Rutcor Research Report

A lower bound for discounting algorithms solving two-person zero-sum limit average payoff stochastic games

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Abstract. It is shown that the discount factor needed to solve an undiscounted mean payoff stochastic game to optimality is exponentially close to 1, even in games with a single random node and polynomially bounded rewards and transition probabilities.

1 Introduction and motivation

We consider two-person zero-sum stochastic games with perfect information and mean payoff: Let $G = (V, E)$ be a digraph whose vertex-set V is partitioned into three subsets $V = V_B \cup V_W \cup V_R$ that correspond to black, white, and random positions, controlled respectively, by two players, BLACK - the *minimizer* and WHITE - the *maximizer*, and by nature. We also fix a *local reward* function $r : E \to \mathbb{R}$, and probabilities $p(v, u)$ for all arcs (v, u) going out of $v \in V_R$. Vertices $v \in V$ and arcs $e \in E$ are called *positions* and *moves*, respectively. In a personal position $v \in V_W$ or $v \in V_B$ the corresponding player WHITE or BLACK selects an arc (v, u), while in a random position $v \in V_R$ a move (v, u) is chosen with the given probability $p(v, u)$. In all cases BLACK pays WHITE the reward $r(v, u)$.

From a given initial position $v_0 \in V$ the game produces an infinite walk (called a *play*). WHITE's objective is to maximize the *limiting mean payoff*

$$c = \liminf_{n\to\infty} \frac{\sum_{i=0}^{n} b_i}{n+1}, \tag{1}$$

where b_i is the expected reward incurred at step i of the play, while the objective of BLACK is the opposite, that is, to minimize $\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} b_i}{n+1}$.

For this class of *BWR-games*, it is known that a *saddle point* exists in *pure positional uniformly optimal* strategies (see, e.g., [BEGM09]). Here "pure" means that the choice of a move (v, u) in a personal position $v \in V_B \cup V_W$ is deterministic; "positional" means that this choice depends solely on v, not on previous positions or moves; finally, "uniformly optimal" means that it does not depend on the initial position v_0, either. This fact was proved by Gillette [Gil57] and Liggett and Lippman [LL69] by considering the *discounted* version, in which the payoff of WHITE is discounted by a factor β^i at step i, giving the effective payoff:

$$a_\beta = (1-\beta)\sum_{i=0}^{\infty} \beta^i b_i,$$

and then proceeding to the limit as the *discount factor* $\beta \in [0, 1)$ goes to 1.

The important special case of BWR-games without random vertices, i.e., $V_R = \emptyset$, is known as *cyclic* or *mean payoff* games [Mou76b, Mou76a, EM79, GKK88]. A BWR-game is reduced to a *minimum mean cycle problem* in case $V_W = V_R = \emptyset$ or $V_B = V_R = \emptyset$, which can be solved in polynomial time [Kar78]. If one of the sets V_B or V_W is empty, we obtain a *Markov decision process* for which polynomial-time algorithms are also known [MO70]. Finally, if both sets are empty $V_B = V_W = \emptyset$, we get a *weighted Markov chain*.

In the special case of a BWR-game, when all rewards are zero except at a single node t (called the terminal), which has a self-loop with reward 1, we obtain the so-called *simple stochastic games* (SSGs), introduced by Condon [Con92, Con93] and considered in several papers (e.g. [GH08, Hal07]). In these games, the objective of White is to maximize the probability of reaching the terminal, while Black wants to minimize this probability. Recently, it was shown that Gillette games (and hence BWR-games by [BEGM09]) are equivalent to SSGs under polynomial-time reductions [AM09]. At the heart of these reductions is the fact, established in [AM09], that it is enough to take

$$\beta = 1 - [O((ND)^{N^2} R)]^{-1} \tag{2}$$

to guarantee that an optimal pair of strategies in the discounted game remains optimal in the undiscounted one. Here, N is the total number of vertices, R is the maximum absolute value of a reward (assuming integral rewards), and D is the common denominator of the transition probabilities (assuming rational transition probabilities).

While there are numerous pseudo-polynomial algorithms known for BW-games (the case when there are no random nodes) [GKK88, Pis99, ZP96], no such algorithm is known for the BWR-case, even if we restrict the number of random vertices. A pseudo-polynomial algorithm was given in [BEGM10] for *ergodic* BWR-games (in which the equilibrium values do not depend on the initial position) with a constant number of random nodes, but a similar result in the non ergodic case was left open.

