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NETWORK SUPPLY SYSTEMS, STABLE FAMILIES OF COALITIONS FOR SUPERADDITIVE TU-GAMES AND BERGE'S NORMAL HYPERGRAPHS

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NETWORK SUPPLY SYSTEMS, STABLE FAMILIES OF COALITIONS FOR SUPERADDITIVE TU-GAMES AND BERGE'S NORMAL HYPERGRAPHS

Vladimir Gurvich Sergei Schreider

Abstract. The very common question appearing in resource management is: what is the optimal way of behaviour of the agents and distribution of limited resources. Is any form of cooperation more preferable strategy then pure competition? How cooperation can be treated in the game theoretic framework: just as one of a set of Pareto optimal solutions or cooperative game theory is a more promising approach? This research is based on results proving the existence of a non-empty K-core, that is, the set of allocations acceptable for the family K of all feasible coalitions, for the case when this family is a set of subtrees of a tree.

A wide range of real situations in resource management, which include optimal water, gas and electricity allocation problems can be modeled using this class of games. Thus, the present research is pursuing two goals: 1. optimality and 2. stability.

Firstly, we suggest to players to unify their resources and then we optimize the total payoff using some standard LP technique. The same unification and optimization can be done for any coalition of players, not only for the total one. However players may object unification of resources. It may happen when a feasible coalition can guarantee a better result for every coalitionist. Here we obtain some stability conditions which ensure that this cannot happen for some family K. Such families were characterized in Boros et al. (1997) as Berge's normal hypergraphs. Thus, we obtain a solution which is optimal and stable. From practical point of view, we suggest a distribution of profit that will not cause the conflict between players.

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'What to suggest?. . . They write and write [...] . It makes my head reel.
Just take everything and share equally. . .'
Michail Bulgakov, ***The heart of a dog***

1 Introduction

We consider the situation when group of agents having the access to the set of limited resources has a target to optimize some objective function. We formulate a question what the optimal cooperation strategy, or coalition structure, should be chosen by the agents in order to optimize the general economic output.

The linear case appears when for each player his strategies x_i are shares of his activities in industry *r*. If

$$A = \|a_{ir}\|$$

be a matrix representing the amount of resource $i \in I$ (I *is* a set of players) needed for producing a unit of product *j*, and

$$b^i = (b_1^i, b_2^i, \ldots, b_{|R|}^i)$$

be a vector of resources available for player *i*.

Then we can formulate the production strategy choice for each player as a LP problem:

Maximize the revenue function $f(x) = \sum_{p \in P} c_p x_p$

Subject to $Ax \le b$, where *b* is a vector of available resources.

This formulation can be settled for each player $i \in I$ or coalition $K \subseteq I$. Respectively, we replace *b* by b^i or b^K. For each coalition *K* vector b^K is additively defined:

$$b^K = \sum_{i \in K} b^i$$

The corresponding LP solution will be denoted x^*.

Let us define function *v* on the set of coalitions as:

$$v(K) = Cx^*(b^K)$$

This function is superadditive, that is, for each two disjoint coalitions K_1 and K_2

$$v(K_1 + K_2) \geq v(K_1) + v(K_2)$$

This result was proven by Johnson (1973), Gomory and Johnson (1973) and further developed by Bliar and Jeroslow (1977), Jeroslow (1977, 1978) and Schrijver (1980). This result is referred in the literature as supperadditive (or subadditive) duality. We can combine it with BGV theorem (see Boros, Gurvich, Vasin (1997) and the originating papers of Gurvich and Vasin (1977, 1978)) stating that family of coalitions $K. \subseteq 2^I$ is stable (that is, has a non empty core) if and only if K is a normal hypergraph, according to Berge (1970).

As an application, we plan to consider an example of particular game, in which the corresponding graph is a water allocation system represented by a network of gravitationally driven water carriers (rivers, canals, pipelines). We show that if a coalition is created by the neighbor players (farmers) whose properties are located along the same water supply carrier or along a tree formed by such carriers then the core of the obtained cooperative game is not empty. In other words, let us now define accurately the concepts considered above.

2 Stable families of coalitions, normal hypergraph and TU-games

For the beginning, let us consider cooparative games with transferable utility (TU-games).

The notation is formalized as follows: Let I be a set of players. Its subsets $K \subseteq I$ are called coalitions. A TU game is defined by a characteristic function $v\colon 2^I \to R$.

Function v (and the corresponding game) is called superadditive if

$$v(K' \cup K'') \geq v(K') + v(K'') \quad \forall K', K'' \subseteq I \text{ such that } K' \cap K'' = \varnothing$$

Vector (x_i: $i \in I$) is called an allocation if $\sum_{i \in I} x_i \leq v(I)$. Here x_i is interpreted as a payoff of player $i \in I$.

Furthermore,

$x_K = \sum_{i \in I} x_i$ is a payoff of coalition $K \subseteq I$.

