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Further generalizations of Wythoff's game and minimum excludant function

Vladimir Gurvich [a]

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RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

[a]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ, 08854; e-mail: gurvich@rutcor.rutgers.edu

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Abstract. Given non-negative integer a and b, let us consider the following game $WYT(a,b)$. Two piles contain x and y matches. Two players take turns. By one move, it is allowed to take x' and y' matches from these piles such that

$0 \leq x' \leq x,\ 0 \leq y' \leq y,\ 0 < x' + y'$, and $[\min(x', y') < b$ or $|x' - y'| < a]$.

The player who takes the last match is the winner (respectively, loser) in the normal (respectively, misere) version of the game.
It is easy to verify that cases $(a = 0, b = 1), (a = b = 1)$, and $(b = 1, \forall\, a)$ correspond to the two-pile NIM, Wythoff, and Fraenkel games, respectively. The concept of the minimum excludant function mex is known to be instrumental in solving the last two games. We generalize this concept by introducing a function mex_b such that $mex = mex_1$ and solve the normal and misere versions of game $WYT(a,b)$.

Keywords: combinatorial games, NIM, Wythoff game, Fraenkel game, minimal excludant, normal and misere versions, Sprague-Grundy function

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1 NIM, Wythoff, Fraenkel, and $WYT(a, b)$ games

As defined in Abstract, in game $WYT(a, b)$, each player can take any number of matches from one pile and at most $b-1$ from the other ($\min(x', y') < b$,) or (s)he can take "almost equal" numbers of matches from both piles ($|x' - y'| < a$); it is not allowed to pass ($x' + y' > 0$).

If $a = 0$ and $b = 1$ we obtain the standard (and trivial) NIM with two piles. Indeed, in this case either $x' = 0$ or $y' = 0$, but not both.

If $a = b = 1$ then a player can take either (i) any positive number of matchings from one pile and nothing from the other, or (ii) the same positive number of matchings from both. Thus, we obtain the classical game introduced in 1907 by Wythoff [11].

In [5, 6] Fraenkel generalized Wythoff's game, replacing (ii) by a weaker restriction (ii') $|x' - y'| < a$. The obtained games are $WYT(a, 1)$, that is, $b = 1$.

In our turn, we replace (i) by a weaker restriction (i') $\min(x', y') < b$.

Remark 1 *Of course, we could generalize further and consider games $WYT(a, b, c)$ in which (i") $(x' < b, \forall\, y')$ or $(y' < c, \forall\, x')$. Yet, in Section 7 we will see that solution of such a game is trivial unless $b = c$. In that setion we will also consider trivial cases when a or b is zero; until then we assume that $b > 0$.*

The positions of $WYT(a, b)$ are pairs (x, y), where x and y denote the numbers of matches in the two piles. By default, we will assume that $x \le y$. Furthermore, (x, y) is called a *P-position* if the player who enters it (the Previous player) can win. Otherwise, (x, y) is called an *N-position*, since in this case the player who leaves it (the Next player) can win.

Clearly, each move from a P-position leads to an N-position and for every N-position there is a move to a P-position. To solve a game, it is sufficient to find all its N- or P-positions.

Due to symmetry of $WYT(a, b)$, a pair (x, y) is a P-position if and only if (y, x) is.

Obviously, there is a a unique terminal position $(0, 0)$ (since $b > 0$). By definition, $(0, 0)$ is a P-position in the normal version of $WYT(a, b)$ and an N-position in its misere version.

In this paper, both the normal and misere versions are recursively solved, namely, we obtain a recursive formula for the P-positions.

2 Solution of Fraenkel's games

Let us start with $b = 1$. In this case game $WYT(a, 1) = WYT(a)$ was solved by Fraenkel; see [5, 6] for the standard and misere versions, respectively.

We will postpone the latter one till Section 6, where it will be considered for $WYT(a, b)$.

As for the standard version of $WYT(a)$, the set of its P-positions $\{(x_n, y_n) \mid n = 0, 1, \ldots\}$ was characterized in [5] by the following recursion:

$$x_n = mex\{x_i, y_i \mid 0 \le i < n\}, \quad y_n = x_n + an; \quad n \ge 0 \tag{1}$$

where the *minimum excludant* function $mex(S)$ is defined for any subset $S \subseteq \mathbb{Z}_+$ of the non-negative integers as the minimum $z \in \mathbb{Z}_+$ such that $z \notin S$; in particular, $mex(\emptyset) = 0$.