One approach towards this end is to consider the β-discounted game, which can be solved in time polynomial in the input size and $\frac{1}{1-\beta}$, and then set β sufficiently close to 1. In the absence of random positions, such approach yields indeed a pseudo-polynomial algorithm: to get the exact solution of an undiscounted BW-game with N positions and maximum absolute reward R, it is enough to solve the corresponding β-discounted game with any $\beta > 1 - 1/(4N^3R)$ [ZP96]. However, such approach requires exponential time in the general case, since one must choose $\beta > 1 - \varepsilon/2^N$ to approximate the value of the game with accuracy ε, as follows follow from an example in [Con92], with only random nodes (that is a weighted Markov chain). We note, however, that the number of random nodes in this example $k = N$, and thus a question that naturally arises is whether one can get a bound similar to (2) in which the exponent N^2 is replaced by some function of k only. If this was the case, it would imply a pseudo-polynomial algorithm for BWR-games with $k = O(1)$. In this short note, we rule-out this possibility by showing that, in general, the discount factor may need to be chosen exponentially close to 1, even for games with a single random node.

Theorem 1. *There exists a BWR-game $\mathcal{G}$ with one random node, $D = O(N)$, and $R = O(N^{20})$, such that solving $\mathcal{G}$ to optimality using discounts requires a discount factor of at least $1 - O(\frac{N^{1/12}}{2^{N/3}})$, where N denotes the total number of nodes.*

2 Notations and basic lemma

Let n be a positive integer, and P be a set of primes p such that $n \leq p \leq n^2$ and $|P| = 2n$. By Chebyshev's prime number theorem we know that the number $\pi(X)$ of primes not larger than X satisfies the inequalities

$$\frac{7}{8}\frac{X}{\ln X} \;\leq\; \pi(X) \;\leq\; \frac{9}{8}\frac{X}{\ln X}$$

if X is large enough, and thus there are more than $2n$ primes between n and n^2 for all large enough integers n.

For a positive integer k let us denote by $\binom{P}{k}$ the family of k element subsets of P. For a subset $I \subseteq P$ we define

$$r(I) = \sum_{p \in I} \frac{1}{p} \quad \text{and} \quad s(I) = \sum_{p \in I} p. \tag{3}$$

Lemma 1. *Let $n > 0$ be a large enough integer. There exist subsets (possibly multisets) of integers $I, J \subseteq \mathbb{Z}_+$ such that $I \cap J = \emptyset$, $|I| = |J| \leq n + 2$, and such that the following inequalities are satisfied:*

$$0 \;<\; r(J) - r(I) \;\leq\; \frac{\sqrt{n}}{2^{2n-1}}, \tag{4}$$

$$s(I) - s(J) \geq 1, \quad \textit{and} \tag{5}$$

$$s(I) \leq 2n^3. \tag{6}$$

Proof. Let us consider the family $\mathcal{F} = \binom{P}{n}$ and observe that by Stirling's approximation we have

$$|\mathcal{F}| \geq \frac{2^{2n-1}}{\sqrt{n}}. \tag{7}$$

Let us observe next that for all subsets $I \in \mathcal{F}$ we have $n\frac{1}{n^2} \leq r(I) \leq n\frac{1}{n}$, and thus in particular

$$0 \leq r(I) \leq 1. \tag{8}$$

We claim that if $I, J \in \mathcal{F}$, $I \neq J$, then $r(I) \neq r(J)$. To this end, let us define $D = \prod_{p \in I \cup J} p$ and let $q \in I \setminus J$. Then we have

$$Dr(I) \mod q = \sum_{p \in I} \frac{D}{p} \mod q = \frac{D}{q} \mod q \neq 0$$

since $\frac{D}{q}$ is a product of primes different from q. On the other hand, for J we have

$$Dr(J) \mod q = \sum_{p \in J} \frac{D}{p} \mod q = 0$$

since $q \notin J$.

Thus, by the above claim the reals $r(I)$ for $I \in \mathcal{F}$ are pairwise distinct, and all belong to the unit interval $[0, 1]$ by (8). Therefore, by (7), we must have two subsets, say $I', J' \in \mathcal{F}$ satisfying

$$0 < r(J') - r(I') \leq \frac{\sqrt{n}}{2^{2n-1}}.$$

Then, the sets $\widetilde{I} = I' \setminus J'$ and $\widetilde{J} = J' \setminus I'$ satisfy (4).