The core of the obtained superadditive TU game $C(v)$ is defined by the system of linear inequalities

$$C(v) = \{x \in R^{|I|} \mid x_I \leq v(I) \text{ and } x_K \geq v(K)\ \forall K \subseteq I\}$$

It can be interpreted as a set of allocations (x_i: $i \in I$) acceptable for all coalitions $K \subseteq I$.

In other words, the core of a game is empty whenever each allocation is rejected by a coalition. The core is a natural and probably the simplest concept of solution in cooperative game theory. Yet it has an important disadvantage: *C(v)* is frequently empty, because it must be acceptable for all $2^{|I|}$ coalitions.

Yet in real life not every coalition has a chance to appear, because some agents may not know or not like each other. The following relaxation of the concept is common in the literature:

Given a family of coalitions $\mathcal{K} \subseteq 2^I$ the $\mathcal{K}$–core is defined as a family of allocations acceptable for all coalitions $K \subseteq \mathcal{K}$, that is

$$C(v, \mathcal{K}) = \{x \in R^{|I|} \mid x_I \leq v(I) \text{ and } x_K \geq v(K)\ \forall K \in \mathcal{K}\}$$

Family $\mathcal{K} \subseteq 2^I$ is called *stable* if the $\mathcal{K}$–core is not empty for any superadditive TU-game $v : 2^I \to R$.

An important result obtained by Boros et al. (1997), which is referred to as the BGV Theorem, claims that family of coalitions $\mathcal{K}$ is stable if and only if $\mathcal{K}$ is a Berge normal hypergraph (Berge (1970)). The definitions of terms 'perfect graph', 'normal hypergraph' and explanation of links between games, coalitions and hypergraphs will be given in Appendix (Section 7).

For the present paper the following example is most important: Let *T* be a tree, assign a player to every vertex of *T*. Then an arbitrary family of subtrees of *T* forms a stable family of coalitions.

This is a special case of the BGV theorem, which is important in applications. As it has been already mentioned above, we consider the family of LP problems whose resource constraints (righthand side) b^K depends on a coalition. Hence, the optimal solution $v(K) = Cx^*(b^K) = Cx^*(K)$ is a superadditive function of *K*. Thus the BGV theorem is applicable. Therefore, it exist such a distribution of the total optimal payoff $v(I)$ among the players that will cause no objection from any player. Which coalitions are feasible will be discussed in the Section 4.

3 Network supply systems and games

Water, electricity and gas supply systems fall into a class of network systems and can be described as a network systems. The common characteristics of all these allocation systems is that they can be represented as a set of nodes (for instance, reservoirs, junctions and users) interconnected by a system of carriers with different capacities (wires, channels, pipelines). The last two decades a number of publications appeared when these types of system were described using the linear and non-linear optimization methods. For water supply system this approach was employed for instance in Perera et al. (2005). Electricity supply systems and associated electricity market, as well as traffic and telecommunication systems were considered in Hu and Ralph (2007) and Ralph (2008). Gas supply system for Europe and North America were also modeled by similar methods of network LP optimization (Egging et al. (2008) and Gabriel et al. (2005)).

In order to be more specific, the REALM (Resource Allocation Model) illustrates how this network LP works for the irrigation water supply systems. More detailed description of this LP network optimization can be found in Dixon et al. (2005).

In stylized, form REALM models can be represented as:

Choose non-negative values for

$$F(i, r, t), \quad \text{for } i \in D,\ r \in D \cup E,\ \ t = 1, 2, \ldots, T,$$

$$S(e, t), \quad \text{for } e \in E,\ ,\ \ t = 1, 2, \ldots, T \quad \text{and}$$

$$W(i, t) \quad \text{for } i \in D,\ \ t = 1, 2, \ldots, T$$

to minimize

$$\sum_{i \in D} \sum_{r \in D \cup E} \sum_{t} c_{i,r}(t) * F(i,r,t) + \sum_{e \in E} \sum_{t} \beta_e(t) * \left| d(e,t) - S(e,t) \right| + \sum_{i \in D} \sum_{t} g_{i,t}[W(i,t) - W_{min}(i,t)]$$

subject to

$$W(i,t+1) \le W(i,t) - \sum_{r \in D \cup E} F(i,r,t) + \sum_{k \in D} F(k,i,t) * \left[1 - l(k,i,t)\right] + X(i,t) - \theta_{i,t}\left[W(i,t)\right]$$

$$\text{for } i \in D,\ t = 1, 2, \ldots, T$$

$$W(i,t) \le C(i) \quad \text{for } i \in D,\ \ t = 1, 2, \ldots, T \quad \text{and}$$

$$S(e,t) = \sum_{i \in D} F(i,e,t) * \left[1 - l(i,e,t)\right] \quad \text{for } e \in E,\ \ t = 1, 2, \ldots, T$$

where:

D is the set of dams. Dams include not only water storage facilities but also junctions in the water network. A junction has either more than one inlet or more than one outlet. It can be treated as a dam with zero capacity;

