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On Cumulative Jump Random Variables

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Michael Tortorella

Abstract. Stochastic models for phenomena that can exhibit sudden changes involve the use of processes whose sample functions may have discontinuities. This report provides some tools for working with such processes. We develop a sample path formula for the cumulative jump height over a given time interval. From this formula an expression for the expected value of the cumulative jump random variable is developed under reasonable conditions. The results are applied to finding the expected number of failures in the separate maintenance model over a stated time interval and to the expected number of occurrences of a regenerative event over a stated time interval.

KEY WORDS: Sample paths, cadlag functions, separate maintenance model, regenerative events.
AMS SUBJECT CLASSIFICATION: 60G17, 90B25.

1 INTRODUCTION

1.1 Rationale

Real-valued stochastic processes whose sample functions are continuous from the right and have (finite) limits from the left at each point in their domains are widely used in many operations research and management science contexts, including reliability engineering, finance, manufacturing systems modeling, and queues. Sample functions with this property are called *cadlag* [18]. Many applications require information about the sizes and locations of the discontinuities (jumps) in the process because discrete events of interest take place at these times. For instance, the two-state stochastic process, although simple in structure, finds wide applicability in modeling the reliability of repairable systems; the jumps in this process are the times at which failures (down-jumps) and repair completions (up-jumps) occur. Many useful results for such processes are known (for example, the formula for availability of an alternating renewal process is so widespread in the application area that it is often used even in cases where it may not be appropriate [12]), but results pertaining to jumps are less common.

1.2 Scope

This paper concerns an integral representation for the cumulative jump random variable in the time interval $[0, t]$ of a stochastic process having cadlag sample paths. The cumulative jump random variable is the sum of the magnitudes of all the jumps in a stated time interval. When the jumps are all of unit size, the cumulative jump random variable reduces to the number of jumps in the interval. A formula for the expected number of jumps has been previously developed [13] for two-state stochastic processes having unit jumps; the main contribution of this paper is to generalize this to arbitrary cadlag sample paths and jumps of other than unit size.

1.3 Background

Jump processes are used describe the effects of cumulative damage in reliability models of physical devices subject to random shocks and deterioration [14], [15], [7], [5]. They also find application in financial models [9], [11] where non-smooth sample paths are not exceptional.

In addition, the use of two-state stochastic processes in modeling the reliability of maintained, or repairable, systems is widespread (see, for example, [1], [3], [4], [16]). In the case of an alternating renewal process, computation of the system availability, that is, the probability that the system is in the up state at time t, is a routine application of the renewal argument [8]. For such a process, the expected number of jumps (a failure corresponds to a down-jump in the process) can be expressed in terms of the primitives of the process (see Section 2.2).

In [13], Marlow develops a sample path formula for the number of up-jumps in a two-state stochastic process, provided the process has only discontinuities of the first kind. He then uses the formula to determine the expected number of jumps in a general two-state process, and then obtains a formula for the expected number of jumps in an alternating renewal process and in a regenerative process.

1.4 Synopsis

In this paper, we generalize the sample path formula of Marlow [13] to stochastic processes having cadlag sample functions and at most a finite number of discontinuities in any finite interval. Our generalization involves an integral representation for the cumulative jump random variable (CJRV) that is similar to the Friedrich mollifier procedure [6]. We express the expected value of the cumulative jump random variable in terms of properties of the underlying process. We give sufficient conditions for a general formula and then show that the alternating renewal process satisfied these conditions, yielding a formula for the expected number of jumps in an alternating renewal process that is the same as is obtained by more elementary means (such as the renewal argument). We apply this to the problem of determining the expected number of failures of a monotone, coherent, separately maintained system during a given time period and, as a further example, develop a formula for the expected number of occurrences of a regenerative event in a given time interval.