If they also satisfy (5) then setting $I = \widetilde{I}$ and $J = \widetilde{J}$ completes our proof, since (6) follows simply by our choice of P.

Otherwise, let us note that we must have

$$s(\widetilde{I}) - s(\widetilde{J}) \geq |\widetilde{I}|n - |\widetilde{J}|n^2 \geq n^2 - n^3, \tag{9}$$

since we have $|\widetilde{I}| = |\widetilde{J}| \leq n$.

Let us then choose an integer a such that

$$n^3 \geq 2a^2 \geq 2(a-1)^2 \geq n^3 - n^2 + 1, \tag{10}$$

and define $I = \widetilde{I} \cup \{a, a(2a-1)\}$ and $J = \widetilde{J} \cup \{2a-1, 2a-1\}$. Note that J became a multiset now in which $2a - 1$ has multiplicity 2. With this, we still have $|I| = |J|$.

Furthermore, since $\frac{1}{a} + \frac{1}{a(2a-1)} = \frac{1}{2a-1} + \frac{1}{2a-1}$ we also have $r(J) - r(I) = r(\widetilde{J}) - r(\widetilde{I}) = r(J') - r(I')$, and hence I and J satisfy (4). Finally, we have

$$s(I) - s(J) = s(\widetilde{I}) + a + a(2a-1) - \Big(s(\widetilde{J}) + 2(2a-1)\Big) = s(\widetilde{I}) - s(\widetilde{J}) + 2(a-1)^2 \geq 1,$$

where the last inequality follows by the lower bounds in (9) and (10).

To see (6) it is enough to note that $s(\widetilde{I}) \leq |\widetilde{I}|n^2 \leq n^3$, and that $a + a(2a-1) = 2a^2 \leq n^3$ by (10).

Thus, the sets I and J satisfy all claimed inequalities, completing our proof. □

3 Construction

Let us choose two subsets $I, J \subseteq \mathbb{Z}_+$ of integers as in Lemma 1. Let $|I| = |J| = k$, and denote $I = \{p_1, p_2, \cdots, p_k\}$ and $J = \{q_1, q_2, \cdots, q_k\}$. Set $N = 3 + \sum_{j=1}^{k}(p_j + q_j)$, and note that by Lemma 1 we have $N = O(n^3)$.

Let us next associate to this input a BWR game on a directed graph $G = (V, A)$ having N vertices, defined as follows:

$$V = \{w_0, w_1, w_2, w_3\} \cup \left(\bigcup_{j=1}^{k}\{u_0^j, u_1^j, \cdots, u_{p_j-1}^j\}\right) \cup \left(\bigcup_{j=1}^{k}\{v_0^j, v_1^j, \cdots, v_{q_j-1}^j\}\right).$$

Let us define cycles $C_j^u = \{u_0^j, u_1^j), (u_1^j, u_2^j), \cdots, (u_{p_j-1}^j, u_0^j)\}$ and $C_j^v = \{(v_0^j, v_1^j), (v_1^j, v_2^j), \cdots, (v_{q_j-1}^j, v_0^j)\}$ and set the arc set as

$$A = \{(w_0, w_1), (w_0, w_2)\} \cup \bigcup_{j=1}^{k} \{(w_1, u_0^j), (w_1, v_0^j)\} \cup \bigcup_{j=1}^{k} (C_j^u \cup C_j^v).$$

Let the initial node w_0 be controlled by the maximizer (WHITE), and have three outgoing arcs, left, right, and bottom. The first two arcs have 0 as local rewards, while the third arc (w_0, w_3) has reward 1. The left and bottom neighbors w_2 and w_3 are controlled by the minimizer (BLACK), and have single loop arcs with local rewards 0 and $-\frac{(1-\beta_0)}{\beta_0}$, respectively, where $\beta_0 := 1 - \frac{1}{36n^{20}}$. The right neighbor of w_0 is w_1, a random node, with $2k$ outgoing arcs to the nodes u_0^j and v_0^j for $j = 1, ..., k$. All these arcs have rewards 0 and have transition probabilities $\frac{1}{2k}$. The cycles C_j^u and C_j^v, $j = 1, ..., k$ are composed of BLACK and WHITE controlled vertices, in an arbitrary distribution. The local rewards on the arcs of these cycles are all 0, except the first arcs of the cycles, where we have

$$r(u_0^j, u_1^j) = 1 \quad \text{and} \quad r(v_0^j, v_1^j) = -1$$

for all $j = 1, ..., k$.