The first ten P-positions of games $WYT(1)$ and $WYT(2)$ are given in Table 1.

n	x_n	y_n
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23

n	x_n	y_n
0	0	0
1	1	3
2	2	6
3	4	10
4	5	13
5	7	17
6	8	20
7	9	23
8	11	27
9	12	30

Table 1: $(a = b = 1)$ and $(a = 2, b = 1)$

Moreover, Fraenkel solved the recursion and got the following explicit formula for (x_n, y_n).

Let $\alpha = \alpha(a) = \frac{1}{2}(2 - a + \sqrt{a^2 + 4})$ be the (unique) positive root of the quadratic equation $\frac{1}{z} + \frac{1}{z+a} = 1$. (In particular, $\alpha(1) = \frac{1}{2}(1 + \sqrt{5})$ is the *golden section* and $\alpha(2) = \sqrt{2}$.) Then

$$x_n = \lfloor \alpha n \rfloor, \quad y_n = x_n + an \equiv \lfloor n(\alpha + a) \rfloor; \quad n \ge 0. \tag{2}$$

As mentioned in [5], the explicit formula (2) solves game $WYT(a)$ in linear time, in contrast to recursion (1), providing only an exponential algorithm.

3 Function mex_b and recursive solution of game $WYT(a, b)$

Function mex can be generalized as follows. Given an integer $b \ge 1$ and a finite subset $S \subseteq \mathbb{Z}_+$ of m non-negative integers, let us order S to get a sequence $s_1 < \cdots < s_m < \infty$ and choose the minimum i such that $s_{i+1} - s_i > b$. Then, by definition, $mex_b(S) = s_i + b$.

It is easily seen that function mex_b is well-defined and $mex_1 = mex$.

Then, recursion (1) can be naturally extended to game $WYT(a,b)$ as follows.

Theorem 1 *One can characterize the set of P-positions $\{(x_n, y_n) \mid n = 0, 1, \ldots\}$ of game $WYT(a,b)$ by the same recursive formula (1), just replacing in it mex by mex_b, that is,*

$$x_n = mex_b\{x_i, y_i \mid 0 \le i < n\}, \quad y_n = x_n + an; \quad n \ge 0. \tag{3}$$

The first ten P-positions of games $WYT(1,2)$ and $WYT(2,3)$ are given in Table 2.

This theorem immediately implies that $(x_n, y_n) \in P$ are uniform functions of a and b.

Corollary 1 *For all non-negative integer a, b and k, n we have:*

$x_n(ka, kb) = kx_n(a,b)$ *and* $y_n(ka, kb) = ky_n(a,b)$.

□

4 Bouton - von Neumann solution algorithm

An algorithm finding all P-positions was suggested in 1901 by Bouton in [2] for the normal and misere versions of NIM (with k piles). Then, it was extended to the games modeled by arbitrary acyclic digraphs in [9].

It works recursively. In step 1, let us find all terminal (that is, of out-degree 0) positions and denote the obtained set by P_1. Furthermore, let N_1 be the set of all positions from which P_1 can be reached by one move. Let us delete $P_1 \cup N_1$ and repeat, that is, obtain P_2 and N_2, etc. Obviously, $(P_1 \cup P_2 \cup \ldots)$ is the set of all P-positions.

In Figure 2 this algorithm is illustrated for $WYT(1,2)$. The only terminal position is $(x_0, y_0) = (0,0)$. Since $a = 1$ and $b = 2$, set N_1 consists of two columns $\{(x,y) \mid x \le 1\}$, two rows $\{(x,y) \mid y \le 1\}$, and the main diagonal $\{(x,y) \mid x = y\}$, excluding position $(0,0)$ itself. After elimination of $P_1 \cup N_1$, we obtain $P_2 = \{(2,3),(3,2)\}$. Then, set N_2 is constructed in a similar way. Position $(2,3)$ can be reached from two columns $\{(x,y) \mid 2 \le x \le 3, 3 \le y\}$, two rows $\{(x,y) \mid 3 \le y \le 4, 2 \le x\}$, and one diagonal $\{(x,y) \mid y = x+1 > 3\}$, excluding position $(2,3)$ itself. Obviously, the symmetric constraints hold for $(3,2)$. The union of the obtained two sets is N_2. Then, after its eliminating, we get $P_3 = \{(5,7),(7,5)\}$, etc.