2 PRIMITIVES AND CONCEPTS

2.1 Cumulative Jump Random Variables for Processes with cadlag Sample Functions

In this paper, we will restrict attention to stochastic processes having, with probability one, cadlag sample functions and only a finite number of discontinuities, none of which are infinite, in any finite interval. Such functions will be called "cadlag+ functions." These conditions are not unreasonably restrictive and permit a wide variety of important applications. The cumulative jump random variables are defined as follows: for $X(\cdot, \omega) : \mathbf{R}^+ \to \mathbf{R}$ a cadlag+ function, define the *jump set* $V_t(X, \omega) = \{x \in [0, t) : X(x, \omega) - X(x^-, \omega) \neq 0\}$, the set of locations in $[0, t)$ at which jumps of X occur. Further define $V_t^-(X, \omega) = \{x \in [0, t) : X(x, \omega) - X(x^-, \omega) < 0\}$, the set of locations in $[0, t)$ at which down-jumps of X occur, and $V_t^+(X, \omega) = \{x \in [0, t) : X(x, \omega) - X(x^-, \omega) > 0\}$, the set of locations in $[0, t)$ at which up-jumps of X occur. The individual jump random variables are defined as $j_\downarrow(u, \omega) = X(u, \omega) - X(u^-, \omega)$ for $u \in V_t^-(X, \omega)$ (down-jumps) and $j_\uparrow(u, \omega) = X(u, \omega) - X(u^-, \omega)$ for $u \in V_t^+(X, \omega)$ (up-jumps). The cumulative up-jump random variable is defined to be $J_\uparrow(t, X, \omega) = \sum_{u \in V_t(X)} \left[X(u, \omega) - X(u^-, \omega) \right]^\wedge$ and the cumulative down-jump random variable is defined to be $J_\downarrow(t, X, \omega) = \sum_{u \in V_t(X)} \left[X(u, \omega) - X(u^-, \omega) \right]^\vee$, where the (slightly unconventional) notation is $a^\vee = a$ if $a < 0$ and $a^\vee = 0$ if $a \geq 0$, and $a^\wedge = a$ if $a > 0$ and $a^\wedge = 0$ if $a \leq 0$. The cumulative jump random variables are defined as $J(t, X, \omega) = J_\downarrow(t, X, \omega) + J_\uparrow(t, X, \omega)$ which aggregates the jump magnitudes without regard to sign and $J_0(t, X, \omega) = J_\uparrow(t, X, \omega) - J_\downarrow(t, X, \omega)$ which aggregates the jump magnitudes with regard to sign.

2.2 Cumulative Jump Random Variables for Two-State Stochastic Processes

A two-state stochastic process is a continuous-time, discrete-state process having only two states usually called 0 and 1 for convenience. We will consider two-state processes that are a special case of the processes described in Section 2.1, so that the sample functions are step functions that are continuous from the right and have limits from the left and at most a finite number of discontinuities (changes of state) in any finite interval. When using such a process as a model for the reliability of a maintainable system, state 0 is usually the "down" state (equipment not working) and state 1 is the "up" state (equipment working). Let the intervals that the process spends in the 1 state be denoted by $U_1, U_2, \ldots$ and the intervals that the process spends in the 0 state be denoted by $D_1, D_2, \ldots$. Then the process alternates between the U-intervals and the D-intervals, giving rise to the synonym "alternating process" for "two-state process." Define $S_1 = U_1$ and $S_n = S_{n-1} + D_{n-1} + U_n$ for $n = 2, 3, \ldots$; these are the times at which down-jumps (failures) occur (for convenience, we define $U_0 = D_0 = S_0 = 0$). A cycle is an interval $D_{n-1} + U_n = S_n - S_{n-1}$; these are the times between failures in the reliability model. Further define

$$Z(t,\omega) = \begin{cases} 0, & t \in \left[S_n(\omega), S_n(\omega) + D_n(\omega)\right),\ n = 1,2,3,\ldots \\ 1, & t \in \left[S_n(\omega) + D_n(\omega), S_{n+1}(\omega)\right),\ n = 1,2,3,\ldots \end{cases}$$

$$= I_{\bigcup_{n=0}^{\infty}\left[S_n(\omega)+D_n(\omega),\, S_{n+1}(\omega)\right)}(t) \quad = \quad \sum_{n=0}^{\infty} I_{\left[S_n(\omega)+D_n(\omega),\, S_{n+1}(\omega)\right)}(t). \tag{2.1}$$

$Z(t, \omega)$ is the indicator that the process is in the up state at time t. In this formulation, we assume the process starts in the 1 state, *i. e.*, $Z(0) = 1$.[1] Because the sample paths are required to be cadlag, the time intervals $U_1, U_2, \ldots$ and $D_1, D_2, \ldots$ comprising the process are closed on the left and open on the right (left-closed, right-open, or LCRO). $J_\downarrow(t, Z, \omega)$ is then the number of down-jumps (*i. e.*, the number of 1-0 transitions) of $Z(t, \omega)$ in the time interval $[0, t)$, $J_\uparrow(t, Z, \omega)$ is the number of up-jumps (*i. e.*, the number of 0-1 transitions) of $Z(t, \omega)$ in the time interval $[0, t)$, and $J(t, Z, \omega)$ is the total number of jumps (both up and down) in $[0, t)$.