For an illustration see Figure 1. We remark that it is enough to have one type of controlling player, i.e., the game can be turned into a Markov decision process. Note also that the number of vertices N satisfies: $n \leq N \leq 4(n^3+1)$, and the common denominator of all probabilities $D = 2k \leq 2(n+2)$. Furthermore, by multiplying all the rewards by $36n^{20}$ we obtain an equivalent game with integral rewards whose maximum absolute value is $R = 36n^{20} - 1$.

4 Game values

Let us denote by $\mu(v)$ the undiscounted game value form initial point v, and denote by $\mu^\beta(v)$ the value from the same initial point of the discounted game with discount factor $0 < \beta < 1$.

Lemma 2. *We have $\mu(w_2) = \mu^\beta(w_2) = 0$ and $\mu(w_3) = \mu^\beta(w_3) = -\frac{(1-\beta_0)}{\beta_0}$, for all $0 < \beta < 1$. Furthermore we have*

$$\mu(w_1) = \frac{1}{2k}\left(\sum_{p \in I} \frac{1}{p} - \sum_{q \in J} \frac{1}{q}\right) = \frac{r(I) - r(J)}{2k}, \quad \textit{and} \tag{11}$$

$$\mu^\beta(w_1) = \frac{1}{2k}\left(\sum_{p \in I} \frac{(1-\beta)\beta}{1-\beta^p} - \sum_{q \in J} \frac{(1-\beta)\beta}{1-\beta^q}\right). \tag{12}$$

If WHITE chooses the left strategy at w_0, then her undiscounted and discounted values are $\mu(w_0) = \mu^\beta(w_2) = 0$; if she chooses the right strategy, her values are $\mu(w_0) = \mu(w_1)$ and $\mu^\beta(w_0) = \beta\mu^\beta(w_1)$; and if she chooses the bottom strategy, her values are $\mu(w_0) = \mu(w_3)$ and $\mu^\beta(w_0) = (1-\beta) + \beta\mu^\beta(w_3)$.

Proof. Straightforward from the definitions. □

5 Proof of Theorem 1

We claim that while $\mu(w_1) < 0$, the discounted value $\mu^\beta(w_1) > 0$ even if β is very close to 1. More precisely, we will prove this claim for $\beta_0 \le \beta < \beta_1$, where $\beta_0 := 1 - \frac{1}{36n^{20}}$, and $\beta_1 := 1 - \frac{6n^{1/4}}{2^n}$. On the other hand, for small values of β, namely for $\beta \in [0, \beta_0)$, the value of $\mu^\beta(w_0)$ will remain strictly positive as long as WHITE chooses the bottom strategy. This will prove Theorem 1 since it implies that WHITE can guarantee a positive value in the discounted game as long as $\beta < \beta_1$, and the corresponding optimal strategy would give a negative value in the undiscounted game; however, the optimal value in the undiscounted game is 0.

Lemma 3. *Suppose that* WHITE *chooses the arc* (w_0, w_3). *Then* $\mu^\beta(w_0) > 0$ *for all* $\beta \in [0, \beta_0)$ *and* $\mu^\beta(w_0) < 0$ *for all* $\beta \in (\beta_0, 1]$.

Proof. Follows from the equation for $\mu^\beta(w_0) = 1 - \frac{\beta}{\beta_0}$ in Lemma 2. □

Lemma 4. *We have* $\mu(w_1) < 0$. *Furthermore, for all discount factors satisfying* $\beta_0 \le \beta < \beta_1$, *we have* $\mu^\beta(w_1) > 0$.

Proof. Since k and β here are positive constants, clearly an equivalent statement is that

$$A = 2k\mu(w_1) \;=\; r(I) - r(J) \;<\; 0$$

while

$$B = \frac{2k}{\beta}\mu^\beta(w_1) \;=\; \sum_{p\in I}\frac{(1-\beta)}{1-\beta^p} \;-\; \sum_{q\in J}\frac{(1-\beta)}{1-\beta^q} \;>\; 0.$$

The first claim follows immediately from (4), so it remains to prove the second claim. To this end let us note that for a positive integer p we have

$$\frac{(1-\beta)}{1-\beta^p} = \frac{1}{1+\beta+\beta^2+\cdots+\beta^{p-1}}.$$

We will use the following fact.