The first ten positions of P (with $x \le y$) are given in Table 2.

n	x_n	y_n
0	0	0
1	2	3
2	5	7
3	9	12
4	11	15
5	14	19
6	17	23
7	21	28
8	25	33
9	27	36

n	x_n	y_n
0	0	0
1	3	5
2	8	12
3	11	17
4	15	23
5	20	30
6	26	38
7	29	43
8	33	49
9	36	54

Table 2: ($a = 1$, $b = 2$) and ($a = 2$, $b = 3$).

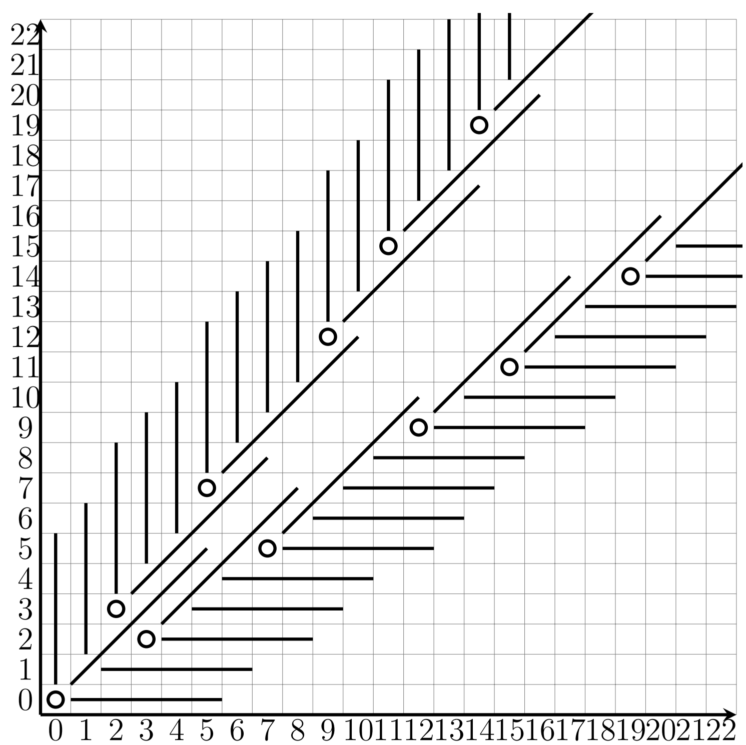

Figure 1: Bouton - von Neumann algorithm for $a = 1$, $b = 2$.

5 Proof of Theorem 2

The above construction easily results in Theorem 2. By symmetry, $P_i = \{(x_i, y_i), (y_i, x_i)\}$ for every $i = 0, 1, \dots$ Each of these two positions can be reached by b rows, b columns, and $2a - 1$ diagonals. For $i = 0$ only a of these diagonals satisfy the restriction $x \le y$. Similarly, for any fixed $i > 0$ only a of them are new, while the remaining $a-1$ were already eliminated before, with N_j for some $j < i$. Thus, in each step, exactly a diagonals are excluded and, hence, $y_n = x_n + an$ for each $n = 0, 1, \dots$

Furthermore, each position (x_i, y_i) eliminates the next b columns $x_i, \dots, x_i + b - 1$. Yet, the symmetric position (y_i, x_i) also eliminates b columns: $y_i, \dots, y_i + b - 1$. (Of course, these two sets may overlap.) Anyway, the desired recursion $x_n = mex_b\{x_i, y_i \mid 0 \le i < n\}$ results from the definition of mex_b and the above construction. □

6 Misere version of game $WYT(a, b)$

Bouton - von Neumann's algorithm can be easily adapted to the misere version of a game modeled by an arbitrary acyclic digraph. Let us just add to this digraph one new position and an arc leading to it from each terminal position of the original game.

