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THE DISCRETE MOMENT METHOD FOR THE NUMERICAL INTEGRATION OF PIECEWISE HIGHER ORDER CONVEX FUNCTIONS

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RUTCOR RESEARCH REPORT

THE DISCRETE MOMENT METHOD FOR THE NUMERICAL INTEGRATION OF PIECEWISE HIGHER ORDER CONVEX FUNCTIONS

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Abstract. A new numerical integration method, termed Discrete Moment Method (DMM), is proposed for univariate functions that are piecewise higher order convex. This means that the interval where the function is defined can be subdivided into non-overlapping subintervals such that in each interval all divided differences of given orders, do not change the sign. The new method uses piecewise polynomial lower and upper bounds on the function, created in connection with suitable dual feasible bases in the univariate discrete moment problem and the integral of the function is approximated by tight lower and upper bounds on them. Numerical illustrations are presented for the cases of the normal, exponential, gamma and Weibull probability density functions.

1 Introduction

Numerical integration methods generally work in such a way that the integrand is evaluated at a finite number of points, called integration points or base points, and a weighted sum of these values approximates the integral. The base points and weights depend on the specific method used and the required accuracy.

An important part of the analysis of any numerical integration method is the study of the approximation error as a function of the number of integrand evaluations. A method which yields a small error for a small number of evaluations is usually considered efficient.

Many integration rules (see, e.g., [1, 2, 3]) use interpolation functions, typically by polynomials, which are easy to integrate. The simplest rules of this type are the midpoint (or rectangle), the trapezoidal and the Simpson's rules, where for a small interval $[c,d]$ the approximations

$$\int_c^d f(x)dx \approx (d-c) f\left(\frac{c+d}{2}\right),$$

$$\int_c^d f(x)dx \approx (d-c)\frac{f(c)+f(d)}{2},$$

$$\int_c^d f(x)dx \approx \frac{d-c}{n}\left(\frac{f(c)+f(d)}{2}+\sum_{k=1}^{n-1} f\left(c+k\frac{d-c}{n}\right)\right),$$

respectively, are used.

Interpolation with polynomials evaluated at equally-spaced points in $[c,d]$ yields the Newton-Cotes formulas, of which the rectangle and the trapezoidal rules are examples. Simpson's rule, which is based on a polynomial of order 2, is also a Newton–Cotes formula. If we allow the intervals between interpolation points to vary in length, we find other integration formulas, such as the Gaussian quadrature formulas. A Gaussian quadrature rule is typically more accurate than a Newton–Cotes rule which requires the same number of function evaluations, if the integrand is smooth. For a large number of variants of Gaussian quadrature the reader is referred to Golub, Meurant [9].

Romberg's method is based upon the approximation of the integral by the trapezoidal rule. Quadrature formulas of higher error order are produced by successive division of the step size by 2 and by an appropriate linear combination of the resulting approximations for the integral.

First, one partitions $[c,d]$ into N_0 subintervals of length $h_0=(c-d)/N_0$ and sets

$$N_i = 2^i N_0,\, h_i = h_0/2^i, \quad i=0,1,\ldots,$$

then the integral is expressed as

$$\int_c^d f(x)dx = L_i^{(k)}(f(x)) + O(h_i^{2(k+1)}),$$

where $L_i^{(k)}(f(x))$ is a quadrature formula with error order $O(h_i^{2(k+1)})$.

Romberg's method provides us with accurate results if the integrand has multiple continuous derivatives, though fairly good results may be obtained if only a few derivatives exist. We also mention numerical methods by Tortorella [8] that are useful when it is impossible or undesirable to use derivatives of the integrand.

In this paper we propose a new univariate numerical integration method. We create lower and upper bounding polynomials for the function on a finite grid but ensure that the integrals of the bounding polynomials provide us with tight lower and upper bounds for the integral of our function in an entire interval. We use Lagrange polynomials for bounding that are natural outcomes of the use of the discrete power moment problem. We illustrate our new method for the functions: $e^{-\frac{x^2}{2}}$, $x^m e^{-\frac{x^2}{2}}$, $\left(\frac{x}{\lambda}\right)^{m-1} e^{-\left(\frac{x}{\lambda}\right)^m}$, and $\lambda e^{-\lambda x}$.

2. Bounding by Lagrange Polynomials

2.1. Summary of the Discrete Moment Problem

In what follows we assume the knowledge of the elements of linear programming. A brief summary of it can be found in Prékopa (1995, 1996). The material in this section is based on Prékopa (1990).