The probability that the process is in the up state at time t is $P\{Z(t) = 1\}$, which in the reliability model is called the *point availability* $A(t)$. Similarly, unavailability is $1 - P\{Z(t) = 1\} = P\{Z(t) = 0\}$. The following relation is immediate:

$$J(t,Z,\omega) = \begin{cases} 2J_\downarrow(t,Z,\omega) & \text{if } Z(t) = 1 \\ 2J_\downarrow(t,Z,\omega) - 1 & \text{if } Z(t) = 0 \end{cases} \quad . \tag{2.2}$$

It follows that $EJ(t, Z) = 2EJ_\downarrow(t, Z) - (1 - A(t))$ and $EJ(t, Z)^2 = 4EJ_\downarrow(t, Z)^2 - (1 - A(t))(4EJ_\downarrow(t, Z) - 1)$. Thus, in the two-state case, the moments of $J(t, Z)$ may be obtained from the moments of $J_\downarrow(t, Z)$.

3 INTEGRAL REPRESENTATION FOR CJRVs

In this Section, we generalize the sample path formula developed by Marlow in [13] to arbitrary stochastic processes whose sample paths are cadlag+ with probability 1 (*i. e.*, without restriction to step functions, two states, or jumps of unit size only). Let (Ω, Φ, P) be a probability space and let $X(t, \omega)$ denote a stochastic process on Ω taking values in the metric space $(\mathbf{R}, d)$ where $\mathbf{R} = (-\infty, \infty)$ and $d(x, y) = |x - y|$. Define $\Omega_0 = \{\omega \in \Omega : X(t, \omega) \in \mathcal{D}_0(t)\}$ where $\mathcal{D}_0(t)$ is the set of cadlag+ functions on $[0, t)$. Assume that $\Omega_0 \in \Phi$ and $P(\Omega_0) = 1$.

3.1 Canonical Representation of Functions in $\mathcal{D}_0(t)$

Let $V_t(X) = \{u_0, u_1, u_2, \ldots, u_N\}$. There is no loss of generality in arranging matters so that $V_t^-(X) = \{u_0, u_1, u_2, \ldots, u_M\}$ and $V_t^+(X) = \{u_{m+1}, u_{m+2}, \ldots, u_N\}$ with $u_0 = 0$ and $u_N = t$ (if necessary, assign $j_\uparrow(u_0) = j_\downarrow(u_0) = j_\uparrow(u_N) = j_\downarrow(u_N) = 0$). Then we may write a function $X \in \mathcal{D}_0(t)$ as

[1] This is not essential; all results continue to hold when $Z(0) = 0$, provided the appropriate changes are made.

$$X(x) = \sum_{i=1}^{N} X_i(x) I_{[u_{i-1}, u_i)}(x) \tag{3.1}$$

where X_i is continuous on $[u_i, u_{i+1})$, $i = 0, 1, \ldots, N-1$. Let $\mathcal{D}_1(t)$ denote the set of functions in $\mathcal{D}_0(t)$ for which each X_i is continuously differentiable on $[u_i, u_{i+1})$, $i = 0, 1, \ldots, N-1$. We will call such functions "cadlag++ function." Define $\Omega_1 = \{\omega \in \Omega : X(t, \omega) \in \mathcal{D}_1(t)\}$ and assume that $\Omega_1 \in \Phi$ and $P(\Omega_1) = 1$.

3.2 CJRVs for Processes With cadlag++ Sample Paths

Choose a nonnegative, continuous function k having support on [0, 1] and whose integral over this interval is one. Define

$$k_h(x) = \frac{1}{h} k\left(\frac{x}{h}\right) \tag{3.2}$$

for $x \in [0, 1]$ and $h > 0$. Note that k_h is nonnegative, continuous, has support $[0, h]$, and the integral of k_h over $[0, h]$ is one. Finally, for $\omega \in \Omega_1$ and $h > 0$, define

$$K_h^{\vee} X(t,\omega) = \frac{1}{h} \iint_{0 \le x \le y \le t} k_h(y-x)\left[X(y,\omega) - X(x,\omega)\right]^{\vee} dy\, dx\,. \tag{3.3}$$

