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Abstract. Given positive integer a, b and p, q, we will consider the following game $\text{NIM}_{a,b}^{p,q}$. Two piles contain x and y matches. Two players take turns. By one move, it is allowed to take x' and y' matches from these two piles such that

$0 \le x' \le x,\ 0 \le y' \le y,\ 0 < x' + y'$, and $[(A)\ \ |x' - y'| < a$ or $(B)\ \ \min(x', y') < b]$.

Furthermore, each player after a move is allowed to block off up to $p-1$ opponent's moves of type (A) and $2(q-1)$ moves of type (B), more precisely, at most $q-1$ moves in each case, when the minimum is realized by x' and y'.

The player who takes the last match is the winner.

Games $\text{NIM}_{1,1}^{1,1}$, $\text{NIM}_{a,1}^{1,1}$, $\text{NIM}_{a,1}^{p,1}$, and $\text{NIM}_{a,b}^{1,1}$. were considered by Wythoff, Fraenkel, Larsson, and the author in 1907, 1982, 2009, and 2010, respectively.

We obtain a simple arithmetic recursion solving $\text{NIM}_{a,b}^{p,q}$ when $p = 1$ or $q = 1$ and get partial results for the general case. The recursion is of a "standard type"

$$x_n = mex_b^q\{x_i, y_i \mid 0 \le i < n\},\ \ y_n = x_n + a\lfloor n/p \rfloor;\ \ n \ge 0,$$

where mex_b^q is a two-parametric generalization of the minimum excludant mex.

Keywords: combinatorial games, NIM, Wythoff's NIM, minimum excludant

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1 Kernels of combinatorial games

Given a digraph $G = (V, E)$, a subset of its vertices $K \subseteq V$ is called a *kernel* if it is *independent* ($(v, v') \in E$ for no $v, v' \in K$) and *absorbing* ($\forall v \notin K \exists v \in K \mid (v, v') \in E$).

Furthermore, let digraph G be *acyclic* (contain no directed cycles) and *locally finite* (only a finite set of vertices can be reached from each fixed $v_0 \in V$). The vertices and arcs of G are called *positions* and *moves*, respectively. Two player take turns moving a token along the arcs of E. The game starts in an initial position $v_0 \in V$ and terminates in a *dead-end* (that is, in a vertex $v_t \in V$ of out-degree 0). The player who cannot move loses.

It was shown in [13] that every locally acyclic digraph has a unique kernel, which can be found by the following recursive algorithm: initialize $i = 0, G_0 = G$, and $K_0 = \emptyset$; then for $i = 0, 1, \ldots,$ for the current subgraph $G_i = (V_i, E_i)$ denote by V_i' the set of all dead-ends and by V_i'' the set of all positions from which a dead-end can be reached by one move; add V_i' to K_i, delete $V_i' \cup V_i''$ from V_i and repeat. Since G is acyclic and locally finite, the (unique) kernel $K = K^m = \cup_{i=1}^{\infty} V_i'$ will be obtained in at most $m = \lceil |V|/2 \rceil$ steps.

A vertex $v \in V$ is called a *P-position* if the player who enters v (the Previous player) wins; otherwise, v is called an *N-position*, since in this case the player who leaves it (the Next player) wins. It is clear that K is exactly the set of all P-positions and, by the definition of K, each move from a P-position leads to an N-position and for every N-position there is a move to a P-position. To solve a game, it is sufficient to find all its N- or P-positions.

2 Minimum excludant and its generalizations

The *minimum excludant* function $mex(S)$ is defined for any proper subset $S \subset \mathbb{Z}_+$ of the non-negative integers as the minimum $z \in \mathbb{Z}_+$ such that $z \notin S$; in particular, $mex(\emptyset) = 0$.

The following generalization was recently introduced in [8]. Given an integer $b \geq 1$ and a finite subset $S \subseteq \mathbb{Z}_+$ of m non-negative integers, let us order S and extend it by $s_{m+1} = \infty$ to get a sequence $0 \leq s_1 < \cdots < s_m < s_{m+1} = \infty$. Let us choose the (unique) minimum $i \in \{1, \ldots, m\}$ such that $s_{i+1} - s_i > b$. By definition, $mex_b(S) = s_i + b$.

We will need another generalization. Given an integer $q \geq 1$ and a finite multi-set $\mathcal{S} : \mathbb{Z}_+ \to \mathbb{Z}_+$, then $mex^q(\mathcal{S})$ is defined as the minimum $z \in \mathbb{Z}_+$ such that $\mathcal{S}(z) < q$.

It is easily seen that both above functions are well-defined and $mex_1 = mex^1 = mex$.

Finally, we will also need the following combination of the above two concepts.

Given integer $b \geq 1, q \geq 1$, and a finite multi-set $\mathcal{S} : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $|S^q| = m$ for $S^q = \{z \in \mathbb{Z}_+ \mid \mathcal{S}(z) \geq q\}$. By definition $mex_b^q(\mathcal{S}) = mex_b(S^q)$.

Obviously, all functions are well-defined and $mex_b^1 = mex_b,\ mex_1^q = mex^q,\ mex_1^1 = mex$.

Furthermore, we assume that $mex_b^q(\emptyset) = 0$ for all b and q, by convention.

3 Wythofff's NIM or "Corner the Queen"

In 1907 Wythoff [16] considered a game $G = (V, E)$, whose positions (x, y) are the pairs of non-negative integers and possible moves (x', y') in (x, y) are defined by the rules:

$0 \le x' \le x,\ 0 \le y' \le y,\ 0 < x' + y'$, and $[x' = 0$ or $y' = 0$ or $x' = y']$.

In other words, there are two piles of matches and, by one move, a player is allowed to take either (B) any positive number of matchings from one pile and nothing from the other ($x' = 0$ or $y' = 0$), or (A) the same positive number of matchings from both $x' = y'$. It is not allowed to pass ($x' + y' > 0$). The player who takes the last match wins.

The following, obviously equivalent, reformulation might be convenient. In a position (x, y) of a (potentially infinite) chess board, there is a Queen. By one move, a player is allowed to move it towards the corner $(0, 0)$, which is a unique terminal position.

The player who corners the Queen wins. It is easily seen that the moves of types (A) and (B) correspond, respectively, to the bishop- and rook-types moves of the Queen.

Due to an obvious symmetry, (x, y) is a P-position if and only if (y, x) is. We assume that $x \le y$ unless it is explicitly said otherwise.

Wythoff proved that the set of P-positions can be obtained by the recursion:

$$x_n = mex\{x_i, y_i \mid 0 \le i < n\}, \quad y_n = x_n + n; \quad n \ge 0. \tag{1}$$

For example, the first seven P-positions are $(0, 0), (1, 2), (3, 5), (4, 7), (6, 10), (8, 13), (9, 15)$.

Also, Wythoff proved that (x, y) is a P-position if and only if $x = x_n = \lfloor n\phi \rfloor$ and $y = y_n = x_n + n$, where n is a non-negative integer and $\phi = (1 + \sqrt{5})/2$ is the golden section.

4 Fraenkel's NIM

In [4, 5] Fraenkel generalized Wythoff's game, replacing $x' = y'$ by a weaker restriction $|x' - y'| < a$. Obviously, Wythoff's NIM corresponds to the case $a = 1$.

Remark 1 *In case $a = 0$ we obtain the standard game of NIM (with two-piles, which is trivial). However, we assume that all four our parameters a, b and p, q are strictly positive.*

Wythoff's recursion is extended to Fraenkel's $NIM(a)$ in the following simple way:

$$x_n = mex\{x_i, y_i \mid 0 \le i < n\}, \quad y_n = x_n + an; \quad n \ge 0. \tag{2}$$

The first seven P-positions for $a = 2$ are $(0, 0), (1, 3), (2, 6), (4, 10), (5, 13), (7, 17), (8, 20)$.

Moreover, Fraenkel solved the recursion and got the following explicit formula for (x_n, y_n).