E is the set of end users;
F(i, r, t) is the flow in period t from dam i to dam or end-use r;
T is the last period of interest. If the model were solved for one year with periods of one month, then T =12;
W(i, t) is the amount of water in dam i at the beginning of period t. W(i,0) is exogenous;
S(e, t) is the amount of water supplied to end-user e in period t;
d(e, t) is the exogenously determined ideal water requirements of end-user e in period t;
$c_{i,r}(t)$ is the cost of sending a unit of water from dam i to dam or end-user r in period t. If it is physically impossible to send water from i to r, then $c_{i,r}(t)$ can be set at an arbitrarily large number;
$\beta_e(t)$ is the penalty or cost per unit of shortfall in meeting the water demands of end-user e in period t;
$W_{min}(i, t)$ is the minimum level of water for dam i that is desirable from an environmental or aesthetic point of view;
$g_{i,t}$ is a penalty function. It takes positive values if W(i, t) – $W_{min}(i, t)$ is negative;
$l(k, i, t)$ is losses per unit of flow from dam k to dam or end-user i in period t (exchange losses);
X(i, t), specified exogenously, is the natural inflow to dam i in period t;
$\theta_{i,t}$ is a function giving evaporation from dam i in period t;
C(i) is the capacity of dam i. If dam i is a junction then C(i) = 0.

Models such as REALM can be used to plan flows in a hydrological area and to decide how these flows should be varied in response to changes in rainfall [reflected in X(i, t)], changes in demands [d(e, t)], and changes in a myriad of technical and cost coefficients.

The similar optimization formulations can be used for electricity and gas supply systems.

4 Superadditive characteristic functions related to linear programming

Here we refer to an important result obtained by Jonson (1973), Gomory and Jonson (1973), Bliar and Jeroslow (1977), Jeroslow (1977, 1978) and Schrijver (1980).

Let a LP formulation be given in the following for:

Maximize the revenue function $f(x) = \sum_{p \in P} c_p x_p = Cx$

Subject to $Ax \le b$

where b is a vector of resources and c is a vector of prices.

The optimal solution is a vector $x^* = (x_i^*, i \in I)$

For each coalition *K*, the function b^K is defined additively:

$$b^K = \sum_{i \in K} b_i$$

Then function *v* is defined on the set of coalitions as $v(K) = Cx^*(b^K)$, which is the optimal value of objective function subject to resources b^K available for the coalition *K*.

It is important that $v(K)$ is a supperadditive function, that is,

$v(K_1 + K_2) \geq v(K_1) + v(K_2)$ for any pair of disjoint coalitions, $K_1 \cap K_2 = \emptyset$

5 Stability of optimal solution

Let us consider a network LP optimization problem for an acyclic graph. Referring to the results of Section 2 , we can state that a family of coalitions is stable if and only if it is a normal hypergraph.. In particular any family subtrees of a tree form a stable family of coalitions. Thus, there exists an allocation corresponding to the optimal LP solution, which is stable. It means there is a distribution of the total income among players which is acceptable for all feasible coalitions.

6 Concluding remarks

In nutshell, this paper suggests a new approach to optimization and fair distribution of total payoff among players. This method is based on integration of two fundamental results, which are

1. Supperadditive duality for LP optimization, and
2. The BGV Theorem claiming that a family of coalitions K is stable (that is the K -core is not empty) if and only if K is a normal hypergraph.

Then BGV theorem can be applied for acyclic network systems. The following two assumptions are crucial: 1. **acyclicity** of corresponding graph and 2. **supperadditivity** of the optimal value. It allows us to conclude that for some 'natural' families of coalitions are stable, that is, admit a non-empty core. Therefore, it is possible to conclude that a fair distribution of payoff among the payers is possible. Thus, the existence of an optimal and stable solution for such class of games is proven.

A few words should be written about possible further continuation of this research. Firstly, we plan to illustrate these results by considering the real network supply system (say, the water allocation system modeled by REALM model) and demonstrate that for the full set of possible coalitions the game is unstable, that is, the core is empty. Then we will select a set of 'natural' coalitions, say, coalitions of farmers whose properties are located on the same water carrier, and demonstrate that such families are stable. The next step of this research is consideration of more sophisticated network supply systems and demonstration what are the optimal and stable solution in this case. The key point is formulation of appropriate managerial advises about redistribution of optimal income between real players constituting these systems.

Finally, we plan to consider the non-linear case related to the objective function in form of Cobb-Douglass' production function:

$$CB(u_1, u_2, ..., u_{|R|}) = \prod_{i=1,..,n} u_i^{\gamma_i}$$ where u_i are production factors and $\gamma_1 + \gamma_2 + ... + \gamma_{|P|} = 1$

Schreider, Zeephongsekul and Abbasi (2010) proved that this function is also supperadditive. This result gives us a potential to employ similar approach for the game on non-cyclic graphs with Cobb-Douglass objective function.