Theorem 1, below, shows that $K_{\downarrow} X(t,\omega) = \lim_{h \to 0^+} K_h^{\vee} X(t,\omega)$ is well defined and that $K_{\downarrow}X(t, \omega)$ is equal to the cumulative down-jump random variable of $X(\cdot, \omega)$ over $[0, t)$ plus additional terms related to the change in the $X_i(t)$ over the $[u_i, u_{i+1})$-intervals. The following heuristic argument shows that this is reasonable. Let Y_h be a random variable having the density k_h (note Y_h goes to 0 as $h \to 0$) and suppose $X_u(\omega)$ is a random variable that has a jump of size 1 at $u(\omega)$, $0 < u(\omega) < t$, and is otherwise constant (*i. e.*, $X_u(\omega)$ is the unit step function at $u(\omega)$). Then $[X_u(x + h, \omega) - X_u(x, \omega)]^{\wedge} = I_{[x,\, x+h]}(u(\omega)) = h \times$ a density; in fact, $\frac{1}{h} I_{[x,x+h]}(u)$ is the density of a uniform random variable $U_{u,h}$ on $[u, u + h]$. Then for $h > 0$, $K_h^{\wedge}X(t, \omega)$ is (nearly) the density of the sum of $U_{u,h}$ and Y_h which converges to the unit step function at u as $h \to 0^+$.

Marlow's kernel function [13] is $\lambda e^{-\lambda x} I_{[0,\, \infty)}(x)$ with $\lambda \to +\infty$, which upon change of variable becomes $k(x) = I_{[0,\, 1]}(x)$, applied to step functions having jumps of size 1 and no discontinuities of the second kind.

We first show the method as applied to functions (3.1) in $\mathcal{D}_1(t)$ for which all X_i have derivatives that do not change sign on their domains.

Theorem 1. Let $X : [0, \infty) \to \mathbf{R}$ be a function in $\mathcal{D}_1(t)$ having the representation (3.1) for which the derivative of X_i does not change sign on $[u_i, u_{i+1})$, $i = 0, 1, \ldots, n-1$. Then

$$K_{\downarrow} X(t,\omega) = J_{\downarrow}(t, X, \omega) + \sum_{i=1}^{N}\left[X_i\left(u_i^-, \omega\right) - X_i\left(u_{i-1}, \omega\right)\right]^{\vee}. \tag{3.4}$$

Proof. Choose $h > 0$ and define $W_h = [0,t) - \bigcup_{i=2}^{N-1} [u_i - h, u_i + h)$. W_h is a union of $N - 1$ LCRO intervals, $W_h = \bigcup_{i=1}^{N+1} W_h^i$. Choose i and any LCRO subinterval $[a, b)$ of $W_h^i = [u_{i-1} + h, u_i - h)$. Then, noting that $k_h(y - x) = 0$ unless $x \le y \le x + h$, we have, for all $h < \min\{\{(u_{i+1} - u_i)/3 : i = 1, \ldots, N-1\}, b - a\}$, and dropping the ω to save space,

$$\frac{1}{h} \iint_{([a,b)\times[0,t])\cap\{0\le x\le y\le t\}} k_h(y-x)[X_i(y) - X_i(x)]^\vee \, dy\, dx = \int_a^b \int_x^t k_h(y-x) \frac{[X_i(y) - X_i(x)]^\vee}{h} \, dy\, dx$$

$$= \int_a^b \int_x^{x+h} k_h(y-x) \left[\left(\frac{dX_i}{dx}(x) \right)^\vee + \frac{[X_i(y) - X_i(x)]^\vee}{h} - \left(\frac{dX_i}{dx}(x) \right)^\vee \right] dy\, dx$$

$$= [X_i(b) - X_i(a)]^\vee + \int_a^b \int_x^{x+h} k_h(y-x) \left[\frac{[X(y) - X(x)]^\vee}{h} - \left(\frac{dX_i}{dx}(x) \right)^\vee \right] dy\, dx .$$

Choose $\varepsilon > 0$. Then, because X_i is continuously differentiable at $x \in [a, b) \subset W_h^i$ and does not change sign on $[a, b)$, there is $\delta > 0$ such that $\left| \frac{[X_i(y) - X_i(x)]^\vee}{h} - \left(\frac{dX_i}{dx}(x) \right)^\vee \right| < \varepsilon$ whenever $h < \delta$ and $y \in [x, x + h]$. Because ε is arbitrary, it follows that the remainder term above tends to zero as $h \to 0^+$ and, finally,

$$\lim_{h\to 0^+} \frac{1}{h} \iint_{([a,b)\times[0,t])\cap\{0\le x\le y\le t\}} k_h(y-x)[X_i(y) - X_i(x)]^\vee \, dy\, dx = [X_i(b) - X_i(a)]^\vee . \tag{3.5}$$