Let $\alpha = \alpha(a) = \frac{1}{2}(2 - a + \sqrt{a^2 + 4})$ be the (unique) positive root of the quadratic equation $\frac{1}{z} + \frac{1}{z+a} = 1$. (In particular, $\alpha(1) = \frac{1}{2}(1 + \sqrt{5})$ is the *golden section* and $\alpha(2) = \sqrt{2}$.) Then

$$x_n = \lfloor \alpha n \rfloor, \quad y_n = x_n + an \equiv \lfloor n(\alpha + a) \rfloor; \quad n \ge 0. \tag{3}$$

As mentioned in [4], the explicit formula (3) solves the game in linear time, in contrast to recursion (2), providing only an exponential algorithm.

5 Larsson's NIM

Further generalizations were suggested by Larsson in his PhD thesis [12]; see also [10, 11].

He introduced one more strictly positive integer p, in addition to a, and defined p-blocking Fraenkel's NIM_a^p as follows: "the rules are the same as in NIM_a, except that before the next player moves, the previous player is allowed to block off (at most) $p-1$ bishop-type - note, not a-bishop-type - options and declare that the next player must refrain from these options. When the next player has moved, any blocked options are forgotten... More precisely, if the current configuration is (x, y) then, before the next move is made, the previous player is allowed to choose up to $p-1$ distinct, positive integers $c_1, \ldots, c_{p-1} \leq \min\{x, y\}$ and declare that the next player may not move to any configuration $(x - c_i, y - c_i)$."

Obviously, $\text{NIM}_a^1 = \text{NIM}_a$. P-positions (x_n, y_n) of NIM_a^p are given by the next recursion:

$$x_n = mex\{x_i, y_i \mid 0 \leq i < n\}, \;\; y_n = x_n + a\lfloor n/p \rfloor; \;\; n \geq 0, \tag{4}$$

which is a natural generalization of (2) for NIM_a. One can try to generalize the explicit formula (3) in a similar way. Let α be the (unique) positive root of the quadratic equation $pz^2 + (a - 2p)x - a = 0$, or equivalently, $\frac{1}{x} + \frac{1}{x+a/p} = 1$. Furthermore, let $\beta = \alpha + a/p$ and

$$x_n = \lfloor n\alpha \rfloor = \left\lfloor n \, \frac{2p - a + \sqrt{a^2 + 4p^2}}{2p} \right\rfloor, \;\; y_n = \lfloor n\beta \rfloor = \left\lfloor n \, \frac{2p + a + \sqrt{a^2 + 4p^2}}{2p} \right\rfloor; \;\; n \geq 0.$$

However, according to [10], although the obtained sequence "is in a certain sense 'very close'", yet, not equal to the set of P-positions of game $\text{NIM}_{a,p}$ when $p > 1$.

6 Game $\text{NIM}_{a,b}$

Another generalization of Frankel's NIM_a was recently suggested in [8]. Given positive integer a and b, two players take turns and, by one move, it is allowed to take x' and y' matches from these piles such that

$$0 \leq x' \leq x, \; 0 \leq y' \leq y, \; 0 < x' + y', \;\; \text{and} \;\; [(A) \;\; |x' - y'| < a \;\; \text{or} \;\; (B) \;\; min(x', y') < b].$$

In other words, a player can take (A) "almost equal" numbers of matches from both piles, or (B) any number of matches from one pile and at most $b-1$ from the other.

The following recursive formula for the P-positions is obtained in [8].

$$x_n = mex_b\{x_i, y_i \mid 0 \leq i < n\}, \;\; y_n = x_n + an; \;\; n \geq 0. \tag{5}$$

Although in this case, it is hardly possible to get an explicit formula for x_n, yet, recently Oudalov in [14] proved that for all positive integer a and b limits $\ell(a, b) = \frac{x_n(a,b)}{n}$ exist and are algebraic numbers; namely, $\ell(a,b) = \frac{a}{r-1}$, where $r > 1$ is the Perron root of the polynomial

$$P_{a,b}(z) = z^{b+1} - z - 1 - \sum_{i=1}^{a-1} z^{\lceil ib/a \rceil},$$

provided a and b are *coprime*, while $\ell(ka, kb) = k\ell(a,b)$, according to [8].

Furthermore, $P(z) = P_{a,b}(z)$ is a characteristic polynomial of a non-negative integer matrix $M(a,b)$, which is primitive and r is its Perron-Frobenius eigenvalue; in particular, r is real, positive (in fact, $r > 1$), and $r > |z^*|$ for any other root z^* of $P(z)$.

7 Game $NIM^{p,q}_{a,b}$

Now let us consider the general case. Two player take turns and by one move in a position (x, y) the Next player can reduce x by x' and y by x' such that

$0 \leq x' \leq x,\ 0 \leq y' \leq y,\ 0 < x' + y'$, and $[(A)\ \ |x' - y'| < a$ or $(B)\ \ \min(x', y') < b]$.

However, some of these moves might be forbidden by the Previous player. Namely, after entering (x, y), (s)he is allowed to choose up to $p + 2q - 3$ positive integers

$c_1, \dots, c_{p-1} \leq \min\{x, y\};\ \ d'_1, \dots, d'_{q-1} \leq x$ and $d''_1, \dots, d''_{q-1} \leq y$

and declare that the next player may not move to any position: $(x - c_i, y - c_i), (x - d'_i, y)$, or $(x, y - d''_i)$. After a move from (x, y) is made all these restrictions are forgotten and the Next player is similarly allowed to block up to $p+2q-3$ new options. In case $\min\{p, q\} = 1$, game $NIM^{p,q}_{a,b}$ is solved by the following statement which we be proven in Section 9.

Theorem 1 *When $p = 1$ or $q = 1$, all P-positions (x_n, y_n) of $NIM^{p,q}_{a,b}$ are given by recursion*

$$x_n = mex^q_b\{x_i, y_i \mid 0 \leq i < n\},\ \ y_n = x_n + a\lfloor n/p \rfloor;\ \ n \geq 0. \tag{6}$$

Remark 2 *By convention, computing $mex^q_b\{x_i, y_i \mid 0 \leq i < n\}$ for each diagonal position (x_i, y_i) with $x_i = y_i$ we count x_i and y_i as only one element, not as two; see $NIM^{1,2}_{1,1}$, $NIM^{1,2}_{1,2}$, $NIM^{4,3}_{1,1}$, $NIM^{3,5}_{1,1}$, $NIM^{1,2}_{2,2}$, $NIM^{2,2}_{2,1}$, $NIM^{2,2}_{1,2}$, $NIM^{2,3}_{2,2}$, and $NIM^{2,3}_{2,2}$ in Tables 1-5.*

Let us also recall that (x, y) is a P-position if and only if (y, x) is.

Conjecture 1 *Limits $\ell(a, b, p, q) = \lim_{n\to\infty} \frac{x_n(a,b,p,q)}{n}$ exist and are algebraic numbers for all positive integer a, b and p, q such that $\min\{p, q\} = 1$.*

Remark 3 *In view of a recent breakthrough by Hadad [9], Fraenkel and Peled [6], such a linear asymptotic may lead to a polynomial algorithm for $NIM^{p,q}_{a,b}$. Yet first, one should extend their results by replacing the standard minimum excludant mex by mex^q_b.*

8 Bouton - von Neumann's algorithm finding kernels of acyclic digraphs and its applications to $\text{NIM}^{p,q}_{a,b}$

An algorithm finding all P-positions was suggested in 1901 by Bouton in [2] for the normal and misere versions of NIM (with k piles). Then, in [13], it was extended to an arbitrary combinatorial game modeled by an acyclic digraph $G = (V, E)$.

It works recursively. In step 1, let us find all terminal (that is, of out-degree 0) positions and denote the obtained set by P_1. Furthermore, let N_1 be the set of all positions from which P_1 can be reached by one move. Let us delete $P_1 \cup N_1$ and repeat, obtain P_2 and N_2, etc. Obviously, $(P_1 \cup P_2 \cup \ldots)$ is the set of all P-positions.