Consider the following linear programming problem:

$$
\begin{aligned}
&\min \max \sum_{i=0}^{n} f(z_i) p_i \\
&\text{s.t.} \quad \sum_{i=0}^{n} z_i^k p_i = \mu_k,\ k = 0,\ldots,m; \\
&\qquad\ \ p_i \geq 0,\ i = 0,\ldots,n,
\end{aligned}
\tag{2.1}
$$

where $f(z), z \in \{z_0, ..., z_n\}$ is a discrete function, $m < n$ and the decision variables are $p_0, p_1, \ldots, p_n$. Problem (2.1) is called discrete power moment problem. The matrix of the equality constraints, its columns, and the right hand side vector will be designated by $\boldsymbol{A}$, $a_0, a_1, \ldots, a_n$ and b, respectively. Thus,

$$
a_h = \begin{pmatrix} 1 \\ z_h \\ \vdots \\ z_h^m \end{pmatrix},\ h = 0, 1, \ldots, n;\ b = \begin{pmatrix} 1 \\ \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}.
$$

We also use the symbol f_i as an alternative notation for $f(z_i)$. Note that the matrix $\boldsymbol{A}$ has full row rank. Let $\boldsymbol{B}$ be a basis and designate by $\boldsymbol{I_B}$ the set of subscripts of those columns of $\boldsymbol{A}$ which form $\boldsymbol{B}$. A basis is said to be dual feasible, relative to the minimization (maximization) problem, if we have

$$
f_B^T B^{-1} a_h \leq f_h \ (f_B^T B^{-1} a_h \geq f_h),
$$

where f_B designates the vector of the basic components of f, $h = 0, 1, \ldots, n$.

Let $L_{I_B}(z)$ be the Lagrange polynomial of degree m, corresponding to the points z_i, $i \in I_B$, i.e.,

$$L_{I_B}(z)=\sum_{i\in I_B} f(z_i)L_{I_B,i}(z),$$

where

$$L_{I_B,i}(z)=\frac{(z-z_0)\dots(z-z_{i-1})(z-z_{i+1})\dots(z-z_m)}{(z_i-z_0)\dots(z_i-z_{i-1})(z_i-z_{i+1})\dots(z_i-z_m)}.$$

Define the vector $b(z)=\left(1, z,\dots, z^m\right)^T$ for every real z. We assert that

$$f_B^T B^{-1}b(z)=L_{I_B}(z).$$

In fact, $b(z_i)=a_i$ for $i\in I_B$, hence

$$f_B^T B^{-1}b(z_i)=f(z_i),\ i\in I_B.$$

From the above discussion a nice characterization follows for the dual feasible bases, in terms of Lagrange polynomials: in the minimization problem, the function $f(z)$ is greater than or equal to the Lagrange polynomial $L_{I_B}(z)$ for every z_i, $i\notin I_B$. In the maximization problem, the function $f(z)$ is smaller than or equal to the Lagrange polynomial $L_{I_B}(z)$ for every z_i, $i\notin I$. In both problems, the function $f(z)$ coincides with $L_{I_B}(z)$ at every z_i, $i\in I_B$. Hence, we readily obtain methodology to find lower (upper) bounding polynomial for the discrete function $f(z)$, $z\in\left\{z_0, z_1,\dots, z_n\right\}$: choose arbitrarily a dual feasible basis for problem (2.1) and create the corresponding Lagrange polynomial with base points z_i, $i\in I_B$. If the lower and upper bounds are close enough, then these polynomials provide us with good approximation for the entire discrete function $f(z)$. Note that the choice of a dual feasible basis is very simple, it does not need any LP algorithm to carry out. The above approximation uses discrete points but, as we will see later, they are enough for the approximation of the numerical integral of a function $f(z)$, defined on an entire interval.

2.2. The Structure of the Dual Feasible Bases

Let us assume that $f(z)$ is defined on the discrete set $z\in\{z_0,..., z_n\}$. The first order divided differences of $f(z)$ are

$$\left[z_i, z_{i+1}\right]f=\frac{f(z_{i+1})-f(z_i)}{z_{i+1}-z_i},\ i=0,1,\dots,n-1.$$

The kth order divided differences are defined recursively by

$$\left[z_i,\dots, z_{i+k}\right]f=\frac{\left[z_{i+1},\dots, z_{i+k}\right]f-\left[z_i,\dots, z_{i+k-1}\right]f}{z_{i+k}-z_i},\ k\ge 2.$$

It is well-known that if $f(z)$ is defined and differentiable on $\left[a,b\right]$ with $f^{(m+1)}(z)\ge 0$, $a\le z\le b$, then all divided differences of order $m+1$ of $f(z)$ in the interval $\left[a,b\right]$ are nonnegative.

If a function has all nonnegative (positive) divided differences of order $m+1$ in its domain of definition (no matter if it is a discrete set or an interval), then the function is called convex (strictly convex) of order $m+1$. The function is concave (strictly concave) of order $m+1$ if its negative is convex (strictly convex) of order $m+1$.

Theorem 1. Suppose that all m+1-order divided differences of the function $f(z)$, $z \in \{z_0, z_1, \ldots, z_n\}$ are positive. Then in Problem (2.1) all bases are dual non-degenerate and the dual feasible bases have the following structures, presented in terms of the subscripts of the basic vectors:

	m+1 even	m+1 odd
min problem	$\{j, j+1, \ldots, k, k+1\}$	$\{0, j, j+1, \ldots, k, k+1\}$
max problem	$\{0, j, j+1, \ldots, k, k+1, n\}$	$\{j, j+1, \ldots, k, k+1, n\}$

where in all parentheses the numbers are arranged in increasing order.