Now choose $\eta > 0$ and let $a = u_{i-1} + \eta$ and $b = u_i - \eta$. Then we obtain

$$\lim_{h\to 0^+} \frac{1}{h} \iint_{([a,b)\times[0,t])\cap\{0\le x\le y\le t\}} k_h(y-x)[X_i(y) - X_i(x)]^\vee \, dy\, dx = \left[X_i(u_i^- - \eta) - X_i(u_{i-1} + \eta) \right]^\vee$$

(3.6)

and since η is arbitrary, (3.6) implies that

$$\lim_{h\to 0^+} \frac{1}{h} \iint_{([u_{i-1},u_i)\times[0,t])\cap\{0\le x\le y\le t\}} k_h(y-x)[X_i(y) - X_i(x)]^\vee \, dy\, dx = \left[X_i(u_i^-) - X_i(u_{i-1}) \right]^\vee .$$

(3.7)

Now define $j_\downarrow(u) = [X(u) - X(u^-)]^\vee$. Note that $j_\downarrow(u) = 0$ when u is a point of continuity of X or $u \in V_t^+(X)$ and $j_\downarrow(u_i) = X(u_i) - X(u_i^-)$ for $u_i \in V_t^-(X)$. Then the integral in (3.3) over the region $\{(x, y)$ s.

t. $x \in [u_i - h, u_i + h)$, $y \in [x, t]\}$ can be written as the sum of the integrals over the following six (disjoint) regions:

I. $\{(x, y)$ s. t. $x \in [u_i - h, u_i), y \in [x, u_i)\}$,
II. $\{(x, y)$ s. t. $x \in [u_i - h, u_i), y \in [u_i, u_i + h)\}$,
III. $\{(x, y)$ s. t. $x \in [u_i - h, u_i), y \in [u_i + h, t]\}$,
IV. $\{(x, y)$ s. t. $x \in [u_i, u_i + h), y \in [x, u_i)\}$,
V. $\{(x, y)$ s. t. $x \in [u_i, u_i + h), y \in [u_i, u_i + h)\}$, and
VI. $\{(x, y)$ s. t. $x \in [u_i, u_i + h), y \in [u_i + h, t]\}$.

The integral over region I tends to zero as $h \to 0^+$, as follows: Choose $\varepsilon > 0$. Then the continuity of X_{i-1} on $[u_{i-1}, u_i)$ entails the existence of $\delta > 0$ for which $\left|\left[X_{i-1}(y) - X_{i-1}(x)\right]^\vee\right| < \varepsilon$ whenever $h < \delta$ and $y \in [x, x + h]$. Then

$$\left|\int_{[u_i-h,u_i)}\int_{[u_i-h,u_i)\cap[x,x+h]} k_h(y-x)\left[X(y)-X(x)\right]^\vee\, dy\, dx\right| = \left|\int_{u_i-h}^{u_i}\int_x^{u_i} k_h(y-x)\left[X_{i-1}(y)-X_{i-1}(x)\right]^\vee\, dy\, dx\right|$$

$$\leq \int_{u_i-h}^{u_i}\int_x^{x+h} k_h(y-x)\left|\left[X_{i-1}(y)-X_{i-1}(x)\right]^\vee\right| dy\, dx < h\varepsilon$$

and division by h and the arbitrariness of ε provides the desired result.

The integrals over regions V and VI also tend to zero by similar reasoning. The integral over region III is zero because $y \notin [x, x + h]$ in this region, so that $k_h(y - x) = 0$ in this region. The integral over region IV is zero because the set $\{(x, y)$ s. t. $x \in [u_i, u_i + h), y \in [x, u_i), x \leq y \leq x + h\}$ is empty. The integral over region II is

$$\int_{[u_i-h,u_i)}\int_{[u_i,u_i+h)} k_h(y-x)\left[X(y)-X(x)\right]^\vee\, dy\, dx = \int_{[u_i-h,u_i)}\int_{[u_i,u_i+h)} k_h(y-x)\, j_\downarrow(u_i)\, dy\, dx$$

$$= j_\downarrow(u_i)\int_{[u_i-h,u_i)}\int_{[u_i,u_i+h)} k_h(y-x)\, dy\, dx + \int_{[u_i-h,u_i)}\int_{[u_i,u_i+h)} k_h(y-x)\left[\left[X_i(y)-X_{i-1}(x)\right]^\vee - j_\downarrow(u_i)\right] dy\, dx\,.$$

The first term reduces to

$$j_\downarrow(u_i)\int_{\{u_i-h\leq x<u_i\}}\int_{\{x\leq y<x+h\}} k_h(y-x)\, dy\, dx = j_\downarrow(u_i)\int_{\{u_i-h\leq x<u_i\}} dx \quad = \quad h j_\downarrow(u_i)\,.$$