In Figure 2 this algorithm is illustrated for $\text{NIM}^{1,1}_{1,2}$. The only terminal position is $(x_0, y_0) = (0, 0)$. Set N_1 consists of two columns $\{(x, y) \mid x \le 1\}$, two rows $\{(x, y) \mid y \le 1\}$, and the main diagonal $\{(x, y) \mid x = y\}$, excluding position $(0, 0)$ itself. Indeed, $(0, 0)$ can be reached by one move from each of the above positions, since $a = 1, b = 2$ and no move can be blocked, since $p = q = 1$. After elimination of $P_1 \cup N_1$, we obtain $P_2 = \{(2, 3), (3, 2)\}$. Then, set N_2 is constructed in a similar way. Position $(2, 3)$ can be reached from two columns $\{(x, y) \mid 2 \le x \le 3, 3 \le y\}$, two rows $\{(x, y) \mid 3 \le y \le 4, 2 \le x\}$, and one diagonal $\{(x, y) \mid y = x + 1 > 3\}$, excluding position $(2, 3)$ itself. Obviously, the symmetric constraints hold for $(3, 2)$. The union of the obtained two sets is N_2. Then, after its eliminating, we get $P_3 = \{(5, 7), (7, 5)\}$, etc.

Let us consider one more example: $\text{NIM}^{2,1}_{1,1}$ in Figure 3. Again $(x_0, y_0) = (0, 0)$ is the unique terminal position. Set N_1 consists of one column $\{(x, y) \mid x = 0\}$ and one rows $\{(x, y) \mid y = 0\}$, excluding position $(0, 0)$ itself. Indeed, $(0, 0)$ can be reached by one move from each of the above positions, since $a = b = 1$ and none of these moves can be blocked, since $q = 1$. Although $(0, 0)$ can be also reached from the diagonal $\{(x, y) \mid x = y > 1\}$, yet, since $p = 2$, such a move can be blocked. Thus, $(1, 1) \in P_2$; in fact, P_2 contains no other positions. Hence, N_2 consists of one column $\{(x, y) \mid x = 1, y > 1\}$, one row $\{(x, y) \mid y = 1, x > 1\}$, and one diagonal $\{(x, y) \mid x = y > 2\}$. Indeed, since $p = 1$, one of two positions $\{(0, 0), (1, 1)\}$ can be blocked, but not both. Hence, $P_3 = \{(2, 3), (3, 2)\}$, etc.

For both above examples, first ten positions of P (with $x \le y$) are given in Table 1.

Let us notice that, in general, all positions of a row $\{(x, y) \mid x = x_0, y > y_0\}$, a column $\{(x, y) \mid x > x_0, y = y_0\}$, or a diagonal $\{(x, y) \mid x - y = x_0 - y_0, x > x_0, y > y_0\}$ go to N_{j_0} as soon as (x_0, y_0) appears in P_{j_0} as the q-th position of $\cup_{j=1}^{j_0} P_j$ in the row or column, or the p-th position of the diagonal, respectively. Moreover, not only this one but next b rows or columns and a diagonals go to N_j in the considered case.

9 Proof of Theorem 1 (sketch)

We will derive Theorem 1 from the Bouton - von Neumann algorithm.

First, let us notice $x_{i+1} > x_i$ if $q = 1$ and $y_{i+1} > y_i$ if $p = 1$ for all j. Hence, $(x_i, y_i) \neq (x_{i+1}, y_{i+1})$ for all $i \geq 1$ whenever $p = 1$ or $q = 1$. Moreover, in this case formula (6) never lists the same position twice, since both x_j and y_j are non-decreasing functions of j.

For simplicity, let us start with the case $p = q = 1$, which was already considered in [8].

By symmetry, $P_i = \{(x_i, y_i), (y_i, x_i)\}$ for every $i \geq 1$. Each of these two positions can be reached by b rows, b columns, and $2a - 1$ diagonals. For $i = 0$ only a of these diagonals satisfy the restriction $x \leq y$. Similarly, for any fixed $i > 0$ only a of them are new, while the remaining $a - 1$ were already eliminated before, with N_j for some $j < i$. Thus, exactly a diagonals are excluded in each step and, hence, $y_n = x_n + an$ for every $n \geq 0$.

Furthermore, each position (x_i, y_i) eliminates the next b columns $x_i, \dots, x_i + b - 1$. Yet, the symmetric position (y_i, x_i) also eliminates b columns: $y_i, \dots, y_i + b - 1$. (Of course, these two sets may overlap; they may also intersect the sets obtained in a previous step $j < i$.) Anyway, the desired recursion $x_n = mex_b\{x_i, y_i \mid 0 \leq i < n\}$ immediately follows.

Case $\min\{p, q\} = 1$ is just a little more sophisticated than $p = q = 1$.

First, let us assume that $q = 1, p \geq 1$; see Figures 3, 6, and 7. In this case, obviously, each diagonal $\{(x, y) \mid y = x + aj, j \geq 0\}$ contains exactly p P-positions, while any other diagonal does not contain them at all. In other words, the second part of the recursion $y_n = x_n + a\lfloor n/p \rfloor$ holds. It is easy to verify that the first part, $x_n = mex_b\{x_i, y_i \mid 0 \leq i < n\}$ holds too, by construction of the algorithm.

Finally, let $p = 1$, while $q \geq 1$; see Figures 4, 5, and 8. In this case, every b-th row or column contains exactly q P-position, while each diagonal contains exactly one of them. Respectively, in the recursion we replace mex_b by mex_b^q, while $y_n = x_n + an$ remains. □

10 Several remarks on the case: $\min\{p, q\} > 1$

This case is considered in Figures 9 - 14 and Tables 3 - 5.

Formally, recursion (6) does not work already for $q > 1$. Yet, in this case the problem can be easily fixed; see remark 2 and the corresponding examples in Tables 1-5.

In case when both $p > 1$ and $q > 1$ we will need several more serious conventions, which sometimes modify significantly the rules of the game, especially, when $\min\{p, q\} > 1$.

First, let us recall that $x_{n+1} > x_n$ whenever $q = 1$ and $y_{n+1} > y_n$ whenever $p = 1$.

In contrast, $y_n - x_n = a\lfloor n/p \rfloor$ may be equal to $y_{n+1} - x_{n+1} = a\lfloor (n+1)/p \rfloor$ when $p > 1$, while equality $x_{n+1} = x_n$ may hold too, when $q > 1$. Hence, applying (6) formally in case when both p and q are larger than 1, we would frequently obtain that $(x_n, y_n) = (x_{n+1}, y_{n+1})$. In this case, by convention, we skip the value $x_{n+1} = x_n$ and, proceed instead with the next value of function mex_b^q. This convention may be applicable several times in a row, until $y_{n+k} = x_{n+k} + a\lfloor (n + k)/p \rfloor$ becomes strictly larger than y_n. Then we return and set $x_{n+k} = x_n$; see, for example, Tables 3 - 5.

If $p > 1$ and $q > 1$ but $a = b = 1$, we can still "save" our main recursion (6). To do so, we keep unchanged its y-part $y_n = x_n + a\lfloor n/p \rfloor$, while in x-part, $x_n = mex_b^q\{x_i, y_i \mid 0 \le i < n\}$, we keep mex_b^q but slightly modify the natural order $n = 0, 1, \dots$ in which the numbers appear. Namely, we introduce a priority set $S = S(p, q) \subset \mathbb{Z}_+$, define (x_i, y_i) for $i \in S$ first, and then set $x_n = mex_b^q\{x_i, y_i \mid i \in [0; n) \cup S\}$; see, for example, bold lines in Table 4.

First, let us assume that $p \ge q$. In this case, we assign numbers $0, 1, \dots, q-1$ to positions $(0, 0), (1, 1), \dots, (q-1, q-1)$, numbers $p, \dots, p+q-1$ to $(0, 1), \dots, (q-2, q-1)$, etc., and in general, $jp, \dots, jp+q-1$ to $(0, j), \dots, (q-1-j, q-1)$, where $j = 0, 1, \dots, q-1$.