In particular, for $WYT(a, b)$ we add one new position $(*, *)$ and one new possible move from $(0, 0)$ to $(*, *)$. By definition, the misere version of the original game is the normal version of the obtained game. Thus, we just apply the standard version of the algorithm to the modified game (rather than develop a "misere version" for the original game) and, for all $b \ge 1$, obtain the following recursion:

For $a = 1$ we have $(x_0, y_0) = (b + 1, b + 1)$, for $n = 0$, and for $n \ge 1$, as before, we have

$$x_n = mex_b\{x_i, y_i \mid 0 \le i < n\}, \quad y_n = x_n + an.$$

From this recursion, it is easy to derive that for the normal and misere versions the sets of P-positions P_N and P_M "almost coincide". More precisely,

$$P_N \setminus P_M = \{(0, 0), (b, b + 1), (b + 1, b)\}; \quad P_M \setminus P_N = \{(0, 1), (1, 0), (b + 1, b + 1)\}.$$

In [1], games of this type (in which P_M and P_M differ "just slightly") are called *tame*. Thus, games $WYT(a, b)$ are tame for $a = 1$ and arbitrary $b \ge 1$. According to [8], another example of a tame game is the game Euclid introduced in [3].

For $a > 1$, we obtain a similar, just slightly different, recursion:

$$x_n = mex_b\{x_i, y_i \mid 0 \le i < n\}, \quad y_n = x_n + an + 1, \quad \forall\, n \ge 0.$$

Yet, in this case, sets P_M and P_N are disjoint. It is well-known, that P_N is the set of zeros of the so-called Sprague-Grundy function [10, 7], and it follows from results of Fergusson [4] that P_M is the set of ones of this function.

We omit the proofs of both recursions, since they go exactly the same lines as in the previous two sections. For $b = 1$ and any a these results were obtained by Fraenkel in [6]. He also derived explicit formulae for x_n and y_n, thus, giving a linear time algorithm that solves the misere version of game $WYT(a)$ for $b = 1$.

Cases $a = 0$ and $b = 0$ will be considered in the next section.

7 Simple cases

Case $a = b = 0$: Then, obviously, there are no moves at all, or in other words, every position is terminal, that is, losing for the normal and winning for the misere version of $WYT(0, 0)$.

Case $a = 0, b \geq 1$: Then it is easily seen that $x_n = y_n = bn$ for all $n \geq 0$ in the normal version and $x_0 = y_0 = (0, 1)$, $x'_0 = y'_0 = (1, 0)$, and $x_n = y_n = bn + 1$ for all $n \geq 1$ in the misere version of $WYT(0, b)$.

Let us notice that the game is tame if $b = 1$, while for $b > 1$, sets P_N and P_M are respectively zeros and ones of the Sprague-Grundy function of $WYT(0, b)$.

Case $b = 0, a \geq 1$: Then, a position (x, y) is terminal if and only if x or y is zero, and there is a move from any non-terminal position to a terminal one.

Finally, let us consider a more general game $WYT(a, b, c)$, in which the set of possible moves (x', y') in a position (x, y) is defined by the following restrictions:

$0 \leq x' \leq x,\ 0 \leq y' \leq y,\ 0 < x' + y'$, and $[\ |x' - y'| < a$, or $x' < b$, or $y' < c]$.

Obviously, $WYT(a, b, c) = WYT(a, b)$ when $b = c$. Let us consider case $b \neq c$.

Applying again the Bouton - von Neumann algorithm we obtain for the P-positions (x_n, y_n) of game $WYT(a, b, c)$ the explicit formula: $x_n = n,\ y_n = n \min(b, c);\ n \geq 0$.

8 Open problems and conjectures

The main open problem is to find a polynomial algorithm solving game $WYT(a, b)$.
Such an algorithm would obviously result from explicit formulae for $x_n(a, b)$ and $y_n(a, b)$. Yet, they are known only for $b = 1, a \geq 0$ (formula (2) by Fraenkel) and for $a = 0, b \geq 1$ (when $x_n = y_n = bn$; see the previous section). Let us also recall that $x_n(ka, kb) = kx_n(a, b)$ and $y_n(ka, kb) = ky_n(a, b)$ for all non-negative integer a, b and k, n, by Corollary 1.
In general, we have recursion (3), which gives only an exponential algorithm for $WYT(a, b)$.