Remark. If the m+1-order divided differences are required to be only nonnegative, then the above basis structures are only sufficient for dual feasibility.

Remark. If the m+1-order divided differences of $f(z)$, $z \in \{z_0, z_1, \ldots, z_n\}$ are negative, then the dual feasible basis structures in the min (max) problem are the same as those in Theorem 1 for the max (min) problem.

The proof of theorem 1 can easily be carried out by the application of a well-known formula in approximation theory. In our case it can be stated as:

$$f(z) - L_{I_B}(z) = [z_i, i \in I_B, z; f] \prod_{i \in I_B} (z - z_i). \tag{2.2}$$

For a different derivation the reader is referred to Prékopa (1990 a, b).

Based on Theorem 1, we can easily find dual feasible bases for problem (2.1), therefore, it is easy to obtain Lagrange polynomials that serve as lower and upper bounds for $f(z)$ on the discrete set $\{z_0, z_1, \ldots, z_n\}$.

3. Conditions on the Base Points to Obtain Bounds on the Integral

Let f be a convex function of order $m+1$ in the interval $[c,d]$ and $Z = \{z_0, z_1, \ldots, z_n\} \subset [a,b]$. Suppose that the set $\{z_{i_0}, z_{i_1}, \ldots, z_{i_m}\} \subset Z$ defines a dual feasible basis in minimization problem (2.1) and let $l(z)$ designate the corresponding Lagrange polynomial (for simplicity we suppress the subscript B). We have the relation

$$f(z) - l(z) = [z_{i_0}, \ldots, z_{i_m}, z; f] \prod_{k=0}^{m} (z - z_{i_k}) \geq 0 \tag{3.1}$$

for any $z \in Z$.. If it is dual feasible in the maximization problem (2.1) and the corresponding Lagrange polynomial is $u(z)$, then we have the relation:

$$f(z) - u(z) = [z_{i_0}, \ldots, z_{i_m}, z; f] \prod_{k=0}^{m} (z - z_{i_k}) \leq 0, \tag{3.2}$$

for any $z \in Z$. In both cases equality holds for $z \in \{z_{i_0}, z_{i_1}, \ldots, z_{i_m}\}$.

Inequalities (3.1) and (3.2) hold true also for $z \in [a, b]$ with the exception of the interiors of consecutive pairs, described in Theorem 1, among the base points $\left\{z_{i_0}, z_{i_1}, \dots, z_{i_m}\right\}$, where the inequalities are reversed. For this reason from (3.1) and (3.2) we cannot immediately derive that

$$\int_c^d l(z)dz \le \int_c^d f(z)dz \le \int_c^d u(z)dz . \tag{3.3}$$

However, the intervals between the consecutive pairs are small and in practice there is a relatively small number of consecutive pairs, hence the integrals of the differences $f(z)-l(z)$, $u(z)-f(z)$ over the union of consecutive pairs are small and allow for the validity of the relations in (3.3).

Figures 3.1 and 3.2 illustrate the situation. In Figure 3.1 the graphs show that if the base points $z \in \left\{z_0, z_j, z_{j+1}, z_k, z_{k+1}\right\}$ are chosen in such a way that z_j, z_{j+1} as well as z_k, z_{k+1} are close to each other, then $l(z) \ge f(z)$ on the small intervals $\left[z_j, z_{j+1}\right]$, $\left[z_k, z_{k+1}\right]$, otherwise we have $l(z) \le f(z)$. The deficiency in the integral $\int_a^b l(z)dz$ caused by $l(z) \ge f(z)$ in $\left(z_j, z_{j+1}\right) \cup \left(z_k, z_{k+1}\right)$ can easily be offset by choosing $z_j, z_{j+1}, z_k, z_{k+1}$ in a suitable way. The same idea applies to the maximization problem (Fig. 3.2).