7 Appendix: Perfect graphs, normal hypergraphs and cores of cooperative games

7.1 Introduction

The core of a game is defined as the set of outcomes acceptable for *all* coalitions. This is probably the simplest and most natural concept of cooperative game theory, yet, there is an important disadvantage: the core is frequently empty, because there are too many coalitions involved. However, some coalitions can hardly be realized, since potential coalitionists may not know (or like) each other, or it is often meaningless, or technically difficult for them to cooperate. For these reasons, the following relaxation of the concept is relevant.

Let K be a fixed family of coalitions. The K-core is defined as the set of outcomes acceptable for all coalitions from K. The family K is called *stable* if the K-core is not empty for any superadditive cooperative TU-game.

Normal hypergraphs can be characterized by several equivalent properties, for example, they are *dual* to the *clique hypergraphs* of *perfect graphs*. In this paper we made use of the following result: any family of subtrees of a tree is a normal hypergraph. In general it is known that a family K of coalitions is stable if and only if K is a normal hypergraph. It is also known that stable families of coalitions remain the same for the cooperative non-transferable utility (NTU) games or games in normal form.

For reader's convenience, in this Appendix we explain the above results in more detail; in particular, we accurately define all involved concepts and explain connections between them. More about perfect graphs and normal hypergraphs can be found in the monograph by Claude Berge (1970); see also the survey Brandstädt et al. (1999) for the properties of chordal graphs and other graph classes.

7.2 Graphs

Given a set V whose elements $v \in V$ are referred to as *vertices*. A *hypergraph* $\mathcal{H} \in 2^V$ is an arbitrary family of vertex-sets. These sets $H \in \mathcal{H}$ are called *edges*. Furthermore, $\mathcal{H}$ is called a *graph* if every its edge consists of exactly two vertices, $|H| = 2$ for all $H \in \mathcal{H}$. For graph we prefer to use the standard notation *G=(V,E)*, where *V=V(G)* and *E=E(G)* are, respectively, vertex- and edge-sets of graph *G*. The numbers of vertices $|H|$ and edges $|V|$ are denoted by *n* and *m*, respectively.

The complementary graph $\overline{G} = (V, \overline{E})$ is defined on the same vertex-set *V* and the complementary edge-set $\overline{E}$, that is, $(v, v') \in \overline{E}$ if and only if $(v, v') \notin E$.

A *subgraph G'= (V',E')*of graph *G = (V,E)* is defined by the subset of vertices $V' \subset V$ and by the subset of edges $E' \subset E$ which join *some* of these vertices. A subgraph is called *induced* if *all* edges of *E* that connect pairs of vertices of *V'* belong *E'*.

A graph is called *complete* if any two vertices are connected by an edge. The complement to a complete graph is called *edge-free*. A complete subgraph is called a *clique*, and an edge-free induced subgraph is called a *stable set*. Each clique (stable set) which does not strictly belong to another clique (stable set) is called *maximal*, and each clique (stable set) of maximal size is called *maximum*. The order of a maximum clique (stable set) is called *clique number* (*stability number*) and will be denoted by $\omega = \omega(G)$ (by $\alpha = \alpha(G)$), respectively. The following relations obviously hold:

$$\alpha(\overline{G}) = \omega(G), \quad \omega(\overline{G}) = \alpha(G), \quad n(\overline{G}) = n(G), \quad m(\overline{G}) = \frac{1}{2}n(n-1) - m(G),$$

A mapping $c : V' \to \{1,2,...j\}$ is called a coloring of the graph *j* colors. A coloring c is called *proper* if any two adjacent vertices are colored by distinct colors. The minimal number of colors that admit a proper coloring is called the chromatic number of the graph and is denoted by $\chi = \chi(G)$. Evidently, $\chi(G) \geq \omega(G)$ for every graph *G*.

7.3 Perfect graphs

A graph G is called *perfect* if equation $\chi(G') = \omega(G')$ holds for any induced subgraph G' of G. There is a well-known simple criterion of perfectness.

Proposition 1 *A graph G is perfect if and only if every its induced subgraph G′ contains a stable set that has (exactly one) vertex in common with every maximum clique G'.*

The following simple generalization is important.

Proposition 2 *A graph is perfect if and only if for each non-negative weighting* $w: V \to R_+$ *of its vertices there is a stable set that has (exactly one) vertex in common with every clique of maximum weight.*

If the weights of vertices take values 0 and 1 only then the above proposition coincide with the previous one. The next criterion is more difficult.

Theorem 1 (Lovász (1972a, 1972b)) *A graph G is perfect if and only if for any induced subgraph G' of G the inequality holds.*

Corollary Graph G is perfect if and only if the complementary graph $\overline{G}$ is perfect.

Perfect graphs were introduced by Berge in 1961 (Berge, 1961a,b). He suggested two conjectures. The last corollary was Berge's weak perfect graph conjecture (WPGC), and since 1972 it is called Lovász perfect graph theorem. (Interestingly, the proof in Lovász (1972a) is based on the concept of normal hypergraph. This concept is central in our work too; see below.)

