Scarf-Scarf oik family

The next subsection demonstrates some useful properties of the rooms of our particular Scarf oik $\mathcal{O}_d$, which arise from a recursive definition of the corresponding Scarf matrix. Then in the following subsection, we exploit some properties of room- and skew-room partitions, which arise from the established properties of rooms, to conclude with the main results of this section.

5.1.1 Properties of Rooms

Let the Scarf oik $\mathcal{O}_d = (V_d, \mathcal{R}_d)$, for $d \geq 2$, be associated with a particular $d \times 2d$ canonical Scarf matrix A_d, which is defined recursively as follows. Let $B_1 = 0$ and $C_1 = 1$, and for $d \geq 2$ define

$$B_d = \begin{bmatrix} 0 & \alpha_{d-1}^T \\ \alpha_{d-1} & B_{d-1} \end{bmatrix} \qquad C_d = \begin{bmatrix} 1 & \delta_{d-1}^T \\ \gamma_{d-1} & C_{d-1} \end{bmatrix}, \tag{5.9}$$

where $\alpha_{d-1} = M \cdot \mathbf{1}_{d-1}$, $\delta_{d-1} = (2, 3, \ldots, d)$, $\gamma_{d-1} = d \cdot \mathbf{1}_{d-1}$, and $\mathbf{1}_k$ is the vector of k 1s. Now let $A_d = [B_d | C_d]$. Clearly A_d, for $d \geq 2$, is a canonical Scarf Matrix, which can be seen by the following constructed matrices.

$$A_2 = \left[\begin{array}{cc|cc} 0 & M & 1 & 2 \\ M & 0 & 2 & 1 \end{array}\right]$$

$$A_3 = \left[\begin{array}{ccc|ccc} 0 & M & M & 1 & 2 & 3 \\ M & 0 & M & 3 & 1 & 2 \\ M & M & 0 & 3 & 2 & 1 \end{array}\right]$$

$$A_4 = \left[\begin{array}{cccc|cccc} 0 & M & M & M & 1 & 2 & 3 & 4 \\ M & 0 & M & M & 4 & 1 & 2 & 3 \\ M & M & 0 & M & 4 & 3 & 1 & 2 \\ M & M & M & 0 & 4 & 3 & 2 & 1 \end{array}\right]$$

$$\vdots$$

$$A_d = \left[\begin{array}{cccccc|cccccc} 0 & M & M & \dots & M & M & 1 & 2 & 3 & \dots & d-1 & d \\ M & 0 & M & \dots & M & M & d & 1 & 2 & \dots & d-2 & d-1 \\ M & M & 0 & \dots & M & M & d & d-1 & 1 & \dots & d-3 & d-2 \\ & \vdots & & \ddots & & \vdots & & \vdots & & \ddots & & \vdots \\ M & M & M & \dots & 0 & M & d & d-1 & d-2 & \dots & 1 & 2 \\ M & M & M & \dots & M & 0 & d & d-1 & d-2 & \dots & 2 & 1 \end{array}\right]$$

We let $a_d(i,j)$ be the (i,j)th entry of A_d, and A_d^J be the submatrix of A_d consisting of columns only with indices $j \in J$. We refer to column $A_d^{\{i\}}$ as A_d^i or simply by its index i. Recall that a set of columns J is *dominating* in A_d (or *dominates* A_d) if for each column $k \in V_d = [2d]$, there exists a row index $i \in [d]$ such that $a_d(i,k) \le a_d(i,j)$ for each column $j \in J$, and furthermore we shall say that column k is *dominated by* J *in row* i.

Remark 13 *By definition of A_d, we have $a_d(i,j) < a_d(i,k)$ for all triples (i,j,k) satisfying $1 \le i \le j-d$, $d < j < k \le 2d$, and $a_d(i,j) > a_d(i,k)$ for all triples (i,j,k) satisfying $j-d < i \le d$ and $d < j < k \le 2d$.*

We define the one-to-one map $\sigma_{d-1} : \mathbb{Z} \to \mathbb{Z}$ as

$$\sigma_{d-1}(i) := \begin{cases} i+1 & \text{if } i \le d-1 \\ i+2 & \text{if } i > d-1 \end{cases} . \tag{5.10}$$

We shall refer to the map as $\sigma(\cdot)$ when the subscript is clear from context. For any set $J \subseteq \mathbb{Z}$, let $\sigma(J) = \{\sigma(j) : \ j \in J\}$. It is easy to see that $\sigma(V_{d-1}) = V_d \setminus \{1, d+1\}$. Essentially, when restricted to the set $V_{d-1} = [2d-2]$, the map σ "shifts" the index i of column A_{d-1}^i to the corresponding index $\sigma(i)$ of column $A_d^{\sigma(i)} = (\kappa, A_{d-1}^i)$, where κ is a scalar (see the above recursive definition of A_d). We implicitly assume hereafter that $d \ge 2$ so that any subscripts $d-1 \ge 1$ yield well-defined matrices.

Lemma 6 *For each room $R \in \mathcal{R}_{d-1}$, the set $\sigma(R)$ is a wall of oik $\mathcal{O}_d$.*

Proof: Choose a room $R \in \mathcal{R}_{d-1}$. This implies that column A_{d-1}^j, $j \in V_{d-1}$, is dominated by R in row $i_j \in [d-1]$. By choice of the map σ and the recursive definition of A_d, columns $A_d^{\sigma(j)}$, $\sigma(j) \in [2d] \setminus \{1, d+1\}$, are dominated by $\sigma(R)$ in row $i_j + 1 \ge 2$. Furthermore, since $a_d(1,1) < a_d(1,d+1) < \min_{k \in [2d]\setminus\{1,d+1\}} a_d(1,k)$, columns A_d^ℓ, $\ell \in \{1, d+1\}$ are dominated by $\sigma(R)$ in row 1. Thus the set $\sigma(R)$ dominates A_d, i.e. $\sigma(R)$ is a wall of $\mathcal{O}_d$. $\square$

Lemma 7 *For any room $R \in \mathcal{R}_d$, if column $1 \in R$ then $R = [d+1] \setminus \{i\}$ for some $i \in [d+1] \setminus \{1\}$. Moreover, for each $i \in [d+1] \setminus \{1\}$, $R = [d+1] \setminus \{i\} \in \mathcal{R}_d$.*

Proof: Choose $R \in \mathcal{R}_d$ such that $1 \in R$. Since R dominates A_d and $a_d(1,1) < a_d(1,d+1)$, column $d+1$ must be dominated in some row $j \geq 2$. By the structure of matrix A_d, we have $a_d(j,d+1) > a_d(j,k)$ for each $k \geq d+2$ and $j \geq 2$. Thus no column $k \geq d+2$ can be contained in the (maximal) dominating set R, i.e. we must have $R \subseteq [d+1]$, proving the first claim.

If $d+1 \notin R$ then, since also $|R| = d$, we must have $R = [d]$, which is our special dominating set. Thus $R = [d+1] \setminus \{i\} \in \mathcal{R}_d$ for $i = d+1$.

Now if $d+1 \in R$, then $R = [d+1] \setminus \{i\}$ for some $i \in [d] \setminus \{1\}$. Clearly $\min_{j \in R} a_d(i,j) = d+1$, which further implies that columns of the set $V_d \setminus R$ and column $d+1$ are dominated by R in row i. For the remaining columns of $V_d = [2d]$, notice that $R \setminus \{d+1\} \subseteq [d]$ is also dominating by (vi) of Section 1.3, which clearly implies that columns $j \in R \setminus \{d+1\}$ are dominated in row j. Thus $R = [d+1] \setminus \{i\} \in \mathcal{R}_d$ for each $i \in [d] \setminus \{1\}$, and our second claim is proved. □

Lemma 8 *For any room $R \in \mathcal{R}_d$, if $\{1, d+1\} \cap R = \emptyset$, then column $2 \in R$.*

Proof: Let us suppose that $2 \notin R$. So we have $\{1, 2, d+1\} \cap R = \emptyset$ and $|R| = d$, which together imply $R \setminus [d] \neq \emptyset$. Furthermore, since also $1 \notin R$, column $j_0 = \min\{j \in R \setminus [d]\} > d+1$ must be the unique column of R which is dominated in the first row. Now since $2 \notin R$, then it can be seen, by Remark 13, that column j_0 is also dominated in row 2, contradicting the choice $R \in \mathcal{R}_d$ (see also (ix) of section 1.3). □

Lemma 9 *For any set $W \subseteq V_d$ of cardinality $d-1$ such that $W = \sigma(R) = \sigma_{d-1}(R)$ for some room $R \in \mathcal{R}_{d-1}$:*

(i) $W \cup \{1\}$ and $W \cup \{2d\}$ are rooms of oik $\mathcal{O}_d$ if $W \subseteq [d]$,

(ii) $W \cup \{d+1\}$ and $W \cup \{j(W)\}$ are rooms of oik $\mathcal{O}_d$ if $W \not\subseteq [d]$, where

$$j(W) = \begin{cases} 2 & \text{if } W \setminus [d] = \{2d\} \text{ or } |W \setminus [d]| > 1 \\ d+3 & \text{if } W \setminus [d] = \{d+2\} \text{ and } 3 \in W, \\ 2d & \text{if } W \setminus [d] = \{d+2\} \text{ and } 3 \notin W. \end{cases}$$

Proof: By Lemma 6, each such set W is a wall of oik $\mathcal{O}_d$. Since $\mathcal{O}_d$ is a 2-adjacent oik (by Scarf's Lemma 3), there are exactly two extensions, per wall, to rooms in $\mathcal{R}_d$. Recall that for any $R \in \mathcal{R}_{d-1}$, $\{1, d+1\} \cap \sigma(R) = \emptyset$.

Case (i): $W \subseteq [d]$. This supposition, along with $W = \sigma(R)$ for some $R \in \mathcal{R}_{d-1}$, implies $W = [d] \setminus \{1\}$. Clearly $W \cup \{1\} = [d] \in \mathcal{R}_d$, i.e. we can extend W to the room $W \cup \{1\}$ in $\mathcal{O}_d$. Now since the diagonal elements $a_d(i,i) = 0 = \min_{j \in [2d]} a_d(i,j)$, for each column $i \in W$, we must choose a column $j_0 > d+1$ such that A_d is dominated in the first row. This implies $j_0 = \arg\max_{j \in [2d] \setminus [d]} a_d(1,j)$. It is easy to see that $j_0 = 2d$ (see Remark 13). Thus $W \cup \{2d\} \in \mathcal{R}_d$.

Case (ii)**:** $|W \setminus [d]| \geq 1$. This implies that $W \setminus [d] \neq \emptyset$, so (i) of Lemma 5 tells us that there exists a unique column $j_0 \in W$ which contains exactly two row minima, say i_1 and i_2. Furthermore, $W = \sigma(R)$ for some $R \in \mathcal{R}_{d-1}$ implies that one such row minima is in row $i_1 = 1$ (see Remark 13), which, moreover, tells us that $j_0 > d$. By Remark 13, we must have $j_0 = \min\{j \in W \setminus [d]\}$. Now, invoking Lemma 5 part (ii), we must find $J_{i_1} = J_1$ and J_{i_2}. Note that $J_1 \subseteq \{1, d+1, \ldots, j_0 - 1\}$ by the structure of the first row of A_d.