For example, for $\text{NIM}_{1,1}^{4,3}$ in Table 4, positions $(0, 0), (1, 1), (2, 2), (0, 1), (1, 2)$, and $(0, 2)$ get numbers $0, 1, 2, 4, 5$, and 8, respectively. Then, by the modified mex_b^q-formula, we obtain $(x_3, y_3) = (3, 3)$, $(x_6, y_6) = (3, 4)$, $(x_7, y_7) = (4, 5)$, $(x_9, y_9) = (3, 5)$, etc.

Second, let us assume that $p < q$. In this case, we assign numbers $0, 1, \dots, p-1$ to positions $(0, 0), \dots, (p-1, p-1)$, numbers $p, p+1, \dots, 2p-1$ to positions $(0, 1), \dots, (p-1, p)$, numbers, etc., $((q-p)p, \dots, (q-p+1)p$ to positions $(0, q-p), \dots, (p-1, q-1)$; then, numbers $(q-p)p+1, \dots, (q-p+1)p-1$ to positions $(0, q), \dots, (p-2, p+q-2)$, etc., finally, number $p(2q-p+1)/2$ to position $(0, q-1)$. For example, for $\text{NIM}_{1,1}^{3,5}$ in Table 4, positions

$(0, 0), (1, 1), (2, 2), (0, 1), (1, 2), (2, 3), (0, 2), (1, 3), (2, 4), (0, 3), (1, 4)$, and $(0, 4)$ get numbers $0 - 10$ and 12, respectively. Then, by the modified mex_b^q-formula, we obtain $(x_{11}, y_{11}) = (3, 6)$, $(x_{13}, y_{13}) = (3, 7)$, etc.

Let us remark that all above conventions modified function mex_b^q but not the rules of the game $\text{NIM}_{a,b}^{p,q}$. In contrast, when $\min\{p, q, a\} > 1$ or $\min\{p, q, b\} > 1$ we have to modify even the rules, to save the recursion, with the function mex_b^q modified as above; see, for example, the last three Figures 12 - 14, where the crossed positions are treated non-standardly.

11 Generalized minimum excludant mex^q and Sprague-Grundy theorem

Let us slightly modify the above algorithm as follows. Let us assign $f(v) = 0$ to every position $v \in P_1$, then $f(v) = 1$ to each $v \in N_1$, and in general, given a position v such that all its (immediate) successors are already evaluated, let us set $f(v) = mex\{f(w) \mid (v, w) \in E\}$. The obtained mapping $f : V \to \mathbb{Z}_+$ is called the *Sprague-Grundy function* of G.

This definition and the definition of *mex* immediately imply that

- (i) $f(v) = 0$ if and only if v is a P-position;
- (ii) for every $k < f(v)$ a position w with $f(w) = k$ can be reached from v;
- (iii) no position w with $f(w) = f(v)$ can be reached from v.

Sprague [15] and Grundy [7] introduced the concept of sum of combinatorial games $G = G_1 \oplus \dots \oplus G_n$ as follows. A position of G is a set $v = (v_1, \dots, v_n)$, where $v_i \in V_i$ is a position of the summand-game $G_i = (V_i, E_i)$ for $i \in [n] = \{1, \dots, n\}$; in other words,

$V = V_1 \times \ldots \times V_n$. Respectively, by one move from $v \in V$ a player is allowed to choose any (but only one) $i \in [n]$, any successor v'_i of v_i and replace v_i by v'_i. The Sprague-Grundy Theorem claims that $f(G) = \bigoplus_{i \in [n]} f(G_i)$, where $\bigoplus$ is the bitwise binary sum, also called the NIM-sum. Thus, as we already mentioned, the P-positions of G are the zeros of $f(G)$.

Making use of mex^q, we obtain a simple generalization of the above theorem as follows. Let before the next player moves, the previous player is allowed to block off (at most) $q_i - 1$ moves in G_i for all $i \in [n]$ and declare that the next player must refrain from these moves. After the next player has moved, all these blocks are forgotten. Obviously, this modification can be solved similarly: just for all $i \in [n]$, replace mex by mex^{q_i}, functions $f(G_i)$ by $f^{q_i}(G_i)$ and, finally, $f(G)$ by $f^{\bar{q}}(G) = \bigoplus_{i \in [n]} f_{q_i}(G_i)$, where $\bar{q} = (q_1, \ldots, q_n)$.

It would be interesting to find a similar application of mex_b^q.

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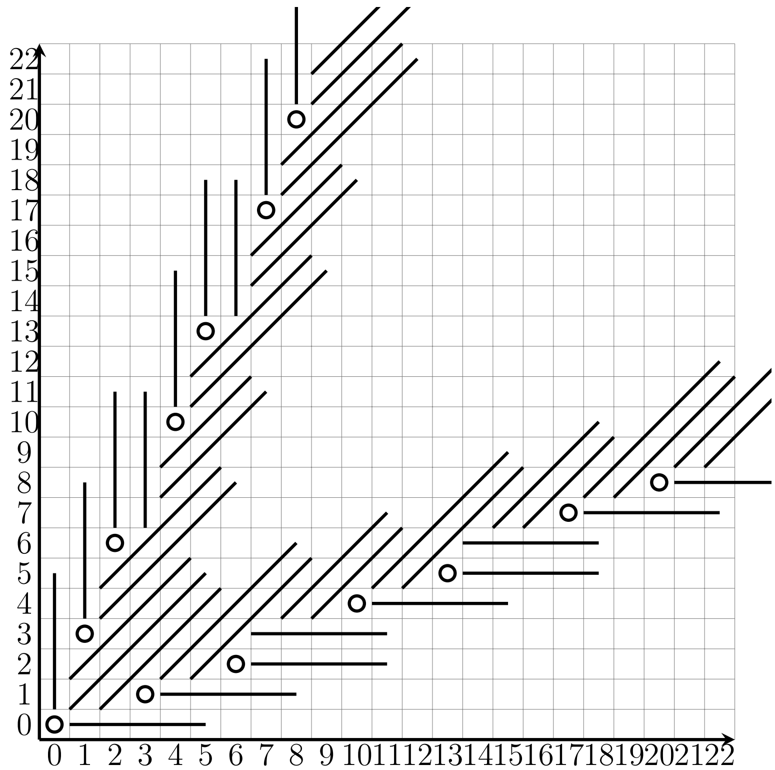