A second, much stronger, conjecture Berge formulated as follows.

A *hole* is a simple chordless cycle of length at least 4, and an *antihole* is the complement of a hole.

Odd holes C_{2i+1} and odd antiholes $\overline{C}_{2i+1}$ satisfy the following equations:

$$\omega(C_{2i+1}) = \alpha(\overline{C}_{2i+1}) = 2, \quad \alpha(C_{2i+1}) = \omega(\overline{C}_{2i+1}) = i.$$

In both cases $\alpha \times \omega = 2i < 2i + 1 = n$. Thus odd holes and odd antiholes are not perfect.

By definition, a graph is not perfect whenever it contains an induced subgraph isomorphic to an odd hole or antihole. The inverse implication is exactly the Berge Strong Perfect Graph Conjecture (SPGC). Obviously, the SPGC implies WPGC, since odd holes and antiholes are complementary, by the definition. SPGC appeared to be very difficult. For more than forty years, it was considered in hundreds of papers and finally resolved in 2002 by Seymour, Chudnovsky, Robertson, and Thomas; see Seymour et al. (2002).

7.4 Hypergraphs

A hypergraph H was defined as an arbitrary family of subsets, called *edges*, of a finite set whose elements are called *vertices*. Let us denote by $E=E(H)$ the set of edges and by $V = V(H)$ the set of vertices. It will be convenient to allow multiple edges.

A hypergraph H can also be given by *incidence matrix A* whose rows and columns are labeled respectively by the edges and the vertices of the hypergraph, and $A(H,j)=1$ if and only if vertex j belongs to edge H. **We assume that each row and each column of *A* contain at least one non-zero.** The transposed matrix A^T defines the dual hypergraph H^d whose edges and vertices are assigned respectively to the vertices and the edges of H that is, $E(H)=V(H^d)$ and $V(H)=E(H^d)$.

Given a hypergraph H ,let us define the *co-occurrence* graph, $G_C(H)$, on the same set of vertices $V(H)$, in which two vertices are connected by an edge if and only if they both belong to an edge H of H. Let us define also the *intersection* graph, $G_i(H)$, with vertex-set is $E(H)$ in which two vertices are connected if and only if the corresponding two edges of H have a non-empty intersection.

Lemma 1 For any hypergraph H graphs $G_C(H)$ and $G_i(H)$ coincide.

Proof This claim results directly from the above definitions. It is easy to verify that two edges of H have a non-empty intersection if and only if the corresponding two vertices of H^d belong to an edge.

Given a graph G, it is easy to obtain a hypergraph $G_C(H)$such that $G = G_C(H)$. Let us consider an arbitrary clique covering of G, that is, a family of cliques that cover all edges of G. Clearly, these cliques form a hypergraph H for which G is the co-ocurrence graph, $G = G_C(H)$. Moreover, all such hypergraphs can be obtained in this way.

Analogously, given a graph G, we can obtain a hypergraph H such that $G = G_i(H)$. According to Lemma 1, this problem is just dual to the previous one. Let us consider an arbitrary clique covering of G again. Then the collections of those cliques, which contain a given vertex of G, form edges of a hypergraph H such that $G = G_i(H)$. Moreover, every such hypergraphs can be obtained in this way from a clique covering.

7.5 Normal hypergraphs

Normal hypergraphs were introduced by Berge in 1970 (Berge, 1970). Soon after this, Loväsz (1972a) made efficient use of this concept and gave a simple proof of the WPGC. The following concepts were also introduced in (Berge, 1970).

A hypergraph H has the *Helly property* if any family of edges whose intersection is empty contains two disjoint edges. For example, intervals of an ordered set or subtrees of a tree have this property.

A hypergraph H is called *clique-maximal* if it contains, as edges, all maximal cliques of its co-ocurrence graph $G_C(H)$. Let us underline that H may contain more edges, which correspond to non-maximal cliques, too.

Lemma 2 A hypergraph H has the Helly property if and only if its dual hypergraph H^d is clique-maximal.

Proof is straightforward; see Berge (1970) or Brandstädt et al. (1999).

Furthermore, we will say that a hypergraph H has the *Berge property* if its intersection graph $G_i(H)$ is perfect.

Finally H is called *normal* if it has both the Helly and Berge properties.

Evidently, H is normal if and only if the graph $G_C(H)$ is perfect and its clique covering hypergraph H^d is clique-maximal.

In other words, any normal hypergraph can be realized as follows: take a perfect graph $G = (V,E)$ and a family of its cliques $H = \{C_i, i \in I\}$ in I that contains all maximal cliques and, perhaps, some other too. Assign a vertex (player) to every i in I. Then, to each vertex v in V we naturally assign a family (coalition) $K_v = \{i \in I \mid v \subseteq C_i\} \subseteq I$ of all cliques that contain v. By this, we assign a hypergraph (a family of coalitions K_W to each vertex-subset $W \subseteq V$ and, in particular, to $W = V$. Any such hypergraph $H = K_W$ is a normal and, conversely, any normal hypergraph can be realized in this way.