First suppose $|W \setminus [d]| = 1$. This implies $W \setminus [d]$ consists only of the column j_0. Since we have found that $i_1 = 1$, we must find i_2, but before doing so, let us determine which values j_0 can assume. Clearly, $W' = W \setminus \{j_0\} \subseteq [d] \setminus \{1\}$, and $|W'| = d - 2$. Furthermore, $R' = \sigma^{-1}(W') \subseteq [d-1]$ is a wall of $\mathcal{O}_{d-1}$ (where $\sigma^{-1}(\cdot)$ is the inverse map). If $1 \in R'$ (and hence in $R' \cup \{\sigma^{-1}(j_0)\}$ since $j_0 \geq d+2$), then Lemma 7 tells us $\sigma^{-1}(j_0) = d$, i.e. $j_0 = d+2$. Now if $1 \notin R'$, then we have $R' = [d-1] \setminus \{1\}$, and so clearly $j_0 = 2d$ in this case (since $R' \cup \{1\}, R' \cup \{2d-2\} \in \mathcal{R}_{d-1}$ are the only two extensions of R', by case (i) above with d replaced with $d-1$).

Now let us suppose that $j_0 = 2d$. In this case $R' = [d-1] \setminus \{1\}$ implies $W' = [d] \setminus \{1, 2\}$, i.e. $W = \{3, 4, \ldots, d, 2d\}$. Thus, by inspection, we have $i_2 = 2$ (and recall $i_1 = 1$), as well as $a_d(1, j_1) = d$ and $a_d(2, j_2) = d - 1$. Thus, we must have $J_1 = \{1, d+1\}$ and $J_2 = \{2\}$, which yields $j^1 = d+1$ and $j^2 = 2$, i.e. $W \cup \{2\}, W \cup \{d+1\} \in \mathcal{R}_d$.

Now we suppose $j_0 = d+2$. Recall that in this case $1 \in R'$, i.e. $2 \in W$. This implies $i_2 \geq 3$ (and whichever value i_2 takes on, *columns* $i_1 = 1$ and i_2 must make up the set $[d] \setminus W$). Clearly, $J_{i_2} = \{d+3, d+4, \ldots, 2d\}$. Remark 13 tells us that if $i_2 = 3$ then $W \cup \{2d\} \in \mathcal{R}_d$, and if $4 \leq i_2 \leq d$, then $W \cup \{d+3\}$. In all cases, $J_1 = \{1, d+1\}$, and so $W \cup \{d+1\} \in \mathcal{R}_d$.

Finally, we suppose $|W \setminus [d]| > 1$. Again, let $j_0 \geq d+2$ be the leftmost column in $W \setminus [d]$, $i_1 = 1$ and $i_2 > 1$ be the two row minima of j_0. Since $|W \setminus [d]| > 1$, there exists a column $j \in W$ such that $j_0 < j \leq 2n$. Remark 13 tells us that the row minimum i_2 is such that $2 \leq i_2 \leq j_0 - d < d$. Clearly, column $i_2 \notin W$, and since i_1 and i_2 are the only row minima, with respect to column j_0, we have $i_2 = \min\{i \in [2d] |\ i \notin W \cup \{1\}\}$. In other words, every other minimum value $a_d(i, j_i) = 0$ for rows $i \neq i_0$ such that $2 \leq i \leq j_0 - d$, attained at some $j_i \in W$. It is easy to see that $J_{i_2} = \{i_2\}$. Indeed, no column j such that $d \leq j < j_0$ satisfies $a_d(1, j) > a_d(1, j_0) = a_d(1, j_1)$ and no $j_0 < j \leq 2d$ satisfies $a_d(i_2, j) < a_d(i_2, j_0) = a_d(i_2, j_{i_2})$.

Now since $W = \sigma(R)$ for some $R \in \mathcal{R}_{d-1}$, we cannot have $1 \in R$. Indeed, otherwise Lemma 7 characterizes $R = \{d\} \setminus \{i\}$ for $i \in \{d\} \setminus \{1\}$, which, by choice of σ, implies $|W \setminus [d]| \leq 1$, a contradiction. Thus $i_2 = 2$, regardless of j_0. Hence, we must have $J_2 = \{2\}$ (as well as $\{3, 4, \ldots, j_0 - d\} \subseteq W$) and so $W \cup \{2\} \in \mathcal{R}_d$. Furthermore, it is easy to see that $J_1 = \{1, d+1\}$, and so $W \cup \{d+1\} \in \mathcal{R}_d$. $\square$

Lemma 10 *Each* $R \in \mathcal{R}_d$ *is such that* $\{1, d+1\} \not\subseteq R$ *if and only if there exists a column* $i \in R$ *such that* $\sigma^{-1}(R \setminus \{i\}) \in \mathcal{R}_{d-1}$.

Proof: Sufficiency is trivial, since $\{1, d+1\} \subseteq R$ implies $\sigma^{-1}(R \setminus \{i\})$ is not well-defined for any $i \in R$.

For necessity, we must prove that for some $i \in R$, $\sigma^{-1}(R \setminus \{i\})$ exists, and each column $j \in V_d \setminus \{1, d+1\}$ is dominated in some row $i(j) \geq 2$.

First suppose $\{1, d+1\} \cap R = \{j^*\}$. If $j^* = 1$ (hence $d+1 \notin R$), then by Lemma 7 we must have $R = [d]$. Clearly, $\sigma^{-1}(R \setminus \{j^*\}) = [d-1] \in \mathcal{R}_{d-1}$. Now if $j^* = d+1$ (hence $1 \notin R$), then we must have $R \setminus [d+1] \neq \emptyset$. Indeed, otherwise $R \setminus \{j^*\} = [d] \setminus \{1\}$, and Lemma 9 tells us that the only two entering columns are 1 and $2d$, contradicting the choice $j^* = d+1$.

Now since column $d+1$ is clearly dominated in row 1, column $j_0 = \min\{R \setminus [d+1]\}$ must be dominated in some row $i^* \geq 2$. Moreover, column j_0 will inherit the row 1 minimum when column $d+1$ is removed to form the wall $W = R \setminus \{d+1\}$. It is easy to check that if $|R \setminus [d+1]|$, then $j_0 = d+2$ and $i^* \geq 3$.

Clearly, all columns $j \in \{2, \ldots, d\}$ are dominated in row j and since $a_d(1, d+1) \leq a_d(1, j')$ for $j' \geq d+2$, each $j \in \{j_0, \ldots, 2d\}$ remains dominated in some row $i(j) \geq 2$ when forming W. Now it remains to show that each $j \in \{d+2, \ldots, j_0 - 1\}$ is dominated in some row $i(j) \geq 2$. If $j_0 = d+2$, then $\{d+2, \ldots, j_0\} = \emptyset$ and we are done, so suppose $j_0 > d+2$. It follows that $i(j_0) = 2$, otherwise column $d+2$ was not dominated in R, a contradiction. It is easy to see now, by 13, that each $j \in \{d+2, \ldots, j_0 - 1\}$ is dominated in row 2.

The case for $\{1, d+1\} \cap R = \emptyset$ is similar, since $|R| = d$ implies that $R \setminus [d+1] \neq \emptyset$ and Lemma 8 tells us that $2 \in R$. It is easy to see then that $\sigma^{-1}(R \setminus \{2\}) \in \mathcal{R}_{d-1}$. □

Lemma 11 *For each oik $\mathcal{O}_d = (V_d, \mathcal{R}_d)$ if we let $D_k^d = \{R \in \mathcal{R}_d | \ |R \setminus [d]| = k\}$ for $k \in [d]$, then we have $|D_k^d| = \binom{d}{k}$. Moreover, $|\mathcal{R}_d| = 2^d$.*

Proof: The second claim follows immediately from the first, due to the well known equality $\sum_{k\in[d]} \binom{d}{k} = 2^d$ and $\mathcal{R}_d = \dot\cup_{k\in[d]} D_k^d$.

We prove the first claim by induction on d. The case $d = 2$ is trivial; by inspection of A_2 it immediately holds. Now we suppose $|D_k^{d-1}| = \binom{d-1}{k}$. By Lemmas 9 and 10, it suffices to exhaust the σ maps and extensions of each room $R \in D_k^{d-1}$ for $k \in [d-1]$, and to characterize which rooms $R \in \mathcal{R}_d$ are such that $\{1, d+1\} \subseteq R$.

First consider $R \in D_0^{d-1}$. Since $|R| = d-1$ and by the induction hypothesis $|D_0^{d-1}| = 1$, we know that the only room is $R = [d-1]$. And so by Lemma 9 the two extensions of $\sigma(R)$ are to rooms $[d] \in D_0^d$ and $[d] \setminus \{1\} \cup \{2d\} \in D_1^d$, which further gives us $|D_0^d| \geq 1 = \binom{d}{0}$ and $|D_1^d| \geq 1$. Furthermore, by Lemma 7, the $d-1$ rooms $R \supseteq \{1, d+1\}$ which cannot be obtained by extensions, are contained in D_1^d. Thus $|D_1^d| \geq 1 + (d-1) = \binom{d}{1}$.

Now consider $R \in D_k^{d-1}$ for $k \geq 2$. By Lemma 9, we have $R^1 = \sigma(R) \cup \{2\} \in D_k^d$ and $R^2 \in \sigma(R) \cup \{d+1\} \in D_{k+1}^d$ as the only two extensions of R. Clearly, since each $R \in D_k^{d-1}$ is distinct for $k \geq 2$, we must have R^1 and R^2 distinct over all such k. This provides us with $|D_k^d| \geq \binom{d-1}{k-1} + \binom{d-1}{k} = \binom{d}{k}$ for $3 \leq k \leq d-1$. But we also have $|D_d^d| \geq \binom{d-1}{d-1} = 1 = \binom{d}{d}$ and $|D_2^d| \geq \binom{d-1}{2}$. Since we want $|D_2^d| = \binom{d}{2}$, we must find $d-1$ other extensions which have not yet been counted.

The only set of $\mathcal{R}_{d-1}$ left to extend is D_1^{d-1}. Clearly, we have $R_0 = [d-1] \setminus \{1\} \cup \{2d-2\} \in D_1^{d-1}$ which extends to the room $[d] \setminus \{1\} \cup \{2d\} \in D_1^d$. But this set has already been counted since $[d] \setminus \{1\} \in D_0^{d-1}$ extends to it. Now let $D' = D_1^{d-1} \setminus \{R_0\}$.

The rest of the sets $R \in D'$, by Lemma 9, extend to $R' = \sigma(R) \cup \{j(R)\}$, where $j(R) = d+3$ if $3 \in R$ and $j(R) = 2d$ otherwise. In either case, we must have $1 \in R$, i.e. $2 \in \sigma(R)$

and $\{1, d+1\} \cap \sigma(R) = \emptyset$. This implies, by Lemma 10, that $\sigma^{-1}(R' \setminus \{2\}) \in D_2^{d-1}$. But we have already counted these rooms. Finally, we see that each room $R \in D_1^{d-1} = D' \cup \{R_0\}$, extends to $\sigma(R) \cup \{d+1\}$, which has not yet been counted. Indeed, otherwise we would have, for some $R'' \neq R$ and $i \neq d+1$, $\sigma(R'') \cup \{i\} = \sigma(R) \cup \{d+1\}$. Clearly, this is impossible, since $d+1 \notin \sigma(R'')$ for any such set. Thus, since $|D_1^{d-1}| = \binom{d-1}{1}$, we have $|D_2^d| \geq \binom{d-1}{2} + \binom{d-1}{1} = \binom{d}{2}$. Furthermore, since we have exhausted all possibilities, the inequalities $D_k^d \geq \binom{d}{k}$ become equalities, and our claim is proved. □

What we have shown in this section is that each of the 2^{d-1} rooms of $\mathcal{O}_{d-1}$ can be mapped (via σ) and extended to two rooms of $\mathcal{O}_d$. Furthermore, each of the $2^d - (d-1)$ rooms of $\mathcal{O}_d$ which do not contain the set $\{1, d+1\}$ can be found via such a mapping and extension of at least one room of $\mathcal{O}_{d-1}$.