Figure 1: $\mathrm{NIM}^{1,1}_{2,1}$; $a = 2$, $b = p = q = 1$

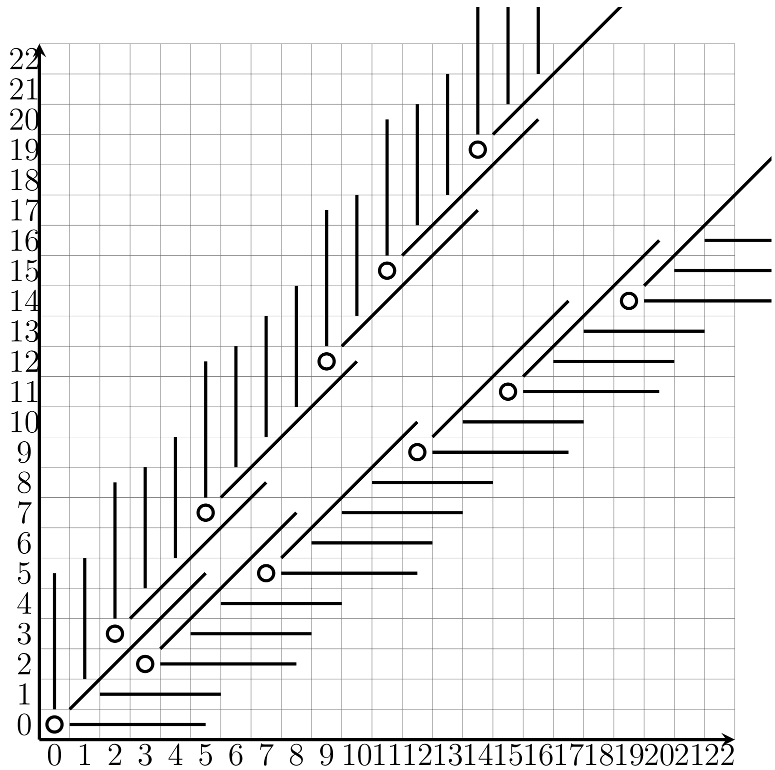

Figure 2: $\text{NIM}^{1,1}_{1,2}$; $a = p = q = 1$, $b = 2$

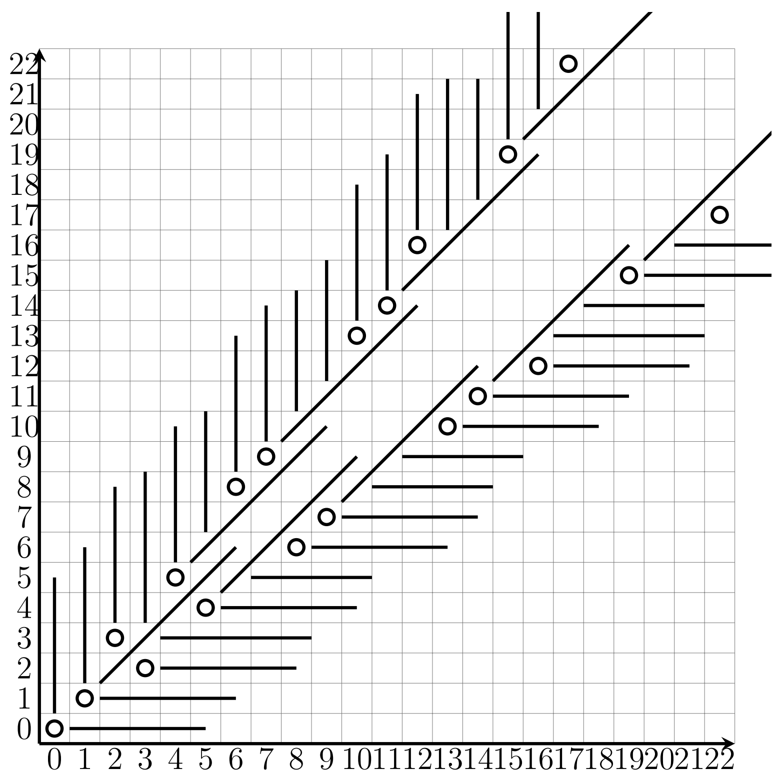

Figure 3: $\text{NIM}^{2,1}_{1,1}$; $a = b = q = 1$, $p = 2$

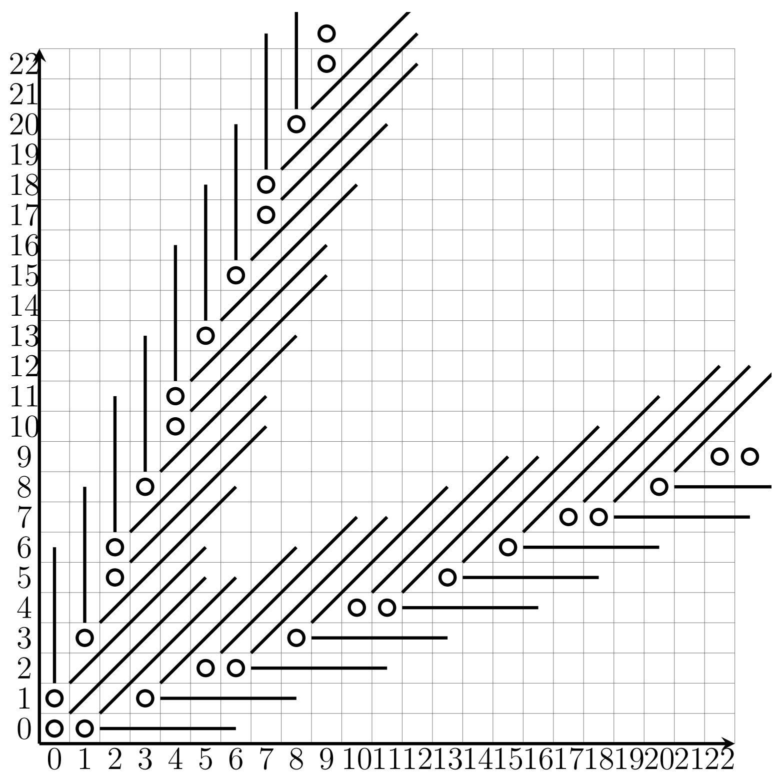

Figure 4: $\mathrm{NIM}_{1,1}^{1,2}$; $a = b = p = 1$, $q = 2$

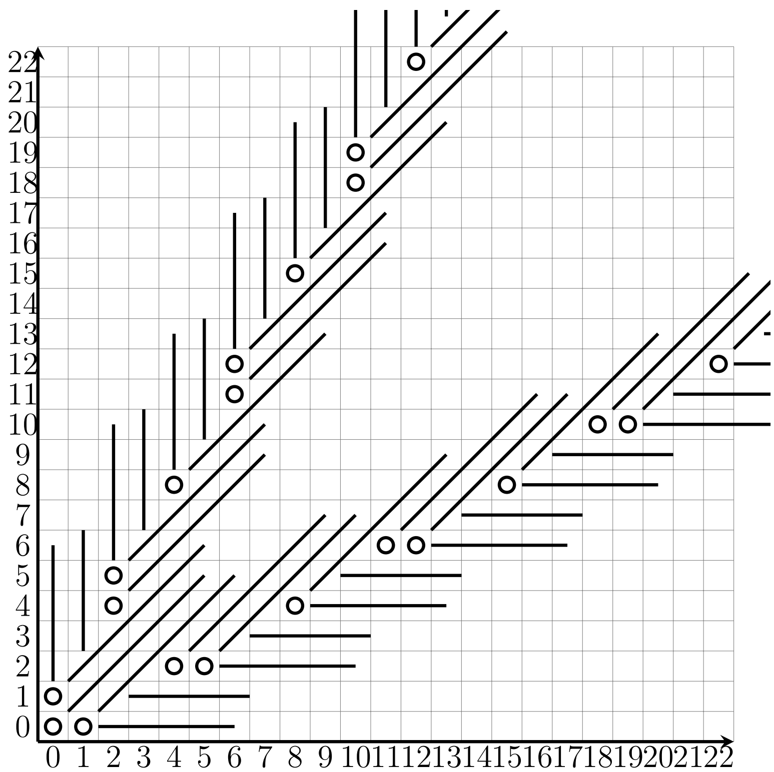

Figure 5: $\mathrm{NIM}^{1,2}_{1,2}$; $a = p = 1$, $b = q = 2$

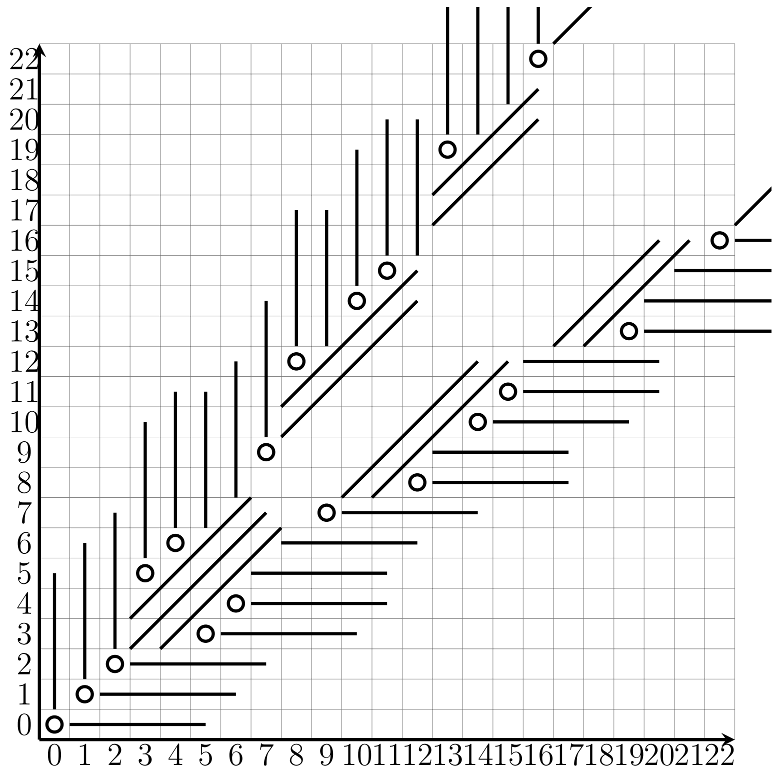

Figure 6: $\text{NIM}_{2,1}^{3,1}$; $a = 2$, $b = q = 1$, $p = 3$

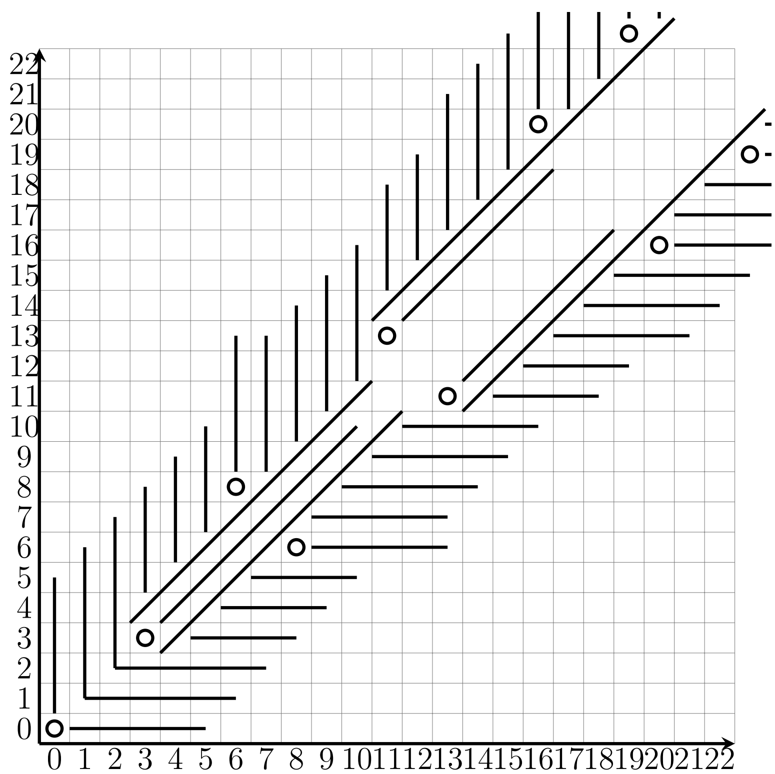

Figure 7: $\mathrm{NIM}^{2,1}_{2,3}$; $a = p = 2$, $b = 3$, $q = 1$

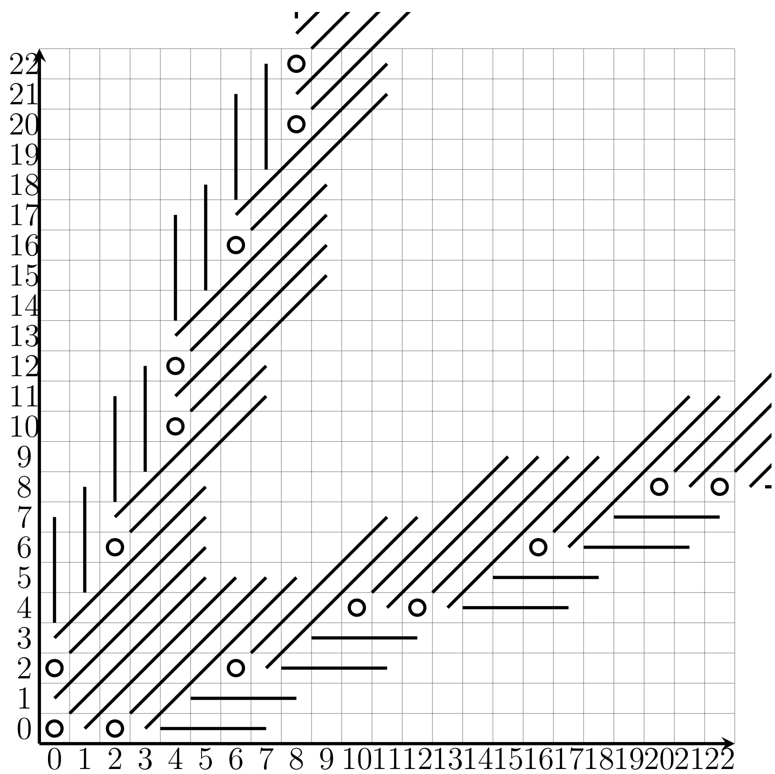

Figure 8: $\text{NIM}^{1,2}_{2,2}$; $a = b = q = 2$, $p = 1$

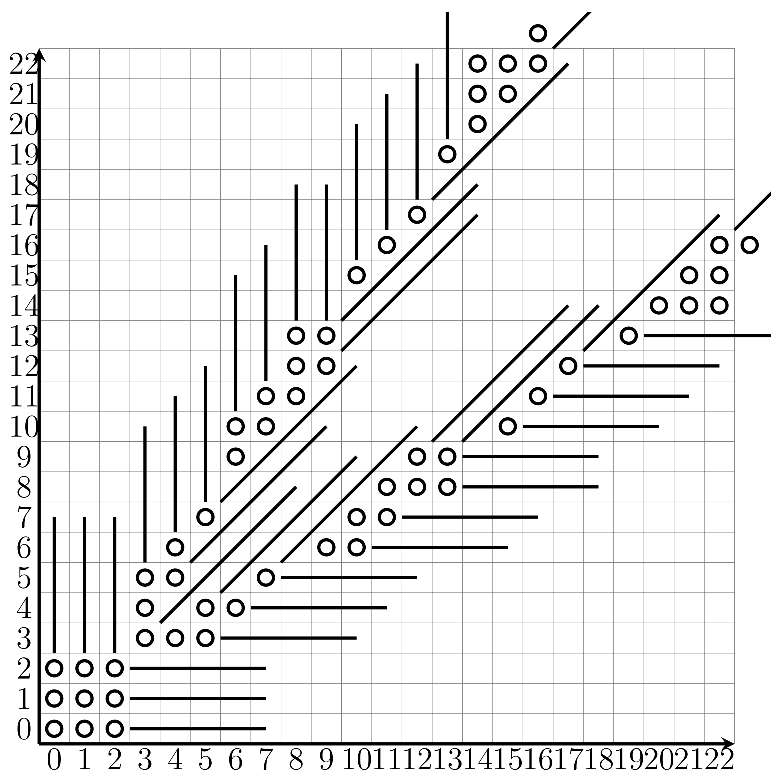

Figure 9: $\mathrm{NIM}^{4,3}_{1,1}$; $a = b = 1$, $p = 4$, $q = 3$

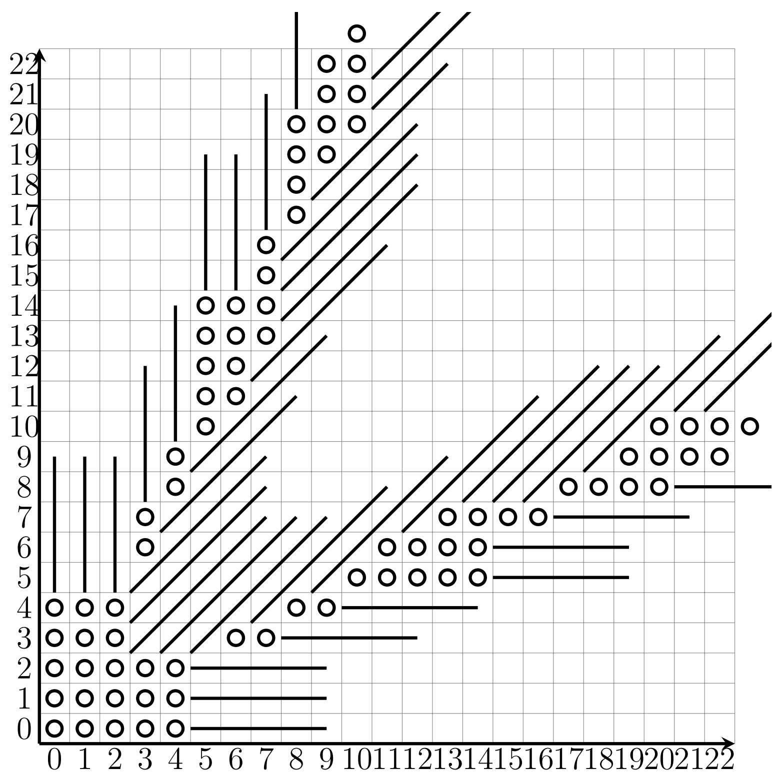

Figure 10: $\mathrm{NIM}_{1,1}^{3,5}$; $a = b = 1$, $p = 3$, $q = 5$

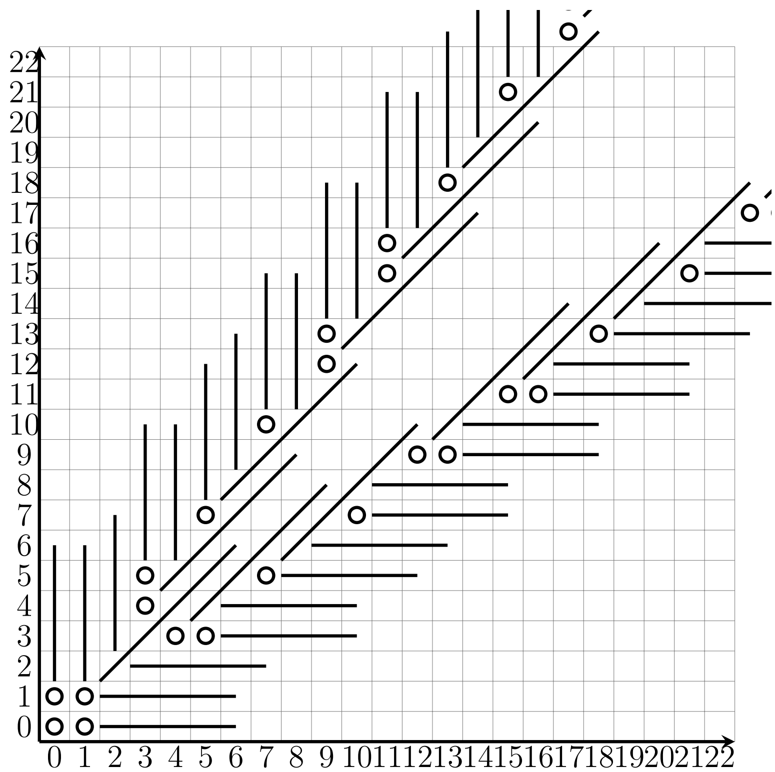

Figure 11: $\mathrm{NIM}^{2,2}_{1,2}$; $a = 1$, $b = p = q = 2$

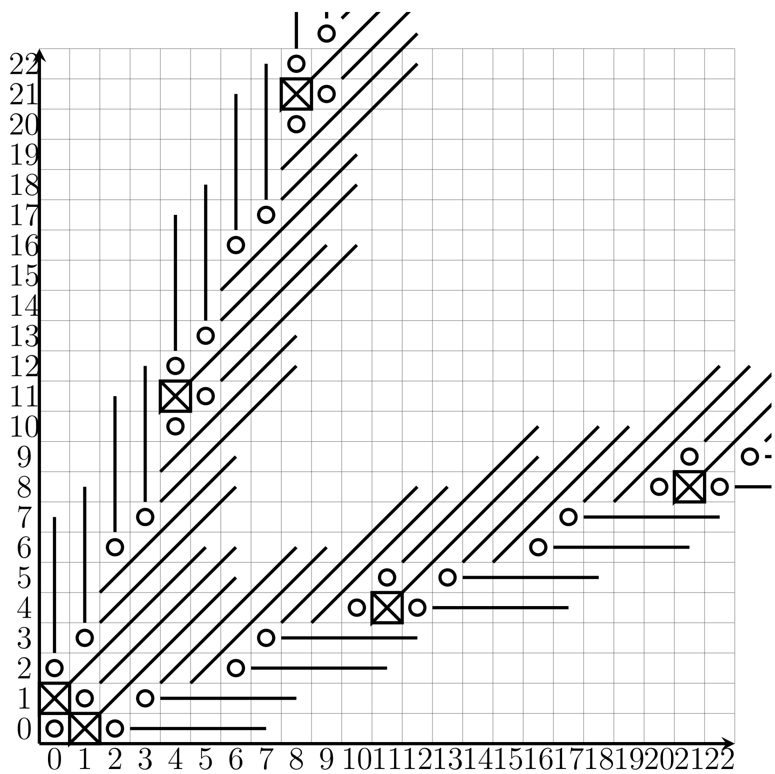