There are many other equivalent characterizations of normality, see, for example, Lovász (1972a) and Berge (1984). Four more are added in Boros et al. (1997), where it is shown that stability of a family of coalitions for several distinct classes of cooperative games; see below.

7.6 Chordal, or triangulated, graphs

A graph $G = (V,E)$ is called chordal, or triangulated if each of its cycles of four or more nodes has a chord, which is an edge joining two nodes that are not adjacent in the cycle; or equivalently, chordal graphs have no holes (induced cycles of length more than three).

An alternative characterization of chordal graphs is due to Gavril (1974):

A chordal graph is the intersection hypergraph $G_i(H)$, where hypergraph H is a family of subtrees of a tree. More precisely, from a collection of subtrees of a tree, one can define a subtree graph, which is an intersection graph that has one vertex per subtree and an edge connecting any two subtrees that overlap in one or more nodes of the tree. As Gavril (1974) showed, the subtree graphs are exactly the chordal graphs.

It is well-known that the chordal hypergraphs are perfect; see, for example, Brandstädt at al. (1999).

7.7 Why subtrees of a tree form a normal hypergraph?

The reason is simple. First, obviously, any family of subtrees satisfies the Helly property: if there is no common vertex then there are two disjoint subtrees.

Second, the intersection graph of such a family is chordal, by Gavril's result Gavril (1974). Hence, this graph is perfect.

7.8 Balanced weightings and partitionable families of coalitions

In game theory we denote our ground-set by I; its elements $i \in I$ and subsets $K \subseteq I$ are called the *players* and *coalitions*, respectively. Given a family of coalition $\mathcal{K} \subseteq 2^I$ a non-negative integer valued function w: $\mathcal{K} \to \mathbb{Z}_+$ is called a *weighting* of $\mathcal{K}$ and it is called a *balanced weighting* with multiplicity m if equation

$$\sum_{K|i\in K} w(K) = m$$

holds for every player $i \in I$, or in other words, if each player participates in the same number of coalitions (taking weights into account).

A family $\mathcal{K}$ is called *balanced* if it has a strictly positive balanced weighting w: $\mathcal{K} \to \mathbb{Z}_+$. For an arbitrary non-negative weighting w, the subfamily $\mathcal{K}_w=\{K \in \mathcal{K} | w(K) > 0\}$ is called the *support* of w. A balanced weighting is called *minimal* if its support does not strictly contain a support of some other balanced weighting.

A balanced weighting w of multiplicity $m = 1$ is called a *partition weighting* (or simply a *partition*), because in this case the support K_w of w is a partition of I. A balanced weighting w is called a *sum of partitions* if there are partitions $w^i: K \to \{0,1\}$ and non-negative integers

$\alpha_i \in Z_+$, $i=1,...,l$, such that $w = \sum_{i=1}^{l} \alpha_i w^i$

Proposition (Shapley, 1965) *The following properties of a family K are equivalent.*

(a) *Any balanced weighting of K is a sum of partitions.*

(b) *Any balanced weighting of K $ contains a partition.*

(c) *Any minimal balanced weighting of K is a partition.*

We call a family K *partitionable* if K satisfies these equivalent conditions (a), (b), (c). Soon, we will see that this list can be extended by other important properties, in terms of perfect graphs and normal hypergraphs; in particular, it will include stability of K with respect to several types of cooperative games.

To illustrate the concept, let us recall several examples of partitionable and non-partitionable families given in seventies by Gurvich and Vasin (1977,1978).

Example 1 *$I = \{1, 2\}$; $K = 2^I = \{(1),(2),(1,2)\}$. If there are only 2 players then the family of all possible coalitions is partitionable.*

Example 2 *$I = \{1,2,3\}$; $K = 2^I - \{(1,3)\} = \{(1),(2),(3),(1,2),(2,3),(1,2,3)\}$. In case of 3 player, the family of all coalitions but one doubleton is partitionable.*

Example 3 *$I = \{1,2,3\}$; $K = C_3 = \{(1,2),(2,3),(3,1)\}$. A 3-cycle is not partitionable. This is the simplest balanced family which does not contain a partition.*

Example 4 *$I = \{1,2,3,4\}$; $K = C_4 = \{(1,2),(2,3),(3,4),(4,1)\}$. A 4-cycle is partitionable. It is easily seen that the above 4-cycle is the sum of 2 partitions: $\{(1,2),(3,4)\}$ and $\{(2,3),(4,1)\}$.*

Example 5 *$I = \{1,2,...,n\}$; $K = C_n = \{(1,2),(2,3),...,(n-1,n),(n,1)\}$, $n \geq 3$. An even cycle is partitionable (clearly, it is the sum of 2 partitions), but an odd cycle is not.*

This is the first reminiscence with perfect graphs.

