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Occupation games on graphs in which the second player takes almost all vertices

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Abstract. Given a connected graph $G = (V, E)$, two players take turns occupying vertices $v \in V$ by putting black and white tokens so that the current vertex sets $B, W \subseteq V$ are disjoint, $B \cap W = \emptyset$, and the corresponding induced subgraphs $G[B]$ and $G[W]$ are connected any time. A player must pass whenever (s)he has no legal move. (Obviously, after this, the opponent will take all remaining vertices, since G is assumed connected.) The game is over when all vertices are taken, $V = B^* \cup W^*$. Then, Black and White get $b = |B^*|/|V|$ and $w = |W^*|/|V|$, respectively. Thus, the occupation game is one-sum, $b + w = 1$, and we could easily reduce it to a zero-sum game by simply shifting the payoffs, $b' = b - 1/2, w' = w - 1/2$. Let us also notice that $b \geq 0$ and $w \geq 0$; moreover, $b > 0$ and $w > 0$ whenever $|V| > 1$.
[Let us remark that the so-called Chinese rules define similar payoffs for the classic game of GO, yet, the legal moves are defined in GO differently.]
Like in GO, we assume that Black begins. It is easy to construct graphs in which Black can take almost all vertices, more precisely, for each $\varepsilon > 0$ there is a graph G for which $b > 1 - \varepsilon$. In this paper we show that, somewhat surprisingly, there are also graphs in which White can take almost all vertices.

Keywords: occupation games, GO

1 Graphs in which Black takes almost all vertices

Let $G = (V, E)$ be a n-star, that is, $V = \{v_0, v_1, \dots, v_n\}$ and $E = \{(v_0, v_1), \dots, (v_0, v_n)\}$. By the first move Black takes v_0. Then, obviously, White will get only 1 vertex, while Black gets the remaining n vertices. Hence, $b = n/(n+1) > 1 - \varepsilon$ whenever $1/(n+1) < \varepsilon$.

2 A graph in which White wins

Let G be the 6-cycle with three extra edges pending at every odd vertex of this cycle.

If by the first move Black takes a pending vertex then White takes the (unique) adjacent one and, after this, the remaining 7 vertices too. If Black takes an odd (respectively, even) vertex of the cycle then White takes the opposite (respectively, an adjacent) vertex of the cycle. It is easy to verify that in both cases White can take 5, while Black only 4 vertices.

3 Reduction of weighted version to standard one

Given a graph $G = (V, E)$ let $z : V \to \mathbb{Z}_+$ be a weight function taking integer strictly positive values. The occupation game for the weighted graph (G, z) is defined in a natural way: we just count the total weight of the taken vertices rather than their cardinality, that is, Black and White get $z(B^*)/z(V)$ and $z(W^*)/z(V)$, respectively, where $z(U) = \sum_{v \in U} z(v)$ for each vertex-subset $U \subseteq V$. Obviously, the weighted occupation game (G, w) is still a one-sum game for any weight w and the unit weights $(w(v) \equiv 1 \; \forall v \in V)$ define the standard occupation game.

Let us show that one can reduce the weighted case to the standard one "for free".

Indeed, given pair (G, z), let us slightly modify graph G attaching to every its vertex $v \in V$ a path $p(v)$ that contains $z(v) - 1$ vertices, in addition to v. In particular, we attach nothing to v whenever $z(v) = 1$. Furthermore, we assume that the $|V| + 1$ vertex sets, V and $\{V(p(v)), v \in V\}$ are pairwise disjoint. It is obvious that the original weighted occupation game (G, z) is equivalent to the standard one defined on the obtained extended graph G_z. It is also clear that the reduction is linear, that is, the size of G_z is linear in size of (G, z).

Remark 1 *Strictly speaking, there are more possible moves in G_z than in G. Indeed, there is no move in G corresponding to taking a vertex of $p(v)$ distinct from v in G_z. However, it is easily seen that such moves can become optimal only when all vertices of G_z corresponding to G are already taken. In particular, for each player, Black or White, it is strictly better by the first move to take v rather than some other vertex of $p(v)$.*

Somewhat similarly, in GO players take "the neutral" points first and only when those are completely shared they proceed with occupying their own territory.

Remark 2 *Let us modify slightly the example of Section 2 replacing in it the three pending edges by three pending paths of the same (very large) length ℓ. It is not difficult to verify* *that in the obtained graph White can win almost* $2:1$*; more precisely, for every* $\varepsilon > 0$ *there is an* ℓ *such that* $w > 2/3 - \varepsilon$ *and* $b < 1/3 + \varepsilon$*; see [1].*

4 Graphs in which White takes almost all vertices

Let $(G = (V, E), z)$ be a weighted graph whose vertices are partitioned into two sets, $V = V' \cup V''$, called *important* and *ordinary*, respectively. For any integer n we will construct an example with $|V'| = n + 1$ and such that White can take n of $n + 1$ important vertices. Then, obviously, we obtain the desired construction by choosing $z(v) = 1$ for $v \in V''$ and some sufficiently large $z(v)$ for $v \in V'$.

Let $V = V_n$ be the set of all non-negative integer $(n + 1)$-vectors whose coordinates sum up to n^2.

Remark 3 *Then,* $|V_n| = \binom{n(n+1)}{n}$*. Indeed, Euler showed that* k *can be presented as an ordered sum of* m *non-negative numbers in exactly* $\binom{m+k-1)}{m-1}$ *ways.*

Let us also notice that any vector of V_n has a coordinate of value at least n. Indeed, otherwise the sum of coordinates would be less than n^2, since $(n + 1)(n - 1) = n^2 - 1 < n^2$.

Furthermore, two vertices of V_n are adjacent if the difference of the corresponding two vectors has one coordinate $(+1)$ one (-1), while the remaining $n - 1$ coordinates are 0.

Finally, let the important vertex $v^i \in V'_n$ have the ith coordinate n^2, while the remaining n are 0; obviously, $|V'_n| = n + 1$.

Let by the first move Black takes a vector $v \in V_n$. Let us choose in it a coordinate i_0 of value at least n, reduce this coordinate by n, enlarge n remaining coordinates by 1 each, denote the obtained vector by v', and let White respond to v by v'. By construction, $v' \in V_n$ and v' is strictly closer than v to n important vertices of V'_n; more precisely, to all $v^i \in V'_n$, except for v^{i_0}.

Furthermore, it is easy for White to maintain the last property any time. Indeed, by one move, Black can reduce, by at most 1, the distance from B to at most one important vertex v^i. Then, White responds by reducing the distance between W and v^i by 1, thus, preserving the strict inequality between the considered two distances. Hence, Black can take only one important vertex v^{i_0}, while White takes the remaining n.

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