For the second term, choose $\varepsilon > 0$. Noting that $X_i(u_i) - X_{i-1}(u_i^-) = j_\downarrow(u_i)$ and that X_{i-1} and X_i are continuous on $[u_{i-1}, u_i)$ and $[u_i, u_{i+1})$, respectively, there is $h > 0$ for which

$$\left|\left[X_i(y)-X_{i-1}(x)\right]^\vee - j_\downarrow(u_i)\right| \leq \left|X_i(y)-X_i(u_i^-)\right| + \left|X_{i-1}(u_i)-X_{i-1}(x)\right| < \varepsilon$$

whenever $(x, y) \in [u_i - h, u_i) \times [u_i, u_i + h)$. We then have

$$\left|\frac{1}{h}\int_{[u_i-h,u_i)}\int_{[u_i,u_i+h)} k_h(y-x)\Big(\big[X_i(y)-X_{i-1}(x)\big]^{\vee}-j_{\downarrow}(u_i)\Big)\,dy\,dx\right|$$

$$\le\frac{1}{h}\int_{[u_i-h,u_i)}\int_{[u_i,u_i+h)} k_h(y-x)\Big|\big[X_i(y)-X_{i-1}(x)\big]^{\vee}-j_{\downarrow}(u_i)\Big|\,dy\,dx<\varepsilon\,.$$

Combining the sum of these over $i = 0, 1, \ldots, n$ with the first part of the proof yields the result. ■

When there are changes of sign in the derivatives of X_i the formula (3.4) needs to be augmented. To illustrate, suppose that $X_i'(v) = 0$ with $\mathrm{sgn}(X_i'(v^-)) \neq \mathrm{sgn}(X_i'(v^+))$for some $v \in [u_i, u_{i+1})$. Let $[a, b)$ be an LCRO subinterval of W_h^i containing v. Then

$$\frac{1}{h}\iint_{([a,b)\times[0,t])\cap\{0\le x\le y\le t\}} k_h\left(y-x\right)\big[X_i(y)-X_i(x)\big]^{\vee}\,dy\,dx$$

$$=\int_a^{v}\int_x^t k_h\left(y-x\right)\frac{\big[X_i(y)-X_i(x)\big]^{\vee}}{h}\,dy\,dx\;+\int_{v}^{b}\int_x^t k_h\left(y-x\right)\frac{\big[X_i(y)-X_i(x)\big]^{\vee}}{h}\,dy\,dx\,.$$

As $h \to 0^+$, the first term tends toward $\left[X_i'(\nu^-)-X_i'(a)\right]^{\vee}$ and the second term tends toward $\left[X_i'(b)-X_i'(\nu)\right]^{\vee}$, so each change of sign in the derivative produces an additional term in (3.4). The reader can supply details for particular examples if needed.

All corresponding results hold for up-jumps, defining

$$K_h^{\wedge}X(t,\omega)=\frac{1}{h}\iint_{0\le x\le y\le t} k_h\left(y-x\right)\big[X\left(y,\omega\right)-X\left(x,\omega\right)\big]^{\wedge}\,dy\,dx \tag{3.8}$$

and $K_{\uparrow}X(t,\omega)=\lim_{h\to 0^+}K_h^{\wedge}X(t,\omega)$.

3.3 Processes With Step-Function Sample Paths in $D_0(t)$

When X is a step function in $D_0(t)$, each X_i is constant and equation (3.4) simplifies to

$$K_{\downarrow}X(t,\omega)=J_{\downarrow}(t,X,\omega)\,. \tag{3.9}$$

For up-jumps, we have

$$K_{\uparrow}X(t,\omega)=J_{\uparrow}(t,X,\omega)\,. \tag{3.10}$$

3.4 Two-State Processes

For the two-state process, because $Z(t)$ assumes only the values 0 or 1, $[Z(y, \omega) - Z(x, \omega)]^{\vee} = Z(x, \omega)[1 - Z(y, \omega)]$ for $y \geq x$. It follows then from (3.3) and Theorem 1 that we may write

$$J_{\downarrow}(t, Z, \omega) = \lim_{h \to 0^+} \frac{1}{h} \iint_{0 \le x \le y \le t} k_h(y - x) Z(x, \omega)\left[1 - Z(y, \omega)\right] dy\, dx \tag{3.11}$$

and

$$J_{\uparrow}(t, Z, \omega) = \lim_{h \to 0^+} \frac{1}{h} \iint_{0 \le x \le y \le t} k_h(y - x)\left[1 - Z(x, \omega)\right] Z(y, \omega)\, dy\, dx\,. \tag{3.12}$$

4 EXPECTATION OF CJRV

In this Section, we generalize Marlow's Theorem 1 [13] to processes $X(t, \omega)$ for $\omega \in \Omega_1$.