Figure 12: $\mathrm{NIM}^{2;2}_{2,1}$; $a = p = q = 2$, $b = 1$

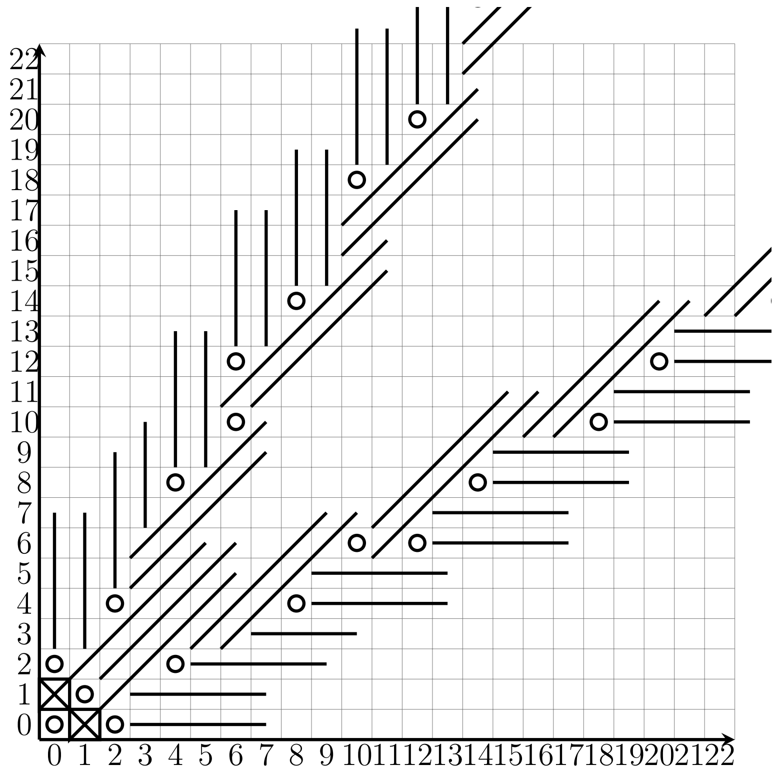

Figure 13: $\text{NIM}^{2,2}_{2,2}$; $a = b = p = q = 2$

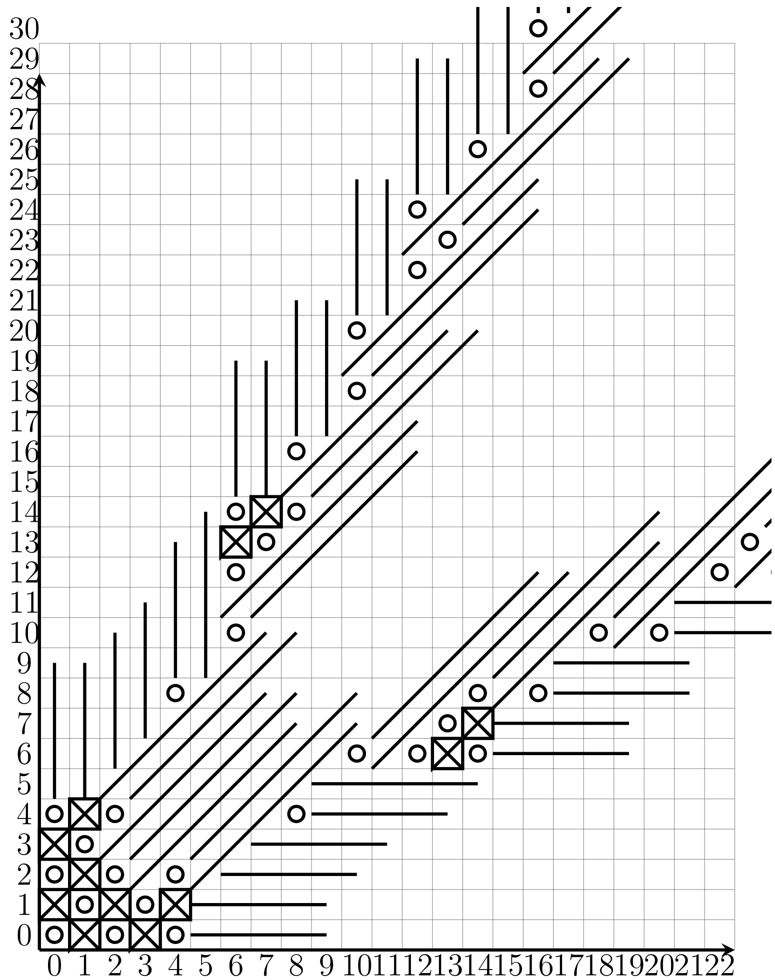

Figure 14: $\mathrm{NIM}^{3,3}_{2,2}$; $a = b = 2$, $p = q = 3$

$NIM^{1\,1}_{2\,1}$

n	x_n	y_n	$y_n - x_n$
0	0	0	0
1	1	3	2
2	2	6	4
3	4	10	6
4	5	13	8
5	7	17	10
6	8	20	12
7	9	23	14
8	11	27	16
9	12	30	18
10	14	34	20

$NIM^{1\,1}_{1\,2}$

n	x_n	y_n	$y_n - x_n$
0	0	0	0
1	2	3	1
2	5	7	2
3	9	12	3
4	11	15	4
5	14	19	5
6	17	23	6
7	21	28	7
8	25	33	8
9	27	36	9
10	30	40	10

$NIM^{2\,1}_{1\,1}$

n	x_n	y_n	$y_n - x_n$
0	0	0	0
1	1	1	0
2	2	3	1
3	4	5	1
4	6	8	2
5	7	9	2
6	10	13	3
7	11	14	3
8	12	16	4
9	15	19	4
10	17	22	5

$NIM^{1\,2}_{1\,1}$

n	x_n	y_n	$y_n - x_n$
0	**0**	**0**	0
1	**0**	**1**	1
2	1	3	2
3	2	5	3
4	2	6	4
5	3	8	5
6	4	10	6
7	4	11	7
8	5	13	8
9	6	15	9
10	7	17	10

Table 1: One of $\{a, b, p, q\}$ is 2, the remaining three equal 1

$\mathrm{NIM}^{3\,1}_{2\,1}$

n	x_n	y_n	$y_n - x_n$
0	0	0	0
1	1	1	0
2	2	2	0
3	3	5	2
4	4	6	2
5	7	9	2
6	8	12	4
7	10	14	4
8	11	15	4
9	13	19	6
10	16	22	6
11	17	23	6
12	18	26	8
13	20	28	8
14	21	29	8
15	24	34	10
16	25	35	10