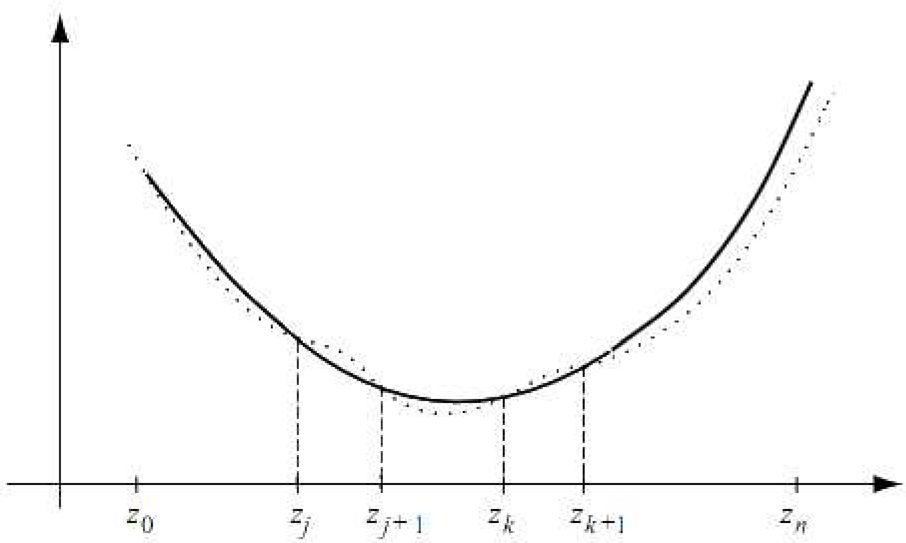

Figure 3.1. Function $f(z)$; Lagrange polynomial (dotted line)
Minimization problem, $m+1$ odd.
Basic subscript set: $z \in \left\{z_j, z_{j+1}, z_k, z_{k+1}, z_n\right\}$

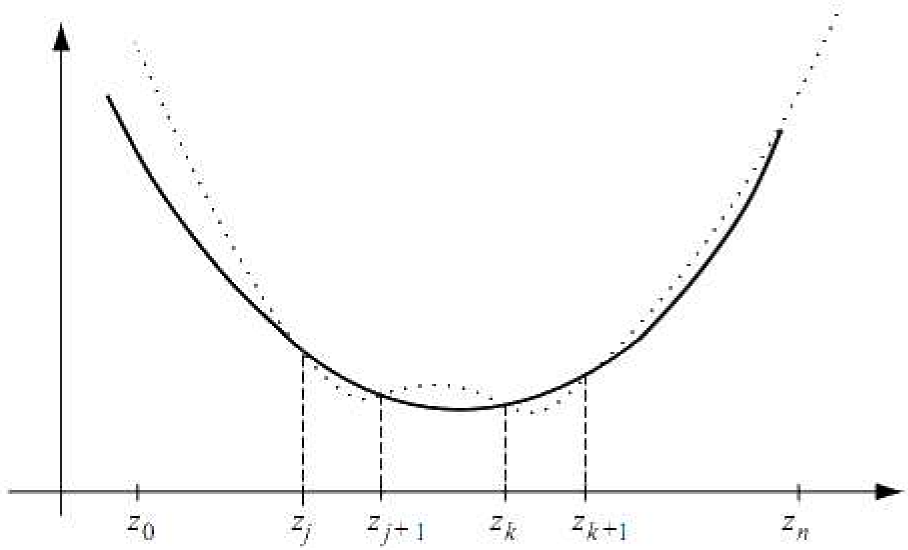

Figure 3.2. Function $f(z)$; Lagrange polynomial (dotted line)
Maximization problem, $m+1$ odd.
Basic subscript set: $z \in \{z_0, z_j, z_{j+1}, z_k, z_{k+1}\}$

Stating it in a different way: under mild conditions on the function and the base points the nonpositivity of the integrals of $f(z)-l(z)$ over the intervals between consecutive pairs is offset by the nonnegativity of the integrals over the much larger set, where $f(z)-l(z)\ge 0$, and a similar assertion holds for $u(z)-f(z)$. This ensures that we in fact have the relations (3.3). But we can say more about it. If that happens then the fact that in some intervals the integral of $f(z)-l(z)$ is negative makes the lower bound tighter and the negativity of $u(z)-f(z)$ in some intervals makes the upper bound tighter.

To support the above statements we prove two theorems but first we mention a simple lemma.

Lemma. If $[u,v]$ is a finite interval of positive length, then for any $y\in[u,v]$ and $z \ge \bar{z} = \frac{u+v}{2} + \frac{\sqrt{2}}{2}(v-u)$ we have the inequality

$$(y-u)(v-y) \le (z-u)(z-v). \tag{3.4}$$

Proof. The largest value on the left-hand side is attained at $y=(u+v)/2$. Thus, (3.4) holds true if under the given condition for z, we have the inequality

$$(z-u)(z-v) \ge \frac{(v-u)^2}{2}.$$

Elementary calculations show the validity of the last inequality for $z \ge \bar{z}$. □

Theorem 2. Let $f(z)$, $a \le z \le b$, be a real valued function that has nonnegative divided differences of orders $m+1$ and $m+2$, where $a < b$. If $m+1$ is even then take $(m+1)/2$ disjoint, equal positive length and equidistant subintervals of $[a,b]$: $[u_1, v_1], [u_2, v_2], \ldots, [u_{(m+1)/2}, v_{(m+1)/2}]$, $a < u_1 < v_1 < \ldots < u_{(m+1)/2} < v_{(m+1)/2} < b$, and let $L(z)$ designate the Lagrange polynomial corresponding to their endpoints, as base points. Suppose that

$$\sum_{i=1}^{(m+1)/2} (v_i - u_i) \le b - \bar{z},$$

where

$$\bar{z} = \frac{u_{(m+1)/2} + v_{(m+1)/2}}{2} + \frac{\sqrt{2}}{2}\left(u_{(m+1)/2} + v_{(m+1)/2}\right).$$

Under these conditions we have the inequality

$$-\sum_{i=1}^{(m+1)/2} \int_{u_i}^{v_i} (f(z) - l(z))dz \le \int_{b-\bar{z}}^{b} (f(z) - l(z))dz. \tag{3.5}$$

If $m+1$ is odd then we take $m/2$ subintervals and create the Lagrange polynomial by the use of all endpoints of them and the left-hand endpoint of the interval $[a,b]$. Then (3.5) remains true, if we replace m for $m+1$ in it and in two relations above it.