5.1.2 Properties of Room and Skew-Room Partitions

Now let $\mathcal{G}_d = \mathcal{G}(\mathcal{O}_d^{ff}, w)$ be the exchange graph of the Scarf-Scarf oik-pair $\mathcal{O}_d^{ff} = (\mathcal{O}_d, \mathcal{O}_d)$ (recall $\mathcal{O}_d = (V_d, \mathcal{R}_d)$), with vertex set $\mathcal{V}_d \cup \mathcal{V}_d^w$, where $\mathcal{V}_d$ consists of all 2-room-partitions $\mathcal{R} = (R_1, R_2)$ and $\mathcal{V}_d^w$ consists of all skew room-partitions (butterflies) $\mathcal{S} = (S_1, S_2)$ with $\delta_{\mathcal{S}}(w) = 0$ (recall $R_i, S_i \in \mathcal{R}_d$ for $i \in \{1, 2\}$). Our edge set is denoted $\mathcal{E}_d$, where again two vertices $\mathcal{R} = (R_1, R_2), \mathcal{S} = (S_1, S_2) \in \mathcal{V}_d \cup \mathcal{V}_d^w$ form an edge if and only if $R_i = S_j$ and $|R_{3-i} \cap S_{3-j}| = d-1$ for some pair $i, j \in \{1, 2\}$. For our exponential length exchange path $\mathcal{P}_d$, we shall search the exchange graph $\mathcal{G}_d(\mathcal{O}_d, 1)$, i.e. we choose $w = 1$.

Remark 14 *Since $\mathcal{O}_d$ is a 2-adjacent Scarf oik, each wall is adjacent to exactly two rooms (see Scarf's Lemma: Theorem 3). Suppose the room selections $\mathcal{R} = (R_1, R_2)$ and $\mathcal{S} = (S_1, S_2)$ form an edge in $\mathcal{G}_d$ on some path $\mathcal{P} \subseteq \mathcal{V}_d \cup \mathcal{V}_d^w$, and that $R_1 = S_1$ and $|R_2 \cap S_2| = d-1$. Since two room partitions cannot be adjacent, either $\mathcal{R}$ or $\mathcal{S}$ has another neighbor $\mathcal{T} = (T_1, T_2)$ on $\mathcal{P}$. Suppose that $(\mathcal{S}, \mathcal{T}) \in \mathcal{E}_d$. The wall $W = R_2 \cap S_2$ of $\mathcal{O}_d$ is adjacent to exactly two rooms of $\mathcal{O}_d$, namely R_2 and S_2. Since $\mathcal{S}$ and $\mathcal{T}$ are adjacent, without loss of generality, $S_i = T_i$ and $|S_{3-i} \cap T_{3-i}| = d-1$ for $i = 1$ or $i = 2$.*

When moving along the arbitrary path from $\mathcal{R}$ to $\mathcal{S}$, we removed the only element $j^1 \in R_1 \cap R_2$ from R_2, forming $W = R_1 \setminus \{j^1\}$, and replaced it with the only element $j^2 \in S_1 \cap S_2$ to get $S_1 = R_1$ and forming $S_2 = W \cup \{j^2\}$. Now, to move from $\mathcal{S}$ to $\mathcal{T}$, we must now remove the only element $j^2 \in S_1 \cap S_2$ from either S_1 or S_2. Clearly, we cannot remove it from S_2, since otherwise we are left with W which is only adjacent to the rooms R_2 and S_2, which brings us back a step in the path to $\mathcal{R}$. Thus only the second case above can hold, i.e. $i = 2$ and $S_2 = T_2$ and $|S_1 \cap T_1| = d-1$.

Repeating this argument from one room partition to another along the path $\mathcal{P}$, we must alternate the "exchange" of rooms. In other words, for any three vertices $\mathcal{R}, \mathcal{S}$, and $\mathcal{T}$ in $\mathcal{P}$ such that $(\mathcal{R}, \mathcal{S}), (\mathcal{S}, \mathcal{T}) \in \mathcal{E}_d$ as above, then, without loss of generality, we must have $R_i = S_i, S_{3-i} = T_{3-i}$ and $|R_{3-i} \cap S_{3-i}| = |S_i \cap T_i| = d-1$ for $i = 1$ or $i = 2$.

Let $E(\mathcal{P}_d)$ be the edge set of path $\mathcal{P}_d$ and let $\ell(d) = |E(\mathcal{P}_d)|$ denote the *length* of $\mathcal{P}_d$. Also let $\mathcal{P}_d = (\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{\ell(d)})$ where $\mathcal{R}^i = (R_1^i, R_2^i) \in \mathcal{P}_d$ for $i \in \{0, \ldots, \ell(d)\}$, and $(\mathcal{R}^{i-1}, \mathcal{R}^i) \in$

$E(\mathcal{P}_d)$ for $i = 1, \ldots, \ell(d)$. We shall call the one endpoint, or room partition, $\mathcal{R}^0 = (R_1^0, R_2^0)$ of $\mathcal{P}_d$ the *starting partition*, where $R_1^0 = \{1, \ldots, d\}$ and $R_2^0 = \{d+1, \ldots, 2d\}$. We also call the other endpoint $\mathcal{R}^{\ell(d)} = (R_1^{\ell(d)}, R_2^{\ell(d)})$ the *ending partition*, where $R_1^{\ell(d)} = \{1, 3, \ldots, d+1\}$ and $R_2^{\ell(d)} = \{2, d+2, \ldots, 2d\}$. For an arbitrary room selection $\mathcal{R} = (R_1, R_2)$, we refer to R_1 as the *first room* and R_2 as the *second room* of $\mathcal{R}$, i.e. the order of representation matters now (see also Remark 14).

When traversing the path $\mathcal{P}_d$ by starting at the starting partition $\mathcal{R}^0$, we first eliminate column 1 from room $R_1^0 = [d]$, which gives us the wall $W^0 = [d] \setminus \{1\} = \{2, \ldots, d\}$. It is not difficult to verify that the entering column is now $2d$ and, hence, the adjacent room is $h = W^1 \cup \{2d\} = \{2, \ldots, d, 2d\}$. Now, since $\mathcal{R}^1 = (R_1^1, R_2^1)$ is a butterfly with $R_2^1 = R_2^0$ and $R_1^1 \cap R_2^1 = \{2d\}$, we have $(\mathcal{R}^0, \mathcal{R}^1) \in \mathcal{E}_d$.

Now we remove column $2d$ from R_2^1. This gives us the wall $W^2 = \{d+1, \ldots, 2d-1\}$. It is not difficult to verify that the entering column is $d-1$, hence, the adjacent room is $R_2^2 = W^2 \cup \{2d\} = \{d-1, d+1, \ldots, 2d-1\}$. Now we see that $\mathcal{R}^2 = (R_1^2, R_2^2)$ is butterfly with $R_1^2 = R_1^1$ and $R_1^2 \cap R_2^2 = \{d-1\}$, thus $(\mathcal{R}^1, \mathcal{R}^2) \in \mathcal{E}_d$.

Next, we eliminate $d-1$ from R_2 giving us the wall $W^3 = \{2, \ldots, d-2, d, 2d\}$. It is not difficult to verify that the entering column is $2(d-1)$. Since the number of nodes in $\mathcal{G}_d$ is finite, we can continue until obtaining another room partition.

Since each room-partition consists of only two rooms, the exchange path is uniquely defined by a sequence of vertices $v_i \in V$ for $i = 1, \ldots, \ell(d) - 1$ satisfying $\delta_{\mathcal{R}^i}(v_i) = 2$. We call this sequence $I_d = (v_1, \ldots, v_{\ell(d)-1})$ and its reverse $I_d^* = (v_{\ell(d)-1}, \ldots, v_1)$. We shall see that constructively proving the existence of a certain exponential path of $\mathcal{G}_d$ by induction, yields the sequences I_d for $d \geq 2$, which can be conveniently represented as follows:

d	I_d
3	$(6, 2, 4, 6)$
4	$(8, 3, 6, 8, 5, 2, 8, 6, 3, 8)$
5	$(10, 4, 8, 10, 7, 3, 10, 8, 4, 10, 6, 2, 10, 4, 8, 10, 3, 7, 10, 8, 4, 10)$
...	...
d	$(\sigma_{d-1}(I_{d-1}), d+1, 2, \sigma_{d-1}(I_{d-1}^*))$

Remark 15 *Similar exponential exchange paths were constructed by Morris [28] and von Stengel [39]; see also [29, 30]. However, at least two distinct oiks are involved in both cases: Sperner and Gale oiks in [28], Sperner and two polytopal oiks (which might be isomorphic) in [39]. In our construction each room-partition consists of two rooms of a single Scarf oik.*

Remark 16 *Let us also remark that no exponential in d example can exist for a pair of d-dimensional Sperner oiks with $2d$ vertices in each. Morris [28] proved that in this case any exchange path between two room-partitions is of length at most $2d$. Moreover, it is not difficult to verify that there are exactly $(d-1)!$ paths of length $2d$. They connect the following room-partitions: Given $2d$ vertices, let us color them $\{1, 2, \ldots, d; 1, 2, \ldots, d\}$ in the first Sperner oik and $\{1, 2, \ldots, d; \sigma(1), \sigma(2), \ldots, \sigma(d)\}$ in the second one, where σ is a* *d-permutation. Let us consider two rooms induced by the first and second d vertices in the first and second oiks, respectively. It is easily seen that an exchange path beginning in this room-partition is of length 2d whenever permutation σ is prime (i.e., formed by a single cycle) and of length $< 2d$ otherwise.*

We shall prove now that the path $\mathcal{P}_d$ of the exchange graph $\mathcal{G}_d = \mathcal{G}(\mathcal{O}_d, 1)$ is of length $\ell(d) = \frac{3}{4}2^d - 1$, inductively, which begins with the starting partition $\mathcal{R}^0 = (\{1, 2, \ldots, d\}, \{d+1, d+2, \ldots, 2n\})$ of $\mathcal{O}_d^{ff}$ and ends with the ending partition $\mathcal{R}^{\ell(d)} = (\{1, 3, \ldots, d+1\}, \{2, d+2, \ldots, 2n\})$. By (i) of Section 1.3, $d \geq 2$. It is easy to verify (see constructed table above) the claim for $d = 2$. So now we prove it for d given it is true for all $2 \leq k < d$.

Lemma 12 *For the Scarf-Scarf oik-pair $\mathcal{O}_d^{ff}$, the starting and ending partitions are the only room partitions.*

Proof: Suppose we have a room partition (R_1, R_2) of $\mathcal{O}_d^{ff}$, i.e. $R_1, R_2 \in \mathcal{R}_d$ with $R_1 \cup R_2 = V_d = [2d]$ and $R_1 \cap R_2 = \emptyset$. Since exactly one set must contain column 1, without loss of generality, suppose $1 \in R_1$. By Lemma 7, we have $R_1 = [d+1] \setminus \{i\}$ for some $i \in [d+1] \setminus \{1\}$. If $i = d+1$, then $R_1 = [d]$, and so $R_2 = [2d] \setminus [d]$, which yields the unique partition (R_1, R_2), i.e. (R_1, R_2) is our starting partition $\mathcal{R}^0$ of $\mathcal{P}_d$.