Theorem 2. Suppose $X \in D_1(t)$ and $E\left|X(x) - X(x^-)\right| < \infty$ for all $x \in [0, t)$. Then

$$EJ_{\downarrow}(t, X) = \lim_{h \to 0^+} \frac{1}{h} \iint_{0 \le x \le y \le t} k_h(y - x) E\left(\left[X(y) - X(x)\right]^{\vee}\right) dy\, dx\,.$$

Proof. Let $\Omega_{m,n} = \left\{\omega \in \Omega \;:\; m - 1 \le \sup_{\substack{x \in [u_{i-1}, u_i) \\ i = 1, \ldots, n}} \left|X_i'(x, \omega)\right| < m,\, N(\omega) = n\right\}$ for m, n = 1, 2, … , and let $Y_{m,n} = I_{\Omega_{m,n}}(\omega)$. Then

$$EJ_{\downarrow}(t, X) = \sum_{m,n=1}^{\infty} \int_{\Omega_{m,n}} \left[\lim_{h \to 0^+} \frac{1}{h} \iint_{0 \le x \le y \le t} k_h(y - x)\left[X(y, \omega) - X(x, \omega)\right]^{\vee} dy\, dx\right] P(d\omega)\,.$$

If $x, y \in [u_{i-1}, u_i)$, then $\exists \zeta$, $x < \zeta < y$, s. t. $X(y) - X(x) = X'(\zeta)(y - x)$. If also $y \in [x, x + h]$ and $\omega \in \Omega_{m,n}$, then $|X(y, \omega) - X(x, \omega)| \le mh$. If $x \in [u_{i-1}, u_i)$ and $y \in [u_i, u_{i+1})$, then

$$X(y) - X(x) = X(y) - X(u_i) + X(u_i) - X(u_i^-) + X(u_i^-) - X(x)$$

$$= X(y) - X(u_i) + j(u_i) + X(u_i^-) - X(x)$$

so that

$$\left|X(y) - X(x)\right| \le \left|X'(\zeta_1)\right|\left|y - u_i\right| + \left|j(u_i)\right| + \left|X'(\zeta_2)\right|\left|u_i - x\right| \le 2mh + \left|j(u_i)\right|$$

for suitable ζ_1, ζ_2. Altogether, then,

$$\left|\frac{1}{h}\iint\limits_{0\le x\le y\le t} k_h(y-x)\left[X(y)-X(x)\right]^{\vee}\,dy\,dx\right| \le$$

$$\le \frac{1}{h}\sum_{i=1}^{n}\int_{W_h^i}\int_x^{x+h} k_h(y-x)\,mh\,dy\,dx + \frac{1}{h}\sum_{i=1}^{n}\int_{u_i-h}^{u_i+h}\int_x^{x+h} k_h(y-x)\left[2mh+\left|j(u_i)\right|\right]dy\,dx$$

$$\le mt + 4m^2h + \sum_{i=1}^{n}\left|j(u_i)\right|$$

which is integrable (P), so

$$E\left[J_{\downarrow}(t,X)\,\middle|\,Y_{m,n}\right] = \lim_{h\to 0^+}\frac{1}{h}\iint\limits_{0\le x\le y\le t} k_h(y-x)\,E\left[\left[X(y)-X(x)\right]^{\vee}\,\middle|\,Y_{m,n}\right]dy\,dx\,, \tag{4.1}$$

the additional exchanges of integrals being permitted by Tonelli's theorem. Then we use the monotone convergence theorem to obtain

$$EJ_{\downarrow}(t,X) = \sum_{m,n=1}^{\infty} E\left[J_{\downarrow}(t,X)\,\middle|\,Y_{m,n}\right]P\left(\Omega_{m,n}\right)$$

$$= \sum_{m,n=1}^{\infty}\lim_{h\to 0^+}\frac{1}{h}\iint\limits_{0\le x\le y\le t} k_h(y-x)\,E\left[\left[X(y)-X(x)\right]^{\vee}\,\middle|\,Y_{m,n}\right]dy\,dx\cdot P\left(\Omega_{m,n}\right)$$

$$= \lim_{h\to 0^+}\frac{1}{h}\iint\limits_{0\le x\le y\le t} k_h(y-x)\sum_{m,n=1}^{\infty} E\left[\left[X(y)-X(x)\right]^{\vee}\,\middle|\,Y_{m,n}\right]P\left(\Omega_{m,n}\right)dy\,dx$$

$$= \lim_{h\to 0^+}\frac{1}{h}\iint\limits_{0\le x\le y\le t} k_h(y-x)\,E\left(\left[X(y)-X(x)\right]^{\vee}\right)dy\,dx\,.$$