By this recursion, $x_n = x_n(a, b)$ is a function of n of a linear order of magnitude.

We conjecture that limits $L(a, b) = \lim_{n \to \infty} \frac{x_n(a,b)}{n}$ exist for all integer $a \geq 0, b \geq 1$.

This conjecture, if true, and recursion (3) would easily result in the following properties.

- (i) $\lim_{n \to \infty} \frac{y_n(a,b)}{n} = L(a, b) + a$, since $y_n(a, b) = x_n(a, b) + an\ \ \forall\ a \geq 0, b \geq 1, n \geq 0$.
- (ii) $b \leq L(a, b) \leq 2b$, since $b \leq [x_{n+1}(a, b) - x_n(a, b)] \leq 2b\ \ \forall\ a \geq 0, b \geq 1, n \geq 0$.

a	b				
	1	2	3	4	5
0	1.	2.	3.	4.	5.
1	1.618	3.080	4.530	5.978	7.418
2	1.414	3.236	4.296	6.159	7.180
3	1.303	2.613	4.854	5.616	6.895
4	1.236	2.828	3.752	6.472	7.016
5	1.193	2.404	3.798	4.847	8.090

Table 3: Hypothetical approximate limits $L(a,b) = \lim_{n\to\infty} \frac{x_n(a,b)}{n}$ for $0 \le a \le 5, 1 \le b \le 5$; these limits do exist for $b = 1$, or $a = b$, or $(a,b) = (4,2)$.

- (iii) $L(ka, kb) = kL(a, b) \ \forall\ a \ge 0, b \ge 1, k \ge 1.$

As we already know, limits $L(a,b)$ do exist when $b = 1$ or $a = 0$; moreover, by (iii),

$L(ka, k) = kL(a, 1) = \frac{k}{2}(2-a+\sqrt{a^2+4})$, $L(0, kb) = kL(0, b) = kb$ for all integer $k \ge 1$.

For small a, b the (hypothetical and approximate) values of $L(a,b)$ are given in Table 3.

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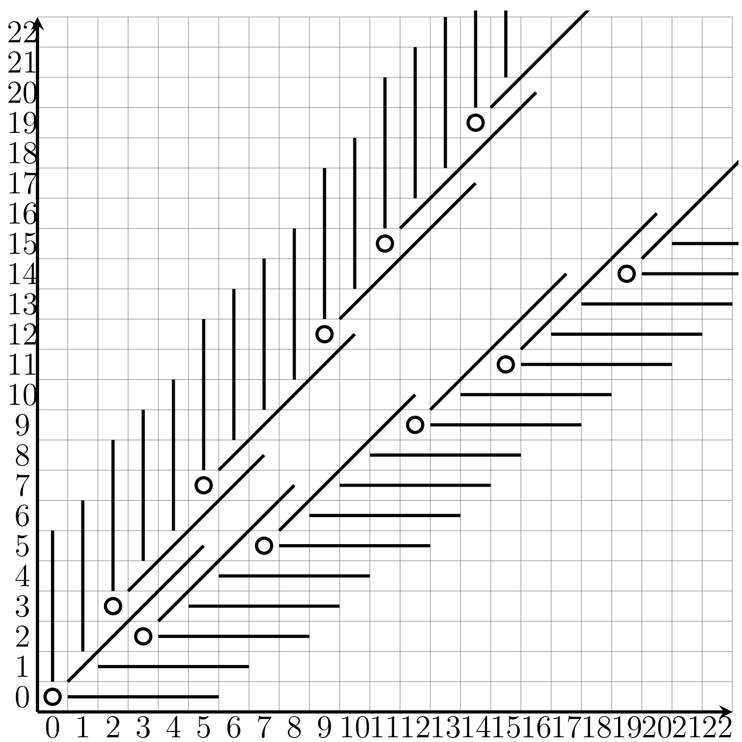

Figure 2: Bouton - von Neumann algorithm for $a = 1$, $b = 2$.