Now suppose $R_1 = [d+1] \setminus \{i\}$ for some $i \in [d] \setminus \{1\}$. This implies $R_2 = \{i, d+2, \ldots, 2d\}$. Since $R_2 \in \mathcal{R}_d$, the set $W = R_2 \setminus \{i\}$ is a wall of $\mathcal{O}_d$, and in particular (by Scarf's Lemma), i can only be one of two values. Since the set $[2d] \setminus [d]$ is a room, one such value of i is clearly $i = d + 1 \notin [d] \setminus \{1\}$, which contradicts the choice of i.

The remaining column which i can be, such that the wall W extends to the room $R_2 \in \mathcal{R}_d$, is $i = 2$. Indeed, since $\{1, d + 1\} \cap R_2 = \emptyset$, Lemma 8 tells us $2 \in R_2$. Thus the partition we find must have $R_1 = [2d] \setminus R_2 = \{1, 3, \ldots, d + 1\}$, which is indeed a room. Hence the ending partition is the only remaining room partition of $\mathcal{O}_d$. $\square$

Lemma 13 *For each vertex $\mathcal{R} = (R_1, R_2) \in \mathcal{P}_{d-1}$, the sets $S_1 = \sigma(R_1)$ and $S_2 = \sigma(R_2)$ can be extended to a vertex of $\mathcal{G}_d$. In particular,*

- (i) *$(S_1 \cup \{1\}, S_2 \cup \{d+1\})$ is the starting partition of $\mathcal{P}_d$ and $(S_1 \cup \{2d\}, S_2 \cup \{d+1\}) \in \mathcal{V}_d^1$ if $\mathcal{R}$ is the starting partition of $\mathcal{P}_{d-1}$.*
- (ii) *$(S_1 \cup \{d+1\}, S_2 \cup \{d+1\})$, $(S_1 \cup \{d+1\}, S_2 \cup \{2\})$, and $(S_1 \cup \{2d\}, S_2 \cup \{d+1\}) \in \mathcal{V}_d^1$ if $\mathcal{R}$ is the ending partition of $\mathcal{P}_{d-1}$.*
- (iii) *$(S_1 \cup \{2\}, S_2 \cup \{d + 1\})$ and $(S_1 \cup \{d + 1\}, S_2 \cup \{2\}) \in \mathcal{V}_d^1$ if $\mathcal{R}$ is not an endpoint of $\mathcal{P}_{d-1}$.*

Proof: Recall for a pair $(R_1, R_2) \in \mathcal{V}_{d-1} \cup \mathcal{V}_{d-1}^1$ to be extended to a room $(T_1, T_2) = (\sigma(R_1) \cup \{r_1\}, \sigma(R_2) \cup \{r_2\}) \in \mathcal{V}_d \cup \mathcal{V}_d^1$, we must have

- (a) $T_1 \cap T_2 = \emptyset$ implies $\{r_1, r_2\} = \{1, d + 1\}$ and $(R_1, R_2) \in \mathcal{V}_{d-1}$, or (*b*) $|T_1 \cap T_2| = 1$ implies $1 \notin T_1 \cup T_2$ and $d+1 \in \{r_1, r_2\}$.

Case (*i*)**:** $\mathcal{R} = (R_1, R_2)$ is the starting vertex of P_{d-1}. By definition, $R_1 = \{1, 2, \ldots, d-1\}$ and $R_2 = \{d, d+1, \ldots, 2d-2\}$. Clearly, $\sigma(R_1) = [d] \setminus \{1\} \subseteq [d]$, $|\sigma(R_2) \setminus [d]| > 1$. Invoking Lemma 9, we see that $\sigma(R_1) \cup \{1\}$, $\sigma(R_1) \cup \{2d\}$, and $\sigma(R_2) \cup \{d+1\}$, $\sigma(R_2) \cup \{2\}$ are rooms in $\mathcal{R}_d$. Clearly, the pair $\mathcal{R}' = (\sigma(R_1) \cup \{1\}, \sigma(R_2) \cup \{d+1\}) \in \mathcal{P}_d \cap \mathcal{V}_d$, since $\mathcal{R}'$ is a room partition of $\mathcal{O}_d$ with $\sigma(R_1) \cup \{1\} = [d]$. Furthermore, $\mathcal{R}'$ is the starting vertex of $\mathcal{P}_d$. It can also be seen by the above two points (*a*) and (*b*) that the only other extension of (R_1, R_2) is $(\sigma(R_1) \cup \{2d\}, \sigma(R_2) \cup \{d+1\}) \in \mathcal{V}_d^1$, in which the intersection of the newly formed rooms is the column $2d \in V_d$.

Case (*ii*)**:** $\mathcal{R} = (R_1, R_2)$ is the ending vertex of $\mathcal{P}_{d-1}$, i.e. Lemma 12 gives us the only remaining room partition $R_1 = \{1, 3, \ldots, d\}$ and $R_2 = \{2, d+1, \ldots, 2d-2\}$ of $\mathcal{R}_{d-1}$. By choice of σ, $|\sigma(R_1) \setminus [d]| = 1$, $d+2 \in \sigma(R_1)$, and $3 \notin \sigma(R_1)$. Furthermore, $|\sigma(R_2) \setminus [d]| > 1$, and thus Lemma 9 tells us that $\sigma(R_1) \cup \{d+1\}$, $\sigma(R_1) \cup \{2d\}$, and $\sigma(R_2) \cup \{d+1\}$, $\sigma(R_2) \cup \{2\}$ are rooms in $\mathcal{R}_d$. By inspection of points (*a*) and (*b*) above, the only pair (of the four possible pairs) which is *not* a vertex of $\mathcal{G}_d$ is $(T_1, T_2) = (\sigma(R_1) \cup \{2d\}, \sigma(R_2) \cup \{2\}) \notin \mathcal{V}_d \cup \mathcal{V}_d^1$, since $|T_1 \cap T_2| > 1$ and $\{1, d+1\} \cap (T_1 \cup T_2) = \emptyset$.

Case (*iii*)**:** $\mathcal{R} = (R_1, R_2)$ is not an endpoint of $\mathcal{P}_{d-1}$. By construction of $\mathcal{G}_{d-1}$, $\mathcal{R}$ is a skew-room partition with $1 \notin R_1 \cup R_2$. By choice of σ, $\sigma(R_1) \cup \sigma(R_2) = [2d] \setminus \{1, 2, d+1\}$ and $|\sigma(R_1) \cap \sigma(R_2)| = 1$. This implies $|\sigma(R_1) \setminus [d]| \geq 1$ and, without loss of generality, $|\sigma(R_2) \setminus [d]| > 1$.

Clearly, the case where $|\sigma(R_1) \setminus [d]| > 1$, implies that the extensions $\sigma(R_1) \cup \{2\}$, $\sigma(R_1) \cup \{d+1\}$, and $\sigma(R_2) \cup \{2\}$, $\sigma(R_2) \cup \{d+1\} \in \mathcal{R}_d$. Clearly, by inspection of the above points (*a*) and (*b*), the only (two) pairs in $\mathcal{G}_d$ are $(\sigma(R) \cup \{2\}, \sigma(R_2) \cup \{d+1\})$ and $(\sigma(R_1) \cup \{d+1\}, \sigma(R_2) \cup \{2\}) \in \mathcal{V}_d^1$.

Now suppose $|\sigma(R_1) \setminus [d]| = 1$. Since $\{1, 2\} \cap \sigma(R_1) = \emptyset$, then $\sigma(R_1) = \{3, 4, \ldots, d, j_0\}$ for some $j_0 \geq d+2$. By choice of σ, $j_0' = \sigma^{-1}(j_0) \geq d$. By choice of room $R \in \mathcal{R}_{d-1}$, and since dominance is hereditary (see (*vi*) of Section 1.3), j_0' can only be one of two extensions of the wall $W = R_1 \setminus \{j_0'\} = \{2, 3, \ldots, d-1\}$ of $\mathcal{O}_{d-1}$. By inspection, $W \cup \{1\}$ and $W \cup \{2d-2\}$ are the only such extensions of the wall W. Clearly now we must have $j_0 = 2d$. Now we see, after invoking Lemma 9 that $\sigma(R_2) \cup \{2\}, \sigma(R_2) \cup \{d+1\}$, and $\sigma(R_1) \cup \{d+1\}$, $\sigma(R_1) \cup \{2\}$ are rooms in $\mathcal{R}_d$. Thus the only two extension are $(\sigma(R_1) \cup \{2\}, \sigma(R_2) \cup \{d+1\})$ and $(\sigma(R_1) \cup \{d+1\}, \sigma(R_2) \cup \{2\}) \in \mathcal{V}_d^1$. □

By the induction hypothesis, each path $\mathcal{P}_k$ has length $\ell(k) = \frac{3}{4}2^k - 1$ for $2 \leq k < d$, and furthermore, it contains an odd number of edges and thus an even number of vertices. Remark 14 implies that $(\mathcal{R}^0, \mathcal{R}^1)$ and $(\mathcal{R}^{\ell(d-1)-1}, \mathcal{R}^{\ell(d-1)}) \in E(\mathcal{P}_{d-1})$ are such that $R_i^0 = R_i^1, R_i^{\ell(d-1)-1} = R_i^{\ell(d-1)}$ for $i = 1$ or $i = 2$.

Theorem 6 *The path* $\mathcal{P}_d$ *is of length* $\ell(d) = \frac{3}{4}2^d - 1$*. In particular,* $\mathcal{P}_d = (\mathcal{S}^0, \ldots, \mathcal{S}^{\ell(d-1)}, \mathcal{T}^{\ell(d-1)}, \ldots, \mathcal{T}^0)$*, where*

(*i*) $\mathcal{S}^0 = (\sigma(R_1^0) \cup \{1\}, \sigma(R_2^0) \cup \{d+1\})$ *and* $\mathcal{T}^0 = (\{1, 3, \ldots, d+1\}, \{2, d+2, \ldots, 2d\})$, (*ii*) $\mathcal{S}^i = (\sigma(R_1^i) \cup \{2\}, \sigma(R_2^i) \cup \{d+1\})$, *and* $\mathcal{T}^i = (\sigma(R_1^i) \cup \{d+1\}, \sigma(R_2^i) \cup \{2\})$, *for* $i = 1, \ldots, \ell(d-1) - 1$, *and*

(*iii*) $\mathcal{S}^{\ell(d-1)} = (\sigma(R_1^{\ell(d-1)}) \cup \{d+1\}, \sigma(R_2^{\ell(d-1)}) \cup \{d+1\})$ *and* $\mathcal{T}^{\ell(d-1)} = (\sigma(R_1^{\ell(d-1)}) \cup \{d+1\}, \sigma(R_2^{\ell(d-1)}) \cup \{2\})$.

Proof: By Lemma 13, it is clear that each $\mathcal{S}^i$ and $\mathcal{T}^i$, as defined in (*i*)-(*iii*), are vertices of $\mathcal{G}_d$, i.e. $\mathcal{S}^0, \mathcal{S}^i, \mathcal{T}^i \in \mathcal{V}_d \cup \mathcal{V}_d^1$ for $i = 1, \ldots, \ell(d-1)$. Since Scarf's Theorem tells us that each $\mathcal{R} \in \mathcal{V}_d$ has degree 1, each $\mathcal{R} \in \mathcal{V}_d^1$ has degree 2, and there are only two room-partitions by Lemma 12, it suffices to show that $(\mathcal{S}^{i-1}, \mathcal{S}^i)$ and $(\mathcal{T}^i, \mathcal{T}^{i-1}) \in \mathcal{E}_d$ for each $i = 1, \ldots, \ell(d-1)$, and $(\mathcal{S}^{\ell(d-1)}, \mathcal{T}^{\ell(d-1)}) \in \mathcal{E}_d$.