Remark. Theorem I tells us that the negativity of the integrals of $f(z) - l(z)$ between the consecutive pairs (over the subintervals) can be offset by the integral of $f(z) - l(z)$ over the interval $[b - \bar{z}, b]$ that is assumed to have length greater than or equal to the sum of the lengths of the subintervals.

Proof of Theorem 2. Assume that $m+1$ is even (the proof for the other case is the same). If $\{z_0, z_1, \ldots, z_m\}$ is the set of base points of the Lagrange polynomial, then we have the equality

$$f(z) - l(z) = [z_0, \ldots, z_m, z; f] \prod_{k=0}^{m} (z - z_k). \tag{3.6}$$

Theorem I tells us that any dual feasible basis consists of consecutive pairs. Let $\{z_0, z_1, \ldots, z_m\} = \{u_1, v_1, u_2, v_2, \ldots, u_{(m+1)/2}, v_{(m+1)/2}\}$. We want to show that if $y \in [u_i, v_i]$ for some $i \in \{1, 2, \ldots, (m+1)/2\}$ and $z \ge \bar{z}$, then we have the inequality

$$\begin{aligned} &-[z_0, \ldots, z_m, y; f] \prod_{i=0}^{m} (y - z_i) \\ &\le [z_0, \ldots, z_m, z; f] \prod_{i=0}^{m} (z - z_i). \end{aligned} \tag{3.7}$$

By assumption the length of the interval $[\bar{z}, b]$ is at least as large as the sum of the subintervals $[u_i, v_i]$, $i = 1, 2, \ldots, (m+1)/2$, thus the negativity of each integral of $f(z) - l(z)$

over a subinterval will be offset by a positive integral over an equal length subinterval of the interval $[b-\bar{z}, b]$. Thus, if we prove (3.7), then the proof of the theorem will be complete.

To prove (3.7) we proceed as follows. First we look at y values from $[u_{(m+1)/2}, v_{(m+1)/2}]$. Since the $m+2$-order divided differences of f are positive, it follows that

$$\left[z_0,\dots,z_m,y;f\right]\le\left[z_0,\dots,z_m,z;f\right]. \tag{3.8}$$

So we have to prove that

$$-\prod_{i=0}^{m}(y-z_i)\le\prod_{i=0}^{m}(z-z_i). \tag{3.9}$$

By the lemma we know that if $z\ge\bar{z}$, then

$$(y-z_{m-1})(y-z_m)\le(z-z_{m-1})(z-z_m). \tag{3.10}$$

On the other hand, we clearly have the inequalities

$$y-z_i\le z-z_i,\ i=1,2,\dots,m-2. \tag{3.11}$$

Inequalities (3.10) and (3.11) imply (3.9) and (3.9) and (3.8) imply (3.7).

If $y\in\left[u_i,v_i\right]$, $i<\dfrac{m+1}{2}$, then since the intervals have equal lengths we have that

$$-(y-z_{2i-1})(y-z_{2i})=-(y-u_i)(y-v_i)\le\left(\frac{v_i-u_i}{2}\right)^2=\left(\frac{v_m-u_m}{2}\right)^2=(z-z_{m-1})(z-z_m).$$

If we look at the distances of the subintervals from y and from z, we easily see that

$$\left|\prod_{j\ne i}(y-u_j)(y-v_j)\right|\le\prod_{j=1}^{m-1}(z-u_j)(z-v_j),$$

provided that $z\ge\bar{z}$ and the assertion follows.

□

Theorem 3. Keep the conditions of Theorem I except for the requirements that the subintervals have equal lengths and are equidistant. Then the assertion of Theorem 2, i.e., inequality (3.5) holds true if the inequalities below are satisfied.

If $m+1$ is even, then

$$\left(\frac{v_i-u_i}{2}\right)^2\prod_{j=1}^{i-1}(v_i-v_j)(v_i-u_j)\prod_{k=i+1}^{(m+1)/2}(v_k-u_i)(u_k-u_i)\le\prod_{i=1}^{(m+1)/2}(\bar{z}-u_i)(\bar{z}-v_i), \tag{3.12}$$

$$i=1,\dots,(m+1)/2,\ \sum_{i=1}^{(m+1)/2}\left(v_i-u_i\right)\le b-\bar{z}, \tag{3.13}$$

where $\bar{z}=\max\limits_{1\le i\le(m+1)/2}\left(\dfrac{u_i+v_i}{2}+\dfrac{\sqrt{2}}{2}\left(v_i-u_i\right)\right)$.