Example 6 *$I = \{1,2,...,n\}$, $K = S_n^{n-1}, n \geq 2$*. The family S_n^{n-1} of all the (n-1)-subsets of I is partitionable if n = 2 and it is not partitionable if n > 2.

Example 7 *$I = \{1,2,...,n\}$, $\mathcal{K} = \{(1),(2),...(n)\} \cup \{K \subseteq I \mid 1 \in K\}$ that is, $\mathcal{K}$ contains all the singletons and all coalitions including player 1. Clearly, $\mathcal{K}$ is partitionable.*

One can interpret player 1 in the above example as a spy, whose duty is to participate in any coalition consisting of more than one player.

Example 8 *$I = \{1,2,...,n\}$, $\mathcal{K} = \{K \subseteq I \mid i', i'' \in K, i' < j < i'' \Rightarrow j \in K\}$. In other words, I is a finite completely ordered set, and $\mathcal{K}$ is the family of all the intervals of I. Clearly, the family $\mathcal{K}$ is partitionable.*

We can interpret the above example such that players are owners of the shops situated along a railway line in a remote region or as farmers whose farms are located along a canal, the main source of water. Then, it is natural to suppose that only a few close neighbors can organize a coalition and effectively coordinate the prices, for example.

This example was considered independently by Greenberg and Weber (1986); see also Greenberg (1993). It was extended by Aharoni et al. (1994) for the case of an infinite ordered set.

The next example is another generalization. It was considered independently by Demange (1990) and Kuipers (1994).

Example 9 Let *$T = (V,E)$ be a finite tree whose vertices $V = I$ are identified with players. Any family $\mathcal{K}$, of subtrees of T is partitionable*.

This example generalizes Examples 7 and 8, in which the corresponding tree T is a star,
$E(T) = \{(1,2),(1,3),...,(1,n)\}$, and a simple path,
$E(T) = \{(1,2),(2,3),...,(n-1,n)\}$, respectively, where *$I = \{1,2,...,n\}$*.

An interpretation, similar to that of Example 8, can be given in terms of a tree structured road, canal, or railway systems. Another interpretation was given by Gurvich and Menshikov (1989). Let us fix an arbitrary vertex $r \in T$. It will be called the root and the corresponding player will be called the "chief or big boss". The obtained rooted tree will be treated as a hierarchical scheme (of a company). It is supposed that if two players $i', i'' \in I$ participate in a coalition then the players along the path connecting the corresponding vertices also have to be included in this coalition. The boss of the minimal rank common for the players i' and i'' and also all the intermediate bosses belong to this path. This path is unique because T is a tree **.**

7.9 TU-games, core, and $\mathcal{K}$ -core

A game with transferrable utilities (also called for brevity, TU-game) is defined by an arbitrary *characteristic function* (CF) $v: 2^I \rightarrow \mathbb{R}$, whose value $v(K)$ is interpreted as a profit that coalition K "could expect". Furthermore, CF v is *superadditive* if

$v(K' \cup K'') \geq v(K') + v(K'') \quad \forall K,' K'' \subseteq I$ such that $K' \cap K'' = \varnothing$

For any $x = (x_1,...,x_n) \in R^n$ and $K \subseteq I$ let $x_K = \sum_{i \in K} x_i$.

The set of outcomes or allocations is defined as A $= \{x \mid x_I \leq v(I)\} \subseteq$ R^I, that is, A consists of all vectors which the total coalition $K = I$ is able to obtain.

An outcome x is called *acceptable* for a coalition $K \subseteq I$ if $x_K \geq v(K)$, that is, if K cannot expect more than given by x.

The *core C(v)* is the set of outcomes acceptable for all coalitions $K \subseteq I$. In other words, it is defined by the following system of linear inequalities

$$C(v) = \{x \in R^{|I|} \mid x_I \leq v(I) \text{ and } x_K \geq v(K)\ \forall K \subseteq I\}$$

Given an arbitrary family of coalitions $\mathcal{K} \subseteq 2^I$, the $\mathcal{K}$ -core $C(v, \mathcal{K})$ is defined as the set of outcomes acceptable for all the coalitions K$\in \mathcal{K}$, i.e.

$$C(v, \mathcal{K}.) = \{x \in R^{|I|} \mid x_I \leq v(I) \text{ and } x_K \geq v(K)\ \forall K \in \mathcal{K} \}$$

According to the above definitions, $C(v, \mathcal{K}) = C(\mathcal{K})$ for $\mathcal{K} = 2^I$.

A set $\mathcal{K}$ is called *v-stable* if $C(\mathcal{K},v) \neq \varnothing$ for any superadditive CF v.

Theorem 2 *(Bondareva 1962, 1963) and Shapley (1965)). The core C(v) is not-empty iff any (minimal) balanced weighting w: $2^I \rightarrow Z_+$ of all the coalitions satisfies the inequality*

$$\sum_{K \subseteq I} w(K)v(K) \leq v(I)$$

Proof follows directly from the classic Farkas Lemma (Farkas, 1901) applied to the system of linear inequalities that define the core; Bondareva refers also to Fan Ku (1956). Theorem 2 was generalized of as follows.