Let $\ell = \ell(d-1)$ for convenience. By the induction hypothesis, Remark 14, and the fact that we must remove column 1 from R_1^0, we must have second rooms $R_2^0 = R_2^1$ and $R_2^{\ell-1} = R_2^\ell$.

First we consider $(\mathcal{S}^0, \mathcal{S}^1)$. So we have $R_2^0 = R_2^1$ and $|R_1^0 \cap R_1^1| = d-2$. This implies $\sigma(R_2^0) \cup \{d+1\} = \sigma(R_2^1) \cup \{d+1\}$. Also $2 \in \sigma(R_1^0) \setminus \sigma(R_1^1)$ and $1 \notin \sigma(R_1^0) \cup \sigma(R_1^1)$ implies $|(\sigma(R_1^0) \cup \{1\}) \cap (\sigma(R_1^1) \cup \{2\})| = |\sigma(R_1^0) \cap \sigma(R_1^1)| + 1 = d-1$. Thus we must have $(\mathcal{S}^0, \mathcal{S}^1) \in \mathcal{E}_d$.

Now consider $(\mathcal{S}^{i-1}, \mathcal{S}^i)$ and $(\mathcal{T}^i, \mathcal{T}^{i-1})$ for $i = 2, \ldots, \ell-1$. So we have $R_1^{i-1} = R_1^i$ if i is even, and $R_2^{i-1} = R_2^i$ otherwise. In the first case, $\sigma(R_1^{i-1}) \cup \{2\} = \sigma(R_1^i) \cup \{2\}$ and $|(\sigma(R_1^{i-1}) \cup \{d+1\}) \cap (\sigma(R_1^i) \cup \{d+1\})| = |\sigma(R_1^{i-1}) \cap \sigma(R_1^i)| + 1 = d-1$. The second case is analogous, and so $(\mathcal{S}^{i-1}, \mathcal{S}^i) \in \mathcal{E}_d$ for $i = 2, \ldots, \ell-1$. Since the roles of columns 2 and $d+1$ are reversed in the definition of $\mathcal{T}^i$, the argument is analogous for $(\mathcal{T}^i, \mathcal{T}^{i-1}) \in \mathcal{E}_d$, $i = 2, \ldots, \ell-1$.

Consider now $(\mathcal{S}^{\ell-1}, \mathcal{S}^\ell)$ and $(\mathcal{T}^\ell, \mathcal{T}^{\ell-1})$. Since $R_2^{\ell-1} = R_2^\ell$, we must have $|R_1^{\ell-1} \cap R_1^\ell| = d-2$. This implies $\sigma(R_2^{\ell-1}) \cup \{d+1\} = \sigma(R_2^\ell) \cup \{d+1\}$. Since we also have $d+1 \notin \sigma(R_1^{\ell-1}) \cup \sigma(R_1^\ell)$ and $\sigma(1) = 2 \in \sigma(R_1^\ell) \setminus \sigma(R_1^{\ell-1})$, then we have $|(\sigma(R_1^{\ell-1}) \cup \{2\}) \cap (\sigma(R_1^\ell) \cup \{d+1\})| = |\sigma(R_1^{\ell-1}) \cap \sigma(R_1^\ell)| + 1 = d-1$. We also notice that $|(\sigma(R_1^\ell) \cup \{d+1\}) \cap (\sigma(R_1^{\ell-1}) \cup \{d+1\})| = d-1$. Thus $(\mathcal{S}^{\ell-1}, \mathcal{S}^\ell), (\mathcal{T}^\ell, \mathcal{T}^{\ell-1}) \in \mathcal{E}_d$.

Now consider the pair $(\mathcal{S}^\ell, \mathcal{T}^\ell)$. By Remark 14, the first rooms of $\mathcal{S}^\ell$ and $\mathcal{T}^\ell$ must coincide, which clearly holds since, by definition, the first room of each is $\sigma(R_1^\ell) \cup \{d+1\}$. Furthermore, $|(\sigma(R_1^\ell) \cup \{d+1\}) \cap (\sigma(R_1^\ell) \cup \{2\})| = |\sigma(R_1^\ell)| = d-1$. Thus $(\mathcal{S}^\ell, \mathcal{T}^\ell) \in \mathcal{E}_d$.

Finally, consider $(\mathcal{T}^1, \mathcal{T}^0)$. By inspection, we have $\sigma(R_1^1) \cup \{d+1\} = \{3, \ldots, d+1, 2d\}$ and $\sigma(R_2^1) \cup \{2\} = \{2, d+2, \ldots, 2d\}$. By Remark 14, the second rooms of $\mathcal{T}^1$ and $\mathcal{T}^0$ must coincide, which in fact holds by definition. Clearly $|\{3, \ldots, d+1, 2d\} \cap \{1, 3, \ldots, d+1\}| = d-1$. All that remains is to show that the wall $W = \{3, \ldots, d+1\}$ of $\mathcal{O}_d$ can be extended to the second room of $\mathcal{T}^0$, proving then that $\mathcal{T}^0 \in \mathcal{V}_d$ and furthermore $(\mathcal{T}^1, \mathcal{T}^0) \in \mathcal{E}_d$. Since A_d^W is a $d \times (d-1)$ dominating submatrix of A_d and $\{3, \ldots, d\} \subseteq W$, column $d+1 \in W$ contains exactly two unique elements which are row minima, namely, rows 1 and 2. Clearly, $J_1 = \{1\}$ and $J_2 = \{2\}$, and so we indeed have $(\mathcal{T}^1, \mathcal{T}^0) \in \mathcal{E}_d$.

Hence, we find that $\mathcal{P}_d = (\mathcal{S}^0, \ldots, \mathcal{S}^\ell, \mathcal{T}^\ell, \ldots, \mathcal{T}^0)$ is a path of $\mathcal{G}_d$, and indeed $\ell(d) = 2\ell + 1 = \frac{3}{4}2^d - 1$, since it is easy to check the base case $\ell(2) = 2$. $\square$

Corollary 1 *For each vertex $\mathcal{R} = (R_1, R_2) \in \mathcal{P}_{d-1}$ and each $i_1, i_2 \in [2d]$ such that the room-selection $\mathcal{R}' = (\sigma(R_1) \cup \{i_1\}, \sigma(R_2) \cup \{i_2\}) \in \mathcal{V}_d \cup \mathcal{V}_d^1$, we must have $\mathcal{R}' \in P_d$.*

Proof: Referring to Lemma 13 and Theorem 6, it is easy to see that we only need to show that $\mathcal{R}' = (\sigma(R_1) \cup \{2d\}, \sigma(R_2) \cup \{d+1\}) \in \mathcal{P}_d$ when $\mathcal{R} = (R_1, R_2)$ is either the starting vertex or the ending vertex of $\mathcal{P}_{d-1}$.

In the first case $\mathcal{R} = (\{1, 2, \ldots, d-1\}, \{d, d+1, \ldots, 2d-2\})$, and so $\mathcal{R}' = (\{2, \ldots, d, 2d\}, \{d+1, d+2, \ldots, 2d\})$, which by inspection is the second vertex, $\mathcal{S}^1$, of $\mathcal{P}_d$.

In the second case $\mathcal{R} = (\{1, 3, \ldots, d\}, \{2, d+1, \ldots, 2d-2\})$, and so $\mathcal{R}' = (\{2, 4, \ldots, d, d+2, 2d\}, \{3, d+1, d+3, \ldots, 2d\})$, which by inspection is $\mathcal{S}^{\ell-1}$ where $\ell = \ell(d-1)$. □

Lemma 14 *For each (skew) room partition (except the last room partition) $\mathcal{R} = (R_1, R_2) \in \mathcal{V}_d \cup \mathcal{V}_d^1$, we can find a pair $(i_1, i_2) \in R_1 \times R_2$ such that $(\sigma^{-1}(R_1 \setminus \{i_1\}), \sigma^{-1}(R_2 \setminus \{i_2\})) \in \mathcal{V}_{d-1} \cup \mathcal{V}_{d-1}^1$.*

Proof: Clearly, in our starting partition $([d], [2d] \setminus [d])$, we can find the pair $(i_1, i_2) = (1, d+1)$ so that we extend back to $([d-1], [2d-2] \setminus [d-1]) \in \mathcal{V}_{d-1}$.

Now consider an arbitrary node $\mathcal{R} = (R_1, R_2) \in \mathcal{V}_d^1$, and we find the pair (i_1, i_2). We know that $R_1 \cap R_2 = \{j_0\}$ and $R_1 \cup R_2 = [2d] \setminus \{1\}$. Clearly, we have $2 \leq |R_1 \cap \{2, d+1\}| + |R_2 \cap \{2, d+1\}| \leq 3$ since columns 2 and $d+1$ must be covered, but R_1 and R_2 intersect in only one column.

Consider the case $|\{2, d+1\} \cap R_1| = |\{2, d+1\} \cap R_2| = 1$. Without loss of generality, let $2 \in R_1$ and $d+1 \in R_2$. Now remove columns 2 from R_1 and $d+1$ from R_2, and call them S_1 and S_2, respectively. Clearly $j_0 \notin \{2, d+1\}$, and so we have $S_1 \cap S_2 = \{j_0\}$. Furthermore, since $|S_1| = |S_2| = d-1$ and $\{1, 2, d+1\} \cap (S_1 \cup S_2) = \emptyset$, we have $(\sigma^{-1}(S_1), \sigma^{-1}(S_2)) \in \mathcal{V}_{d-1}^1$. Indeed Lemma 10 tells us that each inverse is a room of $\mathcal{O}_{d-1}$.

Now consider the case $|\{2, d+1\} \cap R_1| = 2$ and $|\{2, d+1\} \cap R_2| = 1$. The case $|\{2, d+1\} \cap R_1| = 1$ and $|\{2, d+1\} \cap R_2| = 2$ is analogous. Clearly, $j_0 \in \{2, d+1\}$. Now let $S_1 = R_1 \setminus \{d+1\}$ and $S_2 = R_2 \setminus \{j_0\}$. In either case, $2 \in S_1$, $\{1, d+1\} \cap (S_1 \cup S_2) = \emptyset$ and $S_1 \cap S_2 = \emptyset$. By Lemma 10, both sets $\sigma^{-1}(S_1)$ and $\sigma^{-1}(S_2)$ are rooms of $\mathcal{O}_{d-1}$. Also, $1 \in \sigma^{-1}(S_1)$, and thus $(\sigma^{-1}(S_1), \sigma^{-1}(S_2)) \in \mathcal{V}_{d-1}$.