If $m+1$ is odd, then the required relations can be obtained if we replace m for $m+1$.

Proof. The proof is carried out for the case of $m+1$ even and is based on the inequality (3.7). We increase the first line by taking $\left(\frac{v_i-u_i}{2}\right)^2$ as an upper bound for $-(y-u_i)(y-v_i)$ and in the other factors we take u_i or v_i for y, depending on whichever has larger distance from y. In the other hand, the smallest value in the second line is obtained for $z=\overline{z}$. After this substitutions we obtain (3.12), and the value in (3.13) is the same as the value in the second line of (3.7). It follows that (3.12) and (3.13) imply (3.7).

□

Note that inequalities (3.12), (3.13) are easy to check and it is easy to choose the subintervals and $\overline{z}$ in such a way that (3.12) and (3.13) are satisfied.

4. The Discrete Moment Method (DMM) of Univariate Numerical Integration

In this section we briefly describe the new numerical integration method, the main contribution of this paper.

If a function $f(z)$, $a\le z\le b$ is convex of order $m+1$, then for any discrete set of points Z of at least $m+2$ points the discretized function $f(z)$ has all nonnegative divided differences of order $m+1$. As we have seen, we can construct two m-degree polynomials $l(z)$ and $u(z)$ such that

$$l(z)\le f(z)\le u(z),\ z\in Z. \tag{4.1}$$

The bounding polynomials can be obtained by the use of dual feasible bases, corresponding to problem (2.1). Then we approximate

$$\int_a^b f(x)dx \tag{4.2}$$

by the integrals

$$\int_a^b l(z)dz,\ \int_a^b u(z)dz \qquad (4.3).$$

Our intention is not only to approximate the integral (4.2) by the integrals (4.3) but to ensure that the integrals (4.3) serve as lower and upper bounds, respectively for the integral (4.2), i.e.,

$$\int_a^b l(z)dz\le\int_a^b f(z)dz\le\int_a^b u(z)dz\,. \tag{4.4}$$

As we have mentioned in Section 3, inequality (4.4) is not a direct consequence of (4.1) but its validity can be ensured by suitable choices of the points in the base sets, in other words, the dual feasible bases in problem (2.1) that define the polynomials $l(z)$, $u(z)$.

The function $f(z)$ may not be higher order convex (or concave) in the entire interval $[a,b]$. However, it may be true that the interval $[a,b]$ can be subdivided into a finite number of non-overlapping intervals such that on each of them the function is higher (not necessarily always of the same) order convex (concave). If this is the case, then we apply the numerical integration procedure for each subdividing interval and create bounds and approximations of the integral of $f(z)$ on $[a,b]$ by the use of the integrals on the subdividing intervals.

Algorithm.

Initialization: Use Lsum as the notation that the lower bound summation of integral of Lagrange polynomials in subintervals; Use Usum as the notation that the upper bound summation of integral of Lagrange polynomials in subintervals.
Procedure:
Step 1. Determine the subdividing intervals where the *r*th divided difference of the function is positive or negative.
Step 2. For each subdividing interval $[c,d]$,

a) If the *r*th divided difference of the function is positive, repeat
- subdivide it into n subintervals of equal length ($n \geq r$);
- label those endpoints by $z_0, z_1, \ldots, z_n$, evaluate the function at these labeled points.
- Find any dual feasible basis according to Theorem 1 to get its corresponding upper and lower bounding Lagrange polynomials.
- Integrate the upper and lower bounding Lagrange polynomials in the subinterval.
- Lsum = Lsum + Integral of the lower bounding Lagrange polynomial in $[c,d]$.
- Usum = Usum + Integral of the upper bounding Lagrange polynomial in $[c,d]$.

b) If the *r*th divided difference of the function is negative, multiply the function by -1, continue as in part a.

Note that the error in this new numerical integration method can easily be controlled because we provide simultaneous lower and upper bounds for the integral. If the bounds are not close enough then we may increase the number of base points to increase accuracy. The inclusion of new base points increases the lower bound and decreases the upper bound.

5. Illustration for the Case of the Normal Probability Density Function

We pay special attention to the univariate normal probability density function because of the connection to orthogonal polynomials.