17	27	37	10

$\mathrm{NIM}^{1\,2}_{1\,2}$

n	x_n	y_n	$y_n - x_n$
0	**0**	**0**	0
1	**0**	**1**	1
2	2	4	2
3	2	5	3
4	4	8	4
5	6	11	5
6	6	12	6
7	8	15	7
8	10	18	8
9	10	19	9
10	12	22	10
11	14	25	11
12	14	26	12
13	16	29	13
14	16	30	14
15	18	33	15
16	20	36	16
17	20	37	17

Table 2: $\min\{a,p\} > 1$ or $\min\{b,q\} > 1$

$\mathrm{NIM}^{2\,1}_{2\,3}$

n	x_n	y_n	$y_n - x_n$
0	0	0	0
1	3	3	0
2	6	8	2
3	11	13	2
4	16	20	4
5	19	23	4
6	26	32	6
7	29	35	6
8	38	46	8
9	41	49	8

$\mathrm{NIM}^{1\,2}_{2\,2}$

n	x_n	y_n	$y_n - x_n$
0	0	0	0
1	0	2	2
2	2	6	4
3	4	10	6
4	4	12	8
5	6	16	10
6	8	20	12
7	8	22	14
8	10	26	16
9	12	30	18

Table 3: $\min\{a,b,p\} > 1$ or $\min\{a,b,q\} > 1$

$\text{NIM}^{4\;3}_{1\;1}$

n	x_n	y_n	$y_n - x_n$
0	**0**	**0**	0
1	**1**	**1**	0
2	**2**	**2**	0
3	3	3	0
4	**0**	**1**	1
5	**1**	**2**	1
6	3	4	1
7	4	5	1
8	**0**	**2**	2
9	3	5	2
10	4	6	2
11	5	7	2
12	6	9	3
13	7	10	3
14	8	11	3
15	9	12	3
16	6	10	4
17	7	11	4
18	8	12	4
19	9	13	4
20	8	13	5
21	10	15	5
22	11	16	5
23	12	17	5
24	13	19	6
25	14	20	6
26	15	21	6
27	16	22	6

$\text{NIM}^{3\;5}_{1\;1}$

n	x_n	y_n	$y_n - x_n$
0	**0**	**0**	0
1	**1**	**1**	0
2	**2**	**2**	0
3	**0**	**1**	1
4	**1**	**2**	1