Finally, consider the case $|\{2, d+1\} \cap R_1| = 2$ and $|\{2, d+1\} \cap R_2| = 0$. The case $|\{2, d+1\} \cap R_1| = 0$ and $|\{2, d+1\} \cap R_2| = 2$ is analogous. As in the previous case, we remove $d+1$ from R_1 and j_0 ($\notin \{2, d+1\}$) from R_2 and call the new sets S_1 and S_2, respectively. The proof is the same, and yields $(\sigma^{-1}(S), \sigma^{-1}(S')) \in \mathcal{V}_{d-1}$. Thus every node of $\mathcal{G}_d$ (except the last partition, which cannot be an extension of any node) can be found via an extension of a node of $\mathcal{G}_{d-1}$. □

Theorem 7 *The exchange graph $\mathcal{G}_d$ and the exponential length exchange path $\mathcal{P}_d$ coincide, i.e. $\mathcal{G}_d = \mathcal{P}_d$.*

Proof: The claim follows immediately from Corollary 1, Lemma 14, and by induction on $d \geq 2$. □

5.2 Properties of Scarf-Sperner oik-family

In this section we will demonstrate that a Scarf-Sperner oik-family $\mathcal{O}_d^{fr} = (\mathcal{O}'_d, \mathcal{O}''_d)$ defined over a particular $d \times 2d$ canonical Scarf matrix, A'_d yields an exponential path of length $2^d - 1$. The matrix $A'_d = [B_d, C_d]$ where $B_1 = 0, C_1 = 1$, and for $d \geq 2$ we have

$$B_d = \begin{bmatrix} 0 & M \cdot \mathbf{1}_{d-1}^\top \\ M \cdot \mathbf{1} & B'_{d-1} \end{bmatrix} \quad C_d = \begin{bmatrix} 1 & 2 & 3:d \\ C_{d-1}^1 & d \cdot \mathbf{1}_{d-1} & C_{d-1}^{2:d-1} \end{bmatrix}, \tag{5.11}$$

where $i : j$ denotes the interval of all indices $i, i+1, \dots, j$ for $i \leq j$, and $\mathbf{1}_{d-1}$ is the vector of $d-1$ one-entries. Below is a sequence of matrices A'_k for $k \geq 2$.

$$A_2 = \left[\begin{array}{cc|cc} 0 & M & 1 & 2 \\ M & 0 & 1 & 2 \end{array}\right]$$

$$A_3 = \left[\begin{array}{ccc|ccc} 0 & M & M & 1 & 2 & 3 \\ M & 0 & M & 1 & 3 & 2 \\ M & M & 0 & 1 & 3 & 2 \end{array}\right]$$

$$A_4 = \left[\begin{array}{cccc|cccc} 0 & M & M & M & 1 & 2 & 3 & 4 \\ M & 0 & M & M & 1 & 4 & 2 & 3 \\ M & M & 0 & M & 1 & 4 & 3 & 2 \\ M & M & M & 0 & 1 & 4 & 3 & 2 \end{array}\right]$$

$$\vdots$$

$$A_d = \left[\begin{array}{cccccc|cccccc} 0 & M & M & \dots & M & M & 1 & 2 & 3 & \dots & d-1 & d \\ M & 0 & M & \dots & M & M & 1 & d & 2 & \dots & d-2 & d-1 \\ M & M & 0 & \dots & M & M & 1 & d & d-1 & \dots & d-3 & d-2 \\ & \vdots & & \ddots & & \vdots & & \vdots & & \ddots & & \vdots \\ M & M & M & \dots & 0 & M & 1 & d & d-1 & \dots & 3 & 2 \\ M & M & M & \dots & M & 0 & 1 & d & d-1 & \dots & 3 & 2 \end{array}\right]$$

Define $O'_d = (V_d, \mathcal{R}'_d)$ the Scarf oik associated with the columns of A'_d, and let $O''_d = (V_d, \mathcal{R}''_d)$ be the Sperner oik associated with the columns of A'_d with each column j colored j mod d. In other words, columns i and $i + d$ are colored i for each index $i \leq d$.

5.2.1 Properties of rooms

In this subsection, we discuss the properties of $\mathcal{R}'_d$, since $\mathcal{R}''_d$ is trivial. We can easily see that $|\mathcal{R}''_d| = 2^d$ since $\mathcal{R}''_d = \{\{1 + dx_1, 2 + dx_2, \dots, d + dx_d\} | \ x \in \{0, 1\}^d\}$.

We shall prove a relationship between the Scarf oik $\mathcal{O}_d$ defined over the columns of A_d and the new Scarf oik $\mathcal{O}'_d$, which will imply similar results to the previous section. First we record a property similar to that of Remark 13.

Remark 17 *For each tuple* (i, j, k) *such that* $1 \le i \le j < k \le d$ *we have* $a_d(i, j) < a_d(i, k)$*, and if* $1 \le j < k < i \le d$ *then* $a_d(i, j) > a_d(i, k)$*.*

In other words, each row comparison made at or above the diagonal yields an increasing sequence, whereas below the diagonal we have decreasing sequences.

Let $R(i)$ be the ith largest element in the nonempty subset $R \subseteq [2d]$. Now let $\gamma_d^k(R)$ for $k \in [d]$ be defined as

$$\gamma_d^k(R) = \{R(k)\} \cup \{\min\{R(k) + 1, R(k-1) - 1\}\}, \tag{5.12}$$

where we let $R(0) = 2d+1$. Define the *shift operator* as $\beta_d = \beta_d^{\kappa(R)}(R)$ with $\kappa(R) = \max\{i \in [d] |\ R(i) > d\}$ where $\beta_d^i = \circ_{j=1}^i \gamma_d^j$ for any $i \in [d]$, and $\kappa(R) = 0$ if no such i exists (which can only happen if $R = [d]$). Also let $\hat{\kappa}(R) = \max\{i \in [d] |\ \beta_d^i(R) = R, i \le \kappa(R)\}$ where we define $\hat{\kappa}(R) = 0$ if no such i exists. We omit the subscript d from any operator such that no confusion arises.

Further, for any room R of either oik $\mathcal{R}_d$ or $\mathcal{R}'_d$, let us define $i_R(j)$ to be the row in which column $j \in [2d]$ is dominated with respect to matrix A_d or A'_d, respectively. Also let us define $\xi(R) = \max\{j \in V_d \setminus R\}$.

Remark 18 *Column* $d+1$ *cannot be contained in any dominating set* $J \subseteq V_d$*. In particular,* $d + 1 \notin R$ *for each* $R \in \mathcal{R}'_d$*.*

Lemma 15 *Let* $R \in \mathcal{R}_d$*. Then we have* $\beta(R) \in \mathcal{R}'_d$ *if and only if* $\{d-1, d\} \cap R \neq \emptyset$*.*

Proof: We prove necessity by contradiction. Suppose $\beta(R) \in \mathcal{R}'_d$ but $\{d-1, d\} \cap R = \emptyset$. In this case, we must have $\{2d-1, 2d\} \subseteq R$. Indeed, otherwise the rightmost column of R contains at least two rows in which it is dominated (see Remark 13), contradicting $R \in \mathcal{R}_d$. But now we must also have $\{2d-1, 2d\} \subseteq \beta(R)$, which by Remark 17 is such that column $2d$ contains at least two rows in which it is dominated, contradicting $\beta(R) \in \mathcal{R}'_d$.

For sufficiency, suppose $\{d-1, d\} \cap R \neq \emptyset$. First note that it suffices to show $i_{\beta(R)}(j+1) = i_R(j)$ for $j \in \{d+1, \dots, \xi(R)-1\}$ and $i_{\beta(R)}(j) = i_R(j)$ for $j \in [d+1] \cup ([2d] \setminus [\xi(R)])$. Trivially, we have $i_{\beta(R)}(j) = i_R(j)$ for $j \in \{1, \dots, d+1\}$. Also note that for any $j \in R \setminus [d]$, we must have $i_R(j') \le j' - d$, for $d + 1 \le j' < j$. We split the proof depending on the containment of column $2d$.

Case I: Suppose $2d \in R$. Then we have $i_R(j) \le j - d$ for $j \in [2d] \setminus [d]$ (trivially, $i_R(2d) \le d = 2d - d$), and $\xi(R) < 2d$. Clearly $a'_d(i_R(j), j + 1) = a_d(i_R(j), j) + 1$ for $j \in \{d + 1, \dots, \xi(R) - 1\}$, and $a'_d(i_R(j), j) = a_d(i_R(j), j)$ for $j \in \{\xi(R) + 1, \dots, 2d\}$, and so our claim follows if $i_R(j) < j - d$ for each $j \in \{\xi(R) + 1, \dots, 2d - 1\}$ and any value for $i_R(2d)$ (see construction of A'_d and Remark 17). Notice that $2d - 1 \notin R$ if $d \in R$ and $\{2d - 2, 2d - 1\} \not\subseteq R$ if $d - 1 \in R$, otherwise column $2d$ is not dominated. Thus we have $\xi(R) \in \{2d - 2, 2d - 1\}$. Hence we only need to check that $i_R(2d - 1) < 2d - 1 - d = d - 1$ if $\xi(R) = 2d - 2$. This is trivial, since $d - 1 \in R$, $d \notin R$, and $2d \in R$. Thus in this case, $\beta(R) \in R_d$.

Case II: Now suppose $2d \notin R$. Clearly, we have $i_R(j) \le j-d$ for $j \in \{d+1,\ldots,R(1)-1\}$. Furthermore, it can be seen that $i_R(R(1)) > j-d$ for otherwise each column $j > R(1)$ will not be dominated by R. By the construction of A'_d, we also have $a'_d(i_R(R(1)+1),R(1)) > a'_d(i_r(R(1)),j)$ for each $j > R(1)+1$. Thus $i_{\beta(R)}(j) = i_R(R(1))$ completes the claim that $\beta(R) \in R'_d$. □

Lemma 16 *Each room* $R \in \mathcal{R}'_d$ *is such that* $R = \beta(\tilde{R})$ *for some* $\tilde{R} \in \mathcal{R}_d$.

Proof: We divide the proof into two cases, depending on the inclusion of $2d$.

Case I: Suppose column $2d \notin R$. We claim that $\tilde{R} = (R \cap [d]) \cup ((R \setminus [d]) - 1) \in R_d$, i.e. we "shift back" all columns $j \in R \setminus [d]$ to $j-1 \in \tilde{R} \setminus [d]$. Obviously, $R = \beta(\tilde{R})$. It suffices to show that $i_{\tilde{R}}(j-1) = i_R(j)$ for $j \in \{d+2,\ldots,2d\}$, $i_{\tilde{R}}(k) = i_R(k)$ for $k \in [d] \cup \{2d\}$. Clearly, $i_{\tilde{R}}(k) = i_R(k)$ for $k \in [d]$. Notice that for any column $j \in R$, $i_R(j) < j-d$ for $j' \in \{d+1,\ldots,j-1\}$, otherwise $R \notin R'_d$. Since $2d \notin R$, we notice that $j \in \{d+1,\ldots,R(1)\}$ is such that $i_R(j) < j-d$. Furthermore, $i_R(R(1)) \ge R(1)-d$, otherwise $2d$ is not dominated. By Remark 17, we can see that $a_d(i_R(j),j-1) = a'_d(i_R(j),j)$ for $j \in \{d+2,\ldots,R(1)\}$, and $a_d(i_R(R(1)),R(1)) > a_d(i_R(R(1)),j)$ for $R(1) < j \le 2d$. Thus, by definition of $R(1)$, the claim $i_{\tilde{R}}(j-1) = i_R(j)$ for $j \in \{d+2,\ldots,R(1)\}$ and $i_{\tilde{R}}(k) = i_R(k) = i_R(R(1))$ for $k \in \{R(1),\ldots,2d\}$, which implies our claim that $\tilde{R} \in R_d$.

Case II: Suppose now that $2d \in R$. First let $\lambda(R) = \max\{j \in R \cap [d]\}$ (and in fact $\lambda(R) \in \{d-1,d\}$) and $\rho(R) = \max\{j \in [\lambda(R)] \setminus R\} + d$. We claim that $R = (R \cap [d]) \cup (R \setminus [\rho(R)]) \cup ((\{d+1,\ldots,\rho(R)\} \cap R) - 1)$, i.e. we "shift back" only $j \in R \cap \{d+2,\ldots,\rho(R)\}$ to $j-1$ (note by Remark 18 $d+1 \notin R$), if any such columns exist in R.