Fact 1. *For any positive integer p we have*

$$\frac{1}{1+\beta+\beta^2+\cdots+\beta^{p-1}} - \frac{1}{p} \;=\; (1-\beta)\frac{p-1}{2p} + (1-\beta)^2\frac{p^2-1}{12p} + (1-\beta)^3 R(p) \qquad (13)$$

where $R(p) = O(p^6)$.

Proof. Follows from Taylor expansion around $\beta = 1$ and routine calculations (see the appendix). □

To continue with the proof of the lemma, let us write

$$X \;=\; \sum_{p\in I}\frac{p-1}{2p} - \sum_{q\in J}\frac{q-1}{2q} \;=\; \frac{r(J)-r(I)}{2}$$

since we have $|I| = |J|$;

$$Y \;=\; \sum_{p\in I}\frac{p^2-1}{12p} - \sum_{q\in J}\frac{q^2-1}{12q} \;=\; \frac{1}{12}\left[(s(I)-s(J)) + (r(J)-r(I))\right];$$

and

$$Z \;=\; \sum_{p\in I} R(p) - \sum_{q\in J} R(q), \quad |Z| \;=\; O(n^{19})$$

since $p \le n^3$ for $P \in I \cup J$, and $|I| = |J| = k \le n+2$.

The above and Lemma 1 imply that

$$X \ge 0, \qquad Y \ge \frac{1}{12}, \qquad \text{and} \qquad Z \ge -Cn^{19},$$

for a constant C. Thus, we get

$$\begin{aligned} B &= A + (1-\beta)X + (1-\beta^2)Y + (1-\beta^3)Z \\ &\ge A + \tfrac{1}{12}(1-\beta)^2 - (1-\beta)^3 C n^{19} \\ &\ge A + (1-\beta)^2\tfrac{1}{18} \end{aligned}$$

for all discount factors $\beta \ge 1 - \frac{1}{36Cn^{19}}$. Consequently, for all discount factors satisfying

$$1 - \frac{6n^{1/4}}{2^n} > \beta \ge 1 - \frac{1}{36Cn^{19}}$$

we have $B > 0$, proving the lemma. □

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A Proof of Fact 1

Let

$$f(\beta) = \left(\sum_{i=0}^{p-1} \beta^i\right)^{-1} - \frac{1}{p}.$$

Then

$$\begin{aligned}
f'(\beta) &= -\left(\sum_{i=0}^{p-1} \beta^i\right)^{-2} \left(\sum_{i=0}^{p-1} i\beta^{i-1}\right) \\
f''(\beta) &= -\left(\sum_{i=0}^{p-1} \beta^i\right)^{-2} \left(\sum_{i=0}^{p-1} i(i-1)\beta^{i-2}\right) + 2\left(\sum_{i=0}^{p-1} \beta^i\right)^{-3} \left(\sum_{i=0}^{p-1} i\beta^{i-1}\right)^2 \\
f'''(\beta) &= -\left(\sum_{i=0}^{p-1} \beta^i\right)^{-2} \left(\sum_{i=0}^{p-1} i(i-1)(i-2)\beta^{i-3}\right) + 2\left(\sum_{i=0}^{p-1} \beta^i\right)^{-3} \left(\sum_{i=0}^{p-1} i\beta^{i-1}\right)\left(\sum_{i=0}^{p-1} i(i-1)\beta^{i-2}\right) + \\
&\quad 4\left(\sum_{i=0}^{p-1} \beta^i\right)^{-3} \left(\sum_{i=0}^{p-1} i\beta^{i-1}\right)\left(\sum_{i=0}^{p-1} i(i-1)\beta^{i-2}\right) - 6\left(\sum_{i=0}^{p-1} \beta^i\right)^{-4} \left(\sum_{i=0}^{p-1} i\beta^{i-1}\right)^3.
\end{aligned}$$

In particular, $f(1) = 0$, $f'(1) = -\frac{p-1}{2p}$, $f''(1) = \frac{p^2-1}{6p}$ and for any $0 \le \xi \le 1$, $-f'''(\xi) \le 6\left[\binom{p}{4} + \binom{p}{2}^3\right]$. By Taylor expansion of $f(\beta)$ around $\beta = 1$,

$$f(p) = f(1) - f'(1)(1-\beta) + \frac{f''(1)}{2}(\beta-1)^2 - \frac{f'''(\xi)}{6}(1-\beta)^3,$$

for some $\xi \in [\beta, 1]$. We get (13).