Theorem 3 (Gurvich and Vasin, 1977) *The core C(v) is not-empty iff any (minimal) balanced weighting w: $2^I \rightarrow Z_+$ of all the coalitions satisfies the inequality*

$$\sum_{K \in \mathcal{K}} w(K)v(K) \leq v(I)$$

Let us consider 3- and 4-cycles, as an example.

Example 10 *$I = \{1,2,3\}$, $K = C_3 = \{(1,2),(2,3),(3,1)\}$. The K-core is given by*

$$
\begin{aligned}
x_1 + x_2 \quad\quad &\geq v(1,2)\\
x_1 + \quad\; + x_3 &\geq v(1,3)\\
+ x_2 + x_3 &\geq v(2,3)\\
x_1 + x_2 + x_3 &\leq v(1,2,3)
\end{aligned}
$$

This system yields the inequality

$$v(1,2) + v(1,3) + v(2,3) \leq 2v(1,2,3)$$

which is not a consequence of superadditivity. For example, it is violated by the superadditive (and simple) CF v given by

$$v(1)=v(2)=v(3)=0 \quad v(1,2)=v(1,3)=v(2,3)=v(1,2,3)=1$$

Example 11 *$I = \{1,2,3,4\}$, $K = C_4 = \{(1,2),(2,3),(3,4),(4,1)\}$. The K-core is given by*

$$
\begin{aligned}
x_1 + x_2 \quad\quad\quad\quad &\geq v(1,2)\\
+ x_2 + x_3 \quad\quad &\geq v(2,3)\\
+ x_3 + x_4 &\geq v(3,4)\\
x_1 \quad\quad\quad\quad + x_4 &\geq v(1,4)\\
x_1 + x_2 + x_3 + x_4 &\leq v(1,2,3,4)
\end{aligned}
$$

Obviously, this system is consistent if and only if

$$v(1,2) + v(3,4) \leq v(1,2,3,4), \text{ and } v(2,3) + v(1,4) \leq v(1,2,3,4)$$

which is true for any superadditive CF v.

Let us assign a CF v_K to a family of coalitions K. For this, let us first add to K all the unions of pairwise disjoint coalitions from K and let us denote the obtained expanded family by K'. Then let us define $v_K(K)$ by

$$v_K(K) = \max_{K':K \supseteq K' \in K'} |K'|$$

Lemma 3 (*Gurvich and Vasin, 1977) The CF v_K is superadditive, takes nonnegative integer values and satisfies the following relations:*

$v_K(K) = |K| \forall K \in K'$, and $v_K(K) < |K| \forall K \notin K'$

Furthermore, if the family K is not partitionable, then $C(v_K, K)=\emptyset$.

This lemma and last theorem immediately results in the following criterion of stability.

Corollary 2 (*Gurvich and Vasin, 1977; Kaneko and Wooders, 1982). A family of coalitions K is v-stable if and only if K is partitionable*.

7.10 Partitionable and normal hypergraphs are closely related

Let $[I] = \{\{i\}; | i \in I\}$ denote the family of all single-player coalitions of *I*. In particular, $K \cup [I]$ is the extension of K by all singletons.

Theorem 4 (Boros, Gurvich, and Vasin; see Boros et al., 1997) *A family of coalition K is a normal hypergraph if and only if $K \cup [I]$ is partitionable*.

Let us notice that in most of game theoretic papers it is assumed (sometimes implicitly) that $[I] \subseteq K$, that is, any considered family of coalition contains all singletons. Such an assumption is natural for game theory. Indeed, we reduced the family of all coalitions 2^I to some $K \cup [I]$, because some coalitions can hardly be realized, since the player may not know (or like) each other, or it is meaningless, or technically difficult for them to cooperate. However, nobody can prevent a single player to organize "a coalition" that consists only of him- or herself. So, game theorists do not distinguish two conditions: "K is partitionable" and "$K \cup [I]$ is partitionable". Indeed, "mathematically" they differ a lot. The latter one is much stronger, since the set of balanced weightings over $K \cup [I]$ is much larger than of those over K. Also the latter condition is "combinatorially nicer", since, by Theorem 4, it is closely related to normal hypergraphs and perfect graphs.

There is another important reason, too. In Gurvich and Vasin (1978) and Kaneko and Wooders (1982) the concept of stability was extended from TU- to NTU-games, and then also to cooperative games in normal form and in form of the so-called effectivity function in Boros et al. (1997) (see this paper for the precise definitions), where it was shown that all three new concepts of stability are still equivalent to normality.

Since the theory of superadditive duality is developed not only for linear but for integer programming too (see Johnson (1973), Gomory and Johnson (1973), Bliar and Jeroslow (1977), Jeroslow (1977, 1978)), our approach could be also extended for the classes of cooperative games with non-transferrable utilities (NTU) and also games in normal form. Yet, this should become the subject of a further research.

8 References

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