It suffices to show that $R = \beta(\tilde{R})$ by showing that $\{\rho(R)+1,\ldots,\xi(R)\} \cap R = \emptyset$ if $\rho(R) < \xi(R)$ (otherwise we have $\beta(\tilde{R}) = R$, by definition). This can be shown by contradiction; suppose $\exists(j,k)$ such that $\rho(R) < j < k < 2d$ and $j \in R$ but $k \notin R$. Since $2d \in R$, we have $i_R(k) < k-d$. Also, since $\{\rho(R)-d+1,\ldots,\lambda(R)\} \subseteq R$ with $\lambda(R) \in \{d-1,d\}$ and $k > \rho(R)$, we must have $i_R(k) \le \rho(R)-d+1$. But we have $j \in R$ such that $\rho(R) < j < k$, which by Remark 17 implies $a'_d(i,j) < a'_d(i,k)$ for $i \in \{1,\ldots,\rho(R)-d\}$. Thus we contradict $R \in R'_d$.

Now we show $\tilde{R} \in R_d$ by proving $i_{\tilde{R}}(j-1) = i_R(j)$ for $j \in \{d+2,\ldots,\xi(R)\}$ and $i_{\tilde{R}}(k) = i_R(k)$ for $k \in \{\xi(R),\ldots,2d\}$. Since $2d \in R$, we know that each $j \in \{d+2,\ldots,2d-1\}$ is such that $i_R(j) < j-d$. It is clear that $a_d(i_R(j),j-1) = a'_d(i_R(j),j)-1$ for $j \in \{d+1,\ldots,\xi(R)\}$ and $a_d(i_R(k),k) = a'_d(i_R(k),k)$ for $k \in \{\xi(R)+1,\ldots,2d-1\}$ (see the construction of A'_d and Remark 17). Furthermore, $a_d(i_R(2d),2d) = a'_d(i_R(2d),2d)+b$, where $b=1$ if $i_R(2d) = 2d-d = d$ and $b=0$ otherwise. Finally, $a_d(i_R(\xi(R)),\xi(R)) = a_d(i_R(\xi(R)),\xi(R)-1)+1 = a_d(i_R(\xi(R)),\xi(R)+1)-1$. Thus $\tilde{R}$ is indeed dominating in A_d, i.e $\tilde{R} \in R_d$. □

The last two results imply that any property of $R \in \mathcal{R}_d$ is such that a similar property holds for $\beta(R) \in \mathcal{R}'_d$ with the proper transformation. So we define $\alpha_d(R) = \{\alpha_d(i) | \ i \in R\}$ for any $R \subseteq [2d]$ where

$$\alpha_d(i) = \begin{cases} i+1 & \text{if } i \le d+1 \\ i+2 & \text{if } i \ge d+2, \end{cases} \tag{5.13}$$

and omit the subscript d when no confusion can arise.

Lemma 17 *For any room* $R \in \mathcal{R}_{d-1}$*,* $\alpha_{d-1}(\beta_{d-1}(R)) = \beta_d(\sigma_{d-1}(R))$*.*

Proof: The following must hold:

$$\begin{aligned} \sigma_{d-1}(R) &= ((R \cap [d-1]) + 1) \cup ((R \setminus [d-1]) + 2) \\ \beta_d(\sigma_{d-1}(R)) &= ((R \cap [d-1]) + 1) \cup \beta_d((R \setminus [d-1]) + 2) \end{aligned}$$

$$\begin{aligned} \beta_{d-1}(R) &= (R \cap [d-1]) \cup \beta_{d-1}((R \setminus [d-1])) \\ \alpha_{d-1}(\beta_{d-1}(R)) &= ((R \cap [d-1]) + 1) \cup (\beta_{d-1}(R \setminus [d-1]) + 2). \end{aligned}$$

Clearly the order of operations does not change the set, i.e. $\beta_d((R \setminus [d-1]) + 2) = (\beta_{d-1}(R \setminus [d-1]) + 2)$. Thus the two sets are equal, and our claim holds. □

Lemma 18 *For any* $R \in R_{d-1}$ *and* $j(R)$ *such that* $\sigma_{d-1}(R) \cup \{j(R)\} \in R_d$*, we have* $\beta_d(\sigma_{d-1}(R) \cup \{j(R)\}) = \alpha_{d-1}(\beta_{d-1}(R)) \cup \{k(R)\}$ *for some* $k(R) \in V_d$*.*

Proof: Let $\beta^0 = \beta_d(\sigma_{d-1}(R) \cup \{j(R)\})$ and $\beta^1 = \beta_d(\sigma_{d-1}(R))$. Furthermore, let $\alpha^1 = \alpha_{d-1}(\beta_{d-1}(R))$. It can be seen by definitions of β_{d-1}, β_d, α_{d-1}, and $\xi(R)$ for $R \in R_{d-1}$, that $\beta^0 = \beta^1 \cup \{j(R)\}$ if $j(R) \in [d]$ or if $j(R) = \xi(R) + 2$ and $\xi(R) + 1 \notin R$ (recall $\sigma_{d-1}(\xi(R)) = \xi(R) + 2$). Also, $\beta^0 = \beta^1 \cup \{j(R) + 1\}$ if $d < j(R) < \xi(R) + 2$. Thus if we show that for any $j(R) = \xi(R) + 2 > d + 1$, we must have either $j(R) \neq \xi(R) + 2$ or $\xi(R) - 1 \notin R$. Then invoking Lemma 17, we see that $\beta^0 = \alpha^1 \cup \{k(R)\}$ for $k(R) \in \{j(R), j(R) + 1\}$, i.e. the claimed equality immediately follows.

Since $j(R) \geq d + 2$, referring to Lemma 9, we must have $j(R) \in \{d + 3, 2d\}$. In either case, $\sigma_{d-1}(R) \setminus [d] = \{d + 2\}$, and it can easily be seen that either $j(R) \neq \xi(R) + 2$ or $\xi(R) + 1 \notin R$. Thus our claim follows. □

The above lemmas imply that for every room $R \in R_{d-1}$ the β map of an appropriate room extension $\sigma(R) \cup \{j(R)\}$ yields a characterization of the room extensions of $\alpha(\beta(R))$, where the subscripts are ignored when no confusion can arise.

The following lemmas are given without proof, since they following by the above transformation lemmas.

Lemma 19 *For any set* $W \subseteq V_d$ *of cardinality* $d - 1$ *such that* $W = \alpha(R) = \alpha_{d-1}(R)$ *for some room* $R \in \mathcal{R}'_{d-1}$*:*

(*i*) $W \cup \{1\}$ *and* $W \cup \{2d\}$ *are rooms of oik* $\mathcal{O}'_d$ *if* $W \subseteq [d]$*,*

(*ii*) $W \cup \{d + 2\}$ *and* $W \cup \{j(W)\}$ *are rooms of oik* $\mathcal{O}'_d$ *if* $W \not\subseteq [d]$*, where*

$$j(W) = \begin{cases} 2 & \text{if } W \setminus [d] = \{2d\} \text{ or } |W \setminus [d]| > 1 \\ d + 4 & \text{if } W \setminus [d] = \{d + 3\} \text{ and } 3 \in W, \\ 2d & \text{if } W \setminus [d] = \{d + 3\} \text{ and } 3 \notin W. \end{cases}$$

□

Lemma 20 *Each $R \in \mathcal{R}'_d$ is such that $\{1, d+2\} \not\subseteq R$ if and only if there exists a column $i \in R$ such that $\sigma^{-1}(R \setminus \{i\}) \in \mathcal{R}'_{d-1}$.* □

Now let $S_d = \{R \in \mathcal{R}_d | \ \{d-1, d\} \cap R = \emptyset\}$ and $T_d = \{R \in \mathcal{R}_d | \ d \notin R, \{d-1, 2d\} \subseteq R\}$.

Lemma 21 *$R \cap [d] \neq R' \cap [d]$ for any distinct pair $R, R' \in S_d$. Moreover, $|S_d| = \frac{1}{8}2^d$ for $d \geq 3$.*

Proof: We shall prove the claim by induction on d. For S_3 it is trivial since $|S_3| = 1$, and so we suppose the claim is true for S_{d-1}. Let us choose $R \in S_{d-1}$. By Lemma 9, $\sigma(R) \cup \{j^*\}$ only if $j^* \notin \{d-1, d\}$, and so it suffices to show that each such extension of $\sigma(R)$ yields a room in S_d with $R \cap [d]$ unique. But this is trivial since $\{d-1, d\} \cap \sigma(R) = \emptyset$ implies $|\sigma(R) \setminus [d]| > 1$, which by Lemma 9 further implies that $j^* \in \{2, d+1\}$, i.e. each such extension yields a unique sub-characteristic vector $\chi_{\tilde{R}}(2 : d-2)$ where $\tilde{R} = \sigma(R) \cup \{j^*\} \in S_d$. Furthermore, the second claim can be seen since $|S_d| = 2|S_{d-1}|$. □

Lemma 22 *$R \in T_d$ if and only if there exists a unique $R' \neq R$ in $\mathcal{R}_d$ such that $\beta(R') = \beta(R)$. Moreover, $|T_d| = \frac{1}{8}2^d$.*

Proof: For necessity, suppose $R \in T_d$. It can be seen that $2d-1 \in R$, otherwise $2d-1$ is not dominated, a contradiction. Furthermore, for the same reason, $2d-2 \notin R$. Thus $\xi(R) = 2d-2$. It suffices to show that $R' = R \setminus \{2d\} \cup \{2d-2\} \in \mathcal{R}_d$, since clearly no other $R'' \in \mathcal{R}_d$ distinct from R and R' exists such that $\beta(R'') = \beta(R)$. By Remark 13 this must hold, since $i_{R'}(j-1) = i_R(j)$ for $j \in \{2d-1, 2d\}$ and $i_{R'}(2d) = i_R(2d)$.

For sufficiency, suppose that $\beta(R) = \beta(R')$ for a unique $R' \neq R$. If $R \cap \{d-1, d\} = \emptyset$, then this must also hold for R' by definition of β, and so $R, R' \in S_d$. By the previous lemma, $R \cap [d]$ is distinct, contradicting $\beta(R) = \beta(R')$. Thus $\{d-1, d\} \cap R = \emptyset$, which holds also for R'. Furthermore, without loss of generality, $\xi(R) < \xi(R')$. Thus $2d \in R$ and $2d \notin R'$. We prove the claim by contradiction, i.e. we suppose that $\{d-1, 2d\} \not\subseteq R$ or $d \in R$.

Suppose first that $d \in R$, then we have $d \in R'$ by Lemma 21. Since we have $R' \neq R$ such that $\beta(R') = \beta(R)$, then, without loss of generality and $\xi(R) < \xi(R')$. Furthermore, we must have $2d \in R$ and $2d-1 \notin R$ (otherwise column $2d$ is not dominated), which further implies $2d \notin R'$ and $2d-1 \in R'$. But it can now be seen that such an R' is not dominating, as $a_d(d, j_d) = 0$ and $a_0(d-1, j_{d-1}) = 1$

Finally, suppose that $\{d-1, 2d\} \not\subseteq R$ and $d \notin R$. This implies $d-1 \in R$, and so $d-1 \in R'$, clearly $2d \notin R \cap R'$ implies $\xi(R) = \xi(R')$ and $2d-1 \in R \cap R'$, $2d-2 \notin R \cup R'$. Thus either $\beta(R) \neq \beta(R')$ or $R = R'$, a contradiction, and so the claim holds.

The last claim follows by noticing that $|R \setminus [d]| > 1$ implies $\sigma(R) \cup \{j\}$ for $j \in \{2, d+1\}$ and $R \setminus [d]$ being with respect to each $R \in T_d$, and by induction on $d \geq 2$. □

The following lemma is an immediate result of the previous two lemmas, since $R \in S_d$ implies $\beta(R) \notin \mathcal{R}'_d$ and T_d maps to $|T_d|/2$ distinct elements.