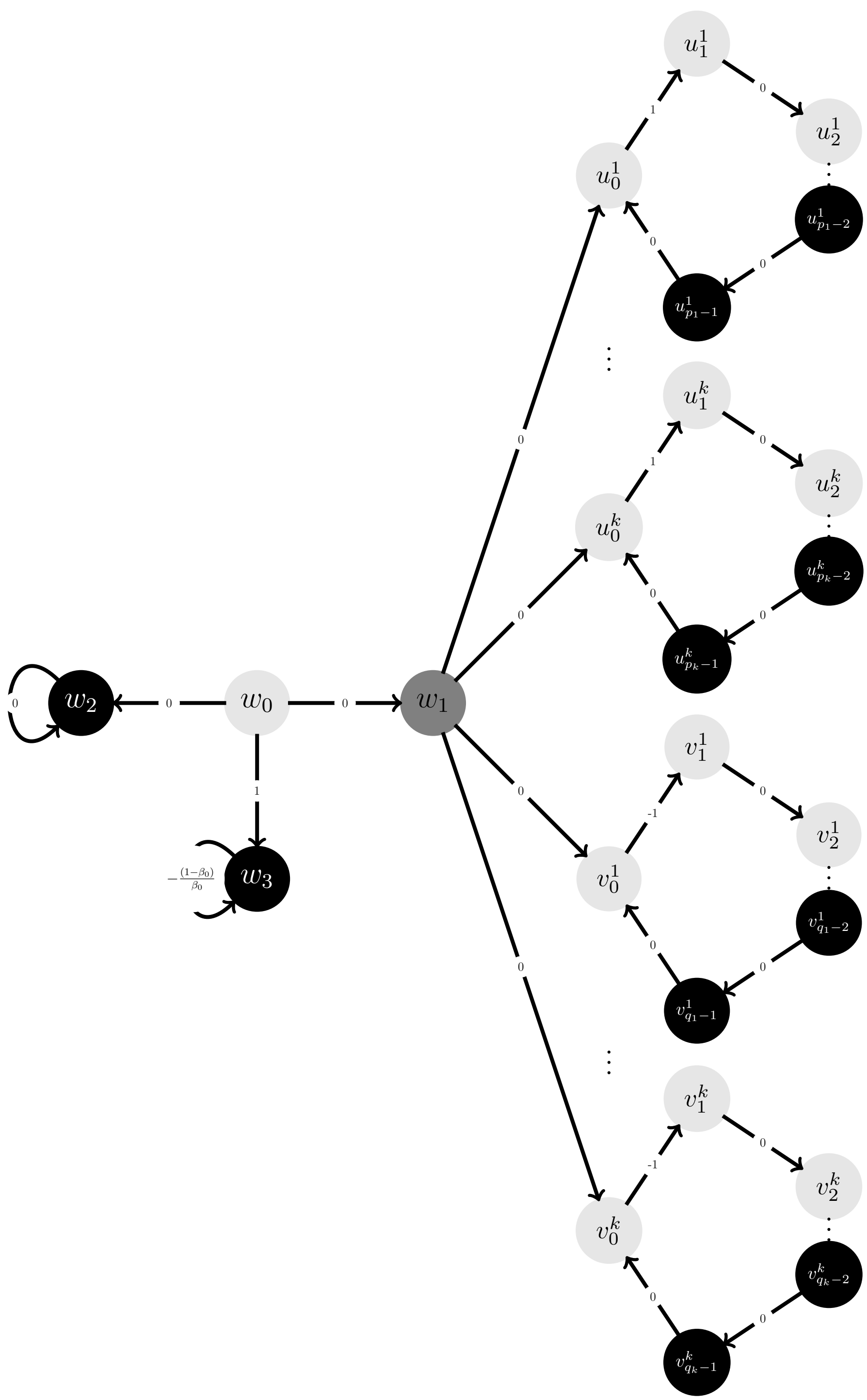

Figure 1: The graph $G = (V, A)$ with some BLACK and WHITE nodes, and a single RANDOM node w_1. The probabilities on the arcs leaving w_1 are all equal to $\frac{1}{2k}$. In fact all nodes could as well be WHITE apart form the single random node w_1.