The Hermite polynomials, a classical sequence of orthogonal polynomials, arise e.g. in probability theory, combinatorics and physics, and can be defined as

$$H_r(x) = (-1)^r e^{\frac{x^2}{2}} \frac{d^r}{dx^r}\left(e^{\frac{-x^2}{2}}\right)$$

or as

$$\tilde{H}_r(x) = (-1)^r e^{x^2} \frac{d^r}{dx^r}\left(e^{-x^2}\right).$$

These two definitions are not exactly equivalent; either is a rescaling of the other, more precisely

$$\tilde{H}_r(x) = 2^{\frac{r}{2}} H_r(\sqrt{2}x) .$$

We use the first definition which is often preferred in probabilistic applications. In fact, $\varphi(x) = 1/\sqrt{2\pi}\exp\left(-x^2/2\right)$ is the probability density function of the standard normal distribution. The first ten Hermite polynomials are:

$$H_0(x) = 1;$$
$$H_1(x) = x;$$
$$H_2(x) = x^2 - 1;$$
$$H_3(x) = x^3 - 3x;$$
$$H_4(x) = x^4 - 6x^2 + 3;$$
$$H_5(x) = x^5 - 10x^3 + 15x;$$
$$H_6(x) = x^6 - 15x^4 + 45x^2 - 15;$$
$$H_7(x) = x^7 - 21x^5 + 105x^3 - 105x;$$
$$H_8(x) = x^8 - 28x^6 + 210x^4 - 420x^2 + 105;$$
$$H_9(x) = x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x.$$

The roots of the Hermite polynomials for $r = 2$ to $r = 10$ have been tabulated to eight decimals and are presented in the table below. Because of symmetry it is enough to present the nonnegative values.

Table 5.1

$r=2$	1.00000000
$r=3$	0.00000000 1.73205081
$r=4$	0.74196378 2.33441422
$r=5$	0.00000000 1.35562618 2.85697001
$r=6$	0.61670659 1.88917588 3.32425743
$r=7$	0.00000000 1.15440539 2.36675941 3.75043971
$r=8$	0.53907981 1.63651904 2.80248586 4.14454719
$r=9$	0.00000000

	1.02325566 2.07684798 3.20542900 4.51274586
$r=10$	0.48493571 1.46600182 2.48432584 3.58182348 4.85946283

Once the roots of $H_r(x)$ are found, it is possible to determine the intervals where $\frac{d^r}{dx^r}\left(e^{\frac{-x^2}{2}}\right)$ is positive or negative. Therefore, for each interval, if $\frac{d^r}{dx^r}\left(e^{\frac{-x^2}{2}}\right)$ is positive, the lower (upper) bound of $e^{-\frac{x^2}{2}}$ is the value at x of the Lagrange polynomial, associated with the minimization (maximization) problem (2.1), where $f(x)=e^{-\frac{x^2}{2}}$. If $\frac{d^r}{dx^r}\left(e^{\frac{-x^2}{2}}\right)$ is negative, the lower (upper) bound of $e^{-\frac{x^2}{2}}$ is the value at x of the Lagrange polynomial, associated with the minimization (maximization) problem.

Hence, we propose the following algorithm to approximate the normal integral in interval $[a,b]$:

Algorithm.

Step 1. Calculate the *r*th roots of the Hermite polynomial in the interval $[a,b]$.

Step 2. Determine the subdividing intervals where the *r*th derivative of $e^{-\frac{x^2}{2}}$ is positive or negative.

Step 3. For each subdividing interval $[c,d]$,

c) If the *r*th derivative of $e^{-\frac{x^2}{2}}$ is positive, repeat
- subdivide it into n subintervals of equal length ($n \geq r$);
- label those endpoints by $z_0, z_1, \ldots, z_n$, evaluate the function at these labeled points and construct the following problem:

$$\min \max \sum_{i=0}^{n} f(z_i) p_i$$

$$\text{s.t.} \quad \sum_{i=0}^{n} z_i^k p_i = \mu_k,\ k = 0,\ldots,m;$$

$$p_i \ge 0,\ i = 0,\ldots,n,$$

that is, problem (2.1).

- Find any dual feasible basis according to Theorem 1 to get its corresponding upper and lower bounding Lagrange polynomials.
- Integrate the upper and lower bounding Lagrange polynomials in the subinterval.
- Lsum = Lsum + Integral of the lower bounding Lagrange polynomial in $[c, d]$.
- Usum = Usum + Integral of the upper bounding Lagrange polynomial in $[c, d]$.

d) If the rth derivative of $e^{-\frac{x^2}{2}}$ is negative, multiply the function by -1, continue as in part a.

Step 4. Multiply the Lsum and Usum by $\frac{1}{\sqrt{2\pi}}$ to get the lower and upper bound for the integral of univariate normal in interval $[a, b]$.

6. Further Numerical Results

We evaluated the probability integrals of the following functions : $e^{-\frac{x^2}{2}}$, $x^m e^{-\frac{x^2}{2}}$, $\left(\frac{x}{\lambda}\right)^{m-1} e^{-\left(\frac{x}{\lambda}\right)^m}$, and $\lambda e^{-\lambda x}$ with different parameters in (a, b).

For a fixed r we consider the zeros of the Hermite polynomials that are in the interval (a, b). Each interval between two zeroes (or between one zero and one endpoint of (a, b)), we divide into k subintervals, and each subinterval - into n smaller intervals of equal length. For each small interval we choose m points to generate two Lagrange polynomials. Integration of these polynomials and summation over all the subintervals yields the final bounds.