5	**2**	**3**	1
6	**0**	**2**	2
7	**1**	**3**	2
8	**2**	**4**	2
9	**0**	**3**	3
10	**1**	**4**	3
11	3	6	3
12	**0**	**4**	4
13	3	7	4
14	4	8	4
15	4	9	5
16	5	10	5
17	6	11	5
18	5	11	6
19	6	12	6
20	7	13	6
21	5	12	7
22	6	13	7
23	7	14	7
24	5	13	8
25	6	14	8
26	7	15	8
27	5	14	9
28	7	16	9

Table 4: $\min\{p, q\} >> 1$

$NIM^{2\,2}_{2\,1}$

n	x_n	y_n	$y_n - x_n$
0	0	0	0
1	1	1	0
2	**0**	**2**	2
3	1	3	2
4	2	6	4
5	3	7	4
6	4	10	6
7	5	11	6
8	**4**	**12**	8
9	5	13	8
10	6	16	10
11	7	17	10
12	8	20	12
13	9	21	12

$NIM^{2\,2}_{1\,2}$

n	x_n	y_n	$y_n - x_n$
0	**0**	**0**	0
1	**1**	**1**	0
2	**0**	**1**	1
3	3	4	1
4	3	5	2
5	5	7	2
6	7	10	3
7	9	12	3
8	9	13	4
9	11	15	4
10	11	16	5
11	13	18	5
12	15	21	6
13	17	23	6

$NIM^{2\,2}_{2\,2}$

n	x_n	y_n	$y_n - x_n$
0	0	0	0
1	1	1	0
2	0	2	2
3	2	4	2
4	4	8	4
5	6	10	4
6	6	12	6
7	8	14	6
8	10	18	8
9	12	20	8
10	14	24	10
11	16	26	10

$NIM^{3\,3}_{2\,2}$

n	x_n	y_n	$y_n - x_n$
0	**0**	**0**	0
2	**2**	**2**	0
3	**0**	**2**	2
4	1	3	2
5	2	4	2
6	0	4	4
7	4	8	4
8	6	10	4
9	6	12	6
10	7	13	6
11	8	14	6
12	6	14	8
13	8	16	8
14	10	18	8
15	10	20	10
16	12	22	10
17	13	23	10
18	12	24	12
19	14	26	12
20	16	28	12

Table 5: $\min\{a, p, q\} > 1$ or $\min\{b, p, q\} > 1$