Lemma 23 *For the oik $\mathcal{O}'_d$, $|\mathcal{R}'_d| = \frac{3}{4}2^d$.* □

5.2.2 Properties of room- and skew room-partitions

Now we show that our Scarf-Sperner oik-pair yields a path $\mathcal{P}'_d$ in $\mathcal{G}'_d = \mathcal{G}_d(\mathcal{O}_d^{fr}, 1)$ (vertex set $\mathcal{V}_d \cup \mathcal{V}_d^1$ and edge set $\mathcal{E}_d$ as before) of length $\ell(d) = 2^d - 1$. As in the previous section, we have a sequence I_d of the unique intersecting element between an oik-pair's two rooms on $\mathcal{P}_d$ which, again, starts with the starting partition $([d], [2d] \setminus [d])$ and ends with the ending partition $(\{1, 3, \ldots, d, d+2\}, \{2, d+1, d+3, \ldots, 2d\})$. Let I_d^* again be the reverse sequence of I_d.

d	I_d
2	$(4, 2)$
3	$(6, 3, 5, 2, 3, 6)$
4	$(8, 4, 7, 3, 4, 8, 6, 2, 8, 4, 3, 7, 4, 8)$
$\ldots$	$\ldots$
d	$(\alpha_{d-1}(I_{d-1}), d+2, 2, \alpha_{d-1}(I_{d-1}^*))$

Lemma 24 *The starting partition and the ending partition of $\mathcal{P}'_d$ are the only room partitions of $\mathcal{O}^{fr}$.*

Proof: Let us choose a partition (R, R') such that $R \in \mathcal{R}''_d$ and $R' \in \mathcal{R}'_d$. Since $R' = (1 + dx_1, \ldots, d + dx_d)$ for some binary vector $x \in \{0, 1\}^d$, we must have $j \in R$ if and only if $2d - j \notin R$. Thus for any $j \in R \cap [d]$ we have $i_R(j) = j$ and $a'_d(i_R(j), j) = 0$, and for any $j \in R \setminus [d]$ we have $i_R(j) = j - d$ and $a'_d(i_R(j), j) = 2d - 2 + j$, by Remark 17. Moreover, we must have $|R \setminus [d]| \leq 1$, otherwise $R \notin R'_d$. Clearly, $R = [d]$ satisfies the conditions, i.e. $([d], [2d] \setminus [d])$ is a room partition. Furthermore, if $|R \setminus [d]| = 1$, then $\{d + 2\} = R \setminus [d]$, for otherwise $R \notin R'_d$. Thus we have $(\{1, 3, \ldots, d, d+2\}, \{2, d+1, d+3, \ldots, 2d\})$ as the only other room partition of $\mathcal{O}_d^{fr}$. □

Lemma 19 and the fact that $\alpha_{d-1}(R) \cup \{j\} \in R''_d$ for any $R \in R''_{d-1}$ if and only if $j \in \{2, d+2\}$, shows that the proofs of the next few lemmas are almost identical to those of the resembling results in the previous section, with the roles of column $d + 1$ replaced with column $d + 2$.

Lemma 25 *For each vertex $\mathcal{R} = (R_1, R_2) \in \mathcal{P}'_{d-1}$, the sets $S_1 = \sigma(R_1)$ and $S_2 = \sigma(R_2)$ can be extended to a vertex of $\mathcal{G}'_d$. In particular,*

(i) $(S_1 \cup \{1\}, S_2 \cup \{d+2\})$ *is the starting partition of* $\mathcal{P}_d$ *and* $(S_1 \cup \{2d\}, S_2 \cup \{d+2\}) \in \mathcal{V}_d^1$ *if* $\mathcal{R}$ *is the starting partition of* $\mathcal{P}'_{d-1}$.

(ii) $(S_1 \cup \{d+2\}, S_2 \cup \{d+2\})$, $(S_1 \cup \{d+2\}, S_2 \cup \{2\})$, *and* $(S_1 \cup \{2d\}, S_2 \cup \{d+2\}) \in \mathcal{V}_d^1$ *if* $\mathcal{R}$ *is the ending partition of* $\mathcal{P}'_{d-1}$.

(iii) $(S_1 \cup \{2\}, S_2 \cup \{d+2\})$ *and* $(S_1 \cup \{d+2\}, S_2 \cup \{2\}) \in \mathcal{V}_d^1$ *if* $\mathcal{R}$ *is not an endpoint of* $\mathcal{P}'_{d-1}$.

Theorem 8 *The path $\mathcal{P}'_d$ is of length $\ell(d) = 2^d - 1$. In particular, $\mathcal{P}'_d = (\mathcal{S}^0, \dots, \mathcal{S}^{\ell(d-1)}, \mathcal{T}^{\ell(d-1)}, \dots, \mathcal{T}^0)$, where*

(i) $\mathcal{S}^0 = (\sigma(R_1^0) \cup \{1\}, \sigma(R_2^0) \cup \{d{+}2\})$ *and* $\mathcal{T}^0 = (\{1, 3, \dots, d, d{+}2\}, \{2, d{+}1, d{+}3, \dots, 2d\})$,

(ii) $\mathcal{S}^i = (\sigma(R_1^i) \cup \{2\}, \sigma(R_2^i) \cup \{d+2\})$, *and* $\mathcal{T}^i = (\sigma(R_1^i) \cup \{d+2\}, \sigma(R_2^i) \cup \{2\})$, *for* $i = 1, \dots, \ell(d-1) - 1$, *and*

(iii) $\mathcal{S}^{\ell(d-1)} = (\sigma(R_1^{\ell(d-1)}) \cup \{d+2\}, \sigma(R_2^{\ell(d-1)}) \cup \{d+2\})$ *and* $\mathcal{T}^{\ell(d-1)} = (\sigma(R_1^{\ell(d-1)}) \cup \{d+2\}, \sigma(R_2^{\ell(d-1)}) \cup \{2\})$.

Corollary 2 *For each vertex $\mathcal{R} = (R_1, R_2) \in \mathcal{P}_{d-1}$ and each $i_1, i_2 \in [2d]$ such that the room-selection $\mathcal{R}' = (\sigma(R_1) \cup \{i_1\}, \sigma(R_2) \cup \{i_2\}) \in \mathcal{V}_d \cup \mathcal{V}_d^1$, we must have $\mathcal{R}' \in P'_d$.*

Lemma 26 *For each (skew) room partition (except the last room partition) $\mathcal{R} = (R_1, R_2) \in \mathcal{V}_d \cup \mathcal{V}_d^1$, we can find a pair $(i_1, i_2) \in R_1 \times R_2$ such that $(\sigma^{-1}(R_1 \setminus \{i_1\}), \sigma^{-1}(R_2 \setminus \{i_2\})) \in \mathcal{V}_{d-1} \cup \mathcal{V}_{d-1}^1$.*

Lemma 27 *For each $d \geq 2$ the exchange graph $\mathcal{G}'_d = \mathcal{P}'_d$.*

Finally, we get the main result of this subsection by applying the above lemmas and by induction on $d \geq 2$.

Theorem 9 *The oik pair $(\mathcal{O}'_d, \mathcal{O}''_d)$ defined in the above section yields a path $\mathcal{P}'_d$ of length $2^d - 1$.*

6 Odd walls and open problems

The concepts of a complex, oik, room, and wall were defined in Section 1.1. A wall of a complex is called *odd* if it is contained in an odd number of rooms. By definition, a complex is an oik if and only if it has no odd walls.

Given a complex $\mathcal{C}$, let us consider the following three decision problems:

Problem 1. Is $\mathcal{C}$an oik or does it contains an odd wall?

Problem 2. Does $\mathcal{C}$ contain a room-partition?

Problem 3. Given a room-partition in $\mathcal{C}$, does $\mathcal{C}$ contain another one?

Thus, the corresponding search problems are: $(Q1)$ Find an odd wall, $(Q2)$ Find a room-partition, and $(Q3)$ Find another room-partition.

The complexity of each problem depends on **how** the complex is specified.

First, let us assume that $\mathcal{C}$ is given explicitly. Then, obviously, Problem 1 is linear. In contrast, Problems 2 and 3 are NP-complete (see the problems ”perfect 3-matching” and ”partition by 3-sets” in [16]).

Secondly, let us assume that $\mathcal{C} = (V, \mathcal{R})$ is given by an abstract membership oracle, which, for any subset $V' \subseteq V$, answers whether V' is a room of $\mathcal{C}$. Let us assume for simplicity that the dimension of $\mathcal{C}$ is known to be d, that is, we can restrict ourselves by testing only sets of cardinality d. Obviously, by testing all $\binom{n}{d}$ such sets we specify $\mathcal{C}$ explicitly. Thus, if d is bounded by a constant then all three problems can be solved in time n^{d+1}, which is polynomial in n. However, this time becomes exponential when dimension d is a part of the input. Moreover, in this case, each of the above three problems might require a number of tests exponential in $n = |V|$, in the worst case.

Proposition 3 *All $\binom{n}{d}$ tests might be needed in case of Problem 1.*

Proof: Let $n - d$ be odd. Then complex $\mathcal{C}$ is (not) an oik of dimension d whenever every d-subset of V (but one) is a room. Hence, to make a decision in Problem 1, we have to verify all $\binom{n}{d}$ d-subsets of V, when only positive answers are given by the oracle. □

Proposition 4 *For Problems 2 and 3 respectively $\binom{n-1}{d-1}$ and $\binom{n-1}{d-1} - 1$ tests might be needed.*

Proof: . Let $\mathcal{P} = \mathcal{P}(n, d)$ be the collection of all partitions of $|V|$ by d-sets. A family $\mathcal{T}$ of d-subsets of V is called a *transversal* to $\mathcal{P}$ if every partition of $\mathcal{P}$ contains a set from $\mathcal{T}$. It is shown in [19] that, by the Erdös Ko-Rado Theorem [14], each minimum transversal consists of $\binom{n-1}{d-1}$ d-sets. For example, the family of all d-sets that contain a fixed vertex $v_0 \in V$ form such a minimal transversal. Hence, when only the negative answers are given by the oracle, at least $\binom{n-1}{d-1}$ (respectively, $\binom{n-1}{d-1} - 1$ tests are needed to exclude the existence of any (respectively, of another) room-partition. □

Now, let us combine Problems 1 and 3 as follows.

Problem 4. Given a complex $\mathcal{C}$ with a room-partition, find in $\mathcal{C}$:

(a) another room-partition or

(b) an odd wall.

By Theorem 1, (a) or (b) (or both) always exist. Moreover, the exchange algorithm obviously generalizes from oiks to complexes and in the latter case finds either (a) or (b) by traversing the exchange graph. A general step is as follows: given a wall W, test by the membership oracle $n - d + 1$ d-sets that contain W and, choose between them a room R such that the pair (W, R) did not appear before. Clearly, an exchange path P constructed in this way can terminate in a skew-partition only if the last wall in P is odd, otherwise P terminates in a partition distinct from the original one.

As we know (see Section 5), P can be exponential in d already for $n = 2d$. Yet, complexity of Problem 4 remains open.

Finally, let us remark that the reduction of Section 2 naturally extends from the oiks to complexes. Namely, an arbitrary complex-family $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ is reduced to the complex-pair $\mathcal{O}' = \{\mathcal{O}_1 + \ldots + \mathcal{O}_k, \mathcal{O}_0\}$ in which sum of complexes and the Sperner oik $\mathcal{O}_0$ are defined as in Section 2. This reduction is exponential in k but it is polynomial in the size of $\mathcal{O}$.

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