The results for different parameters are presented in the tables below.

Table 6.1. $f(x) = e^{-\frac{x^2}{2}}$, $a = 0, b = 2$

M	k	N	Lsum	Usum	Average	Exact value (5 digits accuracy)
3	6	20	1.19513	1.19754	1.19634	1.19629
3	7	20	1.19552	1.19711	1.19632	1.19629
3	8	20	1.19583	1.19682	1.19633	1.19629
3	10	20	1.19604	1.19655	1.19630	1.19629

3	15	20	1.19621	1.19638	1.19623	1.19629
4	5	20	1.19627	1.19633	1.19630	1.19629
5	5	20	1.19629	1.19630	1.19630	1.19629
5	10	20	1.19629	1.19629	1.19629	1.19629

Table 6.2. $f(x)=e^{-\frac{x^2}{2}}$, $a=0, b=3$.

M	k	N	Lsum	Usum	Average	Exact value (5 digits accuracy)
3	6	20	1.24872	1.25018	1.24945	1.24993
3	7	20	1.24917	1.25028	1.24973	1.24993
3	8	20	1.24942	1.25016	1.24979	1.24993
3	10	20	1.24967	1.25005	1.24986	1.24993
3	15	20	1.24985	1.24997	1.24991	1.24993
4	5	20	1.24990	1.24995	1.24993	1.24993
5	5	20	1.24993	1.24993	1.24993	1.24993
5	10	20	1.24993	1.24993	1.24993	1.24993

Table 6.3. $f(x)=x^m e^{-\frac{x^2}{2}}$, $m=3$, $a=0, b=2$

M	k	n	Lsum	Usum	Average	Exact value (5 digits accuracy)
3	6	20	1.18644	1.18930	1.18787	1.18799
3	7	20	1.18692	1.18889	1.18791	1.18799
3	8	20	1.18723	1.18858	1.18791	1.18799
3	10	20	1.18755	1.18832	1.18792	1.18799
3	15	20	1.18779	1.18813	1.18796	1.18799
4	5	20	1.18795	1.18806	1.18801	1.18799
5	5	20	1.18798	1.18801	1.18800	1.18799
5	10	20	1.18799	1.18798	1.18799	1.18799

Table 6.4. $f(x)=x^m e^{-\frac{x^2}{2}}$, $m=3$, $a=0, b=4$

M	k	n	Lsum	Usum	Average	Exact value (5 digits accuracy)
3	6	20	1.99258	1.99539	1.99399	1.99396
3	7	20	1.99302	1.99489	1.99396	1.99396
3	8	20	1.99334	1.99453	1.99394	1.99396
3	10	20	1.99361	1.99425	1.99393	1.99396
3	15	20	1.99380	1.99407	1.99394	1.99396
4	5	20	1.99387	1.99402	1.99395	1.99396
5	5	20	1.99393	1.99398	1.99396	1.99396
5	10	20	1.99395	1.99397	1.99396	1.99396

Table 6.5. $f(x)=\left(\frac{x}{\lambda}\right)^{m-1} e^{-\left(\frac{x}{\lambda}\right)^m}$, $m=3$, $\lambda=4$, $a=0, b=4$

M	k	n	Lsum	Usum	Average	Exact value (5 digits accuracy)
3	6	20	0.84150	0.84410	0.84280	0.84283
3	7	20	0.84194	0.84368	0.84281	0.84283
3	8	20	0.84225	0.84336	0.84281	0.84283
3	10	20	0.84253	0.84311	0.84282	0.84283
3	15	20	0.84271	0.84293	0.84282	0.84283
4	5	20	0.84276	0.84287	0.84282	0.84283
5	5	20	0.84280	0.84284	0.84282	0.84283
5	10	20	0.84282	0.84283	0.84283	0.84283

Table 6.6. $f(x)=\lambda e^{-\lambda x}$, $\lambda=-1$, $a=0, b=2$

M	k	n	Lsum	Usum	Average	Exact value (5 digits accuracy)
3	6	20	0.86439	0.86479	0.86459	0.86466
3	7	20	0.86444	0.86471	0.86458	0.86466
3	8	20	0.86465	0.86466	0.86466	0.86466

7. Conclusion

We have introduced a new numerical integration method that can be applied if the integrand is a univariate piecewise higher order convex or concave function of the same or different orders. Using the theory of the discrete moment problem, we easily obtain Lagrange polynomials that serve as lower and upper bounds of the function in suitable subintervals. The bounds hold true in the entire interval, except for a few very small intervals, where the differences of the function and the bounding polynomials have opposite signs than required. However, the negative effects of the integrals on these small subintervals are offset by the integrals of the polynomials on the other parts of the interval. Two theorems serve as guidelines how to choose the base points to allow for the offsetting effect. The new numerical integration technique is illustrated on four special functions and it is shown that in the numerical integration high accuracy can be obtained with a relatively few base points.

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