This completes the proof. ■

Corollary 3. For a two-state process Z, the expected value of $J_{\downarrow}(t, Z)$ is given by

$$EJ_{\downarrow}(t,Z) = \lim_{h\to 0^+}\frac{1}{h}\iint\limits_{0\le x\le y\le t} k_h(y-x)\,P\left\{Z(x)=1,\,Z(y)=0\right\}dy\,dx\,. \tag{4.2}$$

Proof. $E([X(y)-X(x)]^{\vee}) = P\{Z(x)=1, Z(y)=0\}$ for $x \le y$. ■

If the expectation in Theorem 2 is sufficiently regular, we may obtain a simpler formula that resembles the fundamental theorem of calculus.

Theorem 4. Suppose that $\lim_{h\to 0^+}\frac{1}{h}E\left[X(x+h)-X(x)\right]^{\vee} := \eta(x)$ exists uniformly in *x*, is finite for almost all $x \in [0, t]$, and is integrable on [0, *t*]. Then $EJ_{\downarrow}(t, X) = \int_0^t \eta(x)\, dx$.

Proof. For $h > 0$, we may write

$$\frac{1}{h}\int_0^t\int_x^{x+h} k_h(y-x)E\left(\left[X(y)-X(x)\right]^{\vee}\right) dy\, dx = \int_0^t\int_x^{x+h} k_h(y-x)\eta(x)\, dy\, dx +$$

$$+\int_0^t\int_x^{x+h} k_h(y-x)\left[\frac{1}{h}E\left(\left[X(y)-X(x)\right]^{\vee}\right)-\eta(x)\right] dy\, dx.$$

The second term is bounded above in absolute value by

$$\int_0^t\int_x^{x+h} k_h(y-x)\left|\frac{1}{h}E\left(\left[X(y)-X(x)\right]^{\vee}\right)-\eta(x)\right| dy\, dx.$$

Choose $\varepsilon > 0$. Then there is a $\delta > 0$ such that for all $h < \delta$ and $y \in [x, x + h]$, $\left|\frac{1}{h}E\left(\left[X(y)-X(x)\right]^{\vee}\right)-\eta(x)\right| < \varepsilon / t$. Then the remainder term is bounded above by $\int_0^t\int_x^{x+h} k_h(y-x)\left(\varepsilon/t\right) dy\, dx = \varepsilon$. Because ε is arbitrary, the remainder term tends to zero as $h \to 0^+$. It follows that

$$\lim_{h\to 0^+}\frac{1}{h}\int_0^t\int_x^{x+h} k_h(y-x)E\left(\left[X(y)-X(x)\right]^{\vee}\right) dy\, dx = \lim_{h\to 0^+}\int_0^t\int_x^{x+h} k_h(y-x)\eta(x)\, dy\, dx = \int_0^t \eta(x)\, dx.$$

■

For a two-state process *Z*, define $\xi_{\downarrow}(x, h) = P\{Z(x) = 1, Z(x + h) = 0\}$ for $0 \le x \le t - h$ and $\xi_{\downarrow}(x, h) = 0$ for $t - h \le x \le t$. The analogous result is

Corollary 5. Suppose that $L_{\downarrow}(x) = \lim_{h\to 0^+}\frac{\xi_{\downarrow}(x,h)}{h}$ exists uniformly for almost all $x \in [0, t]$ and that $\lim_{h\to 0^+}\frac{1}{h}\int_0^{t-h}\xi_{\downarrow}(x,h)\, dx$ exists, is finite, and is integrable on [0, *t*]. Then

$$EJ_{\downarrow}(t) = \int_0^t L_{\downarrow}(x)\, dx.$$

Proof. $\eta(x) = L_{\downarrow}(x)$. ■

Analogous results hold for the cumulative up-jump random variable, using

$$K_{\uparrow}X(t,\omega) = \lim_{h\to 0^+} \frac{1}{h} \iint_{0\le x\le y\le t} k_h\left(y-x\right)\left[X\left(y,\omega\right)-X\left(x,\omega\right)\right]^{\wedge} dy\, dx \,. \tag{4.3}$$

4.1 Expected Number of Jumps for an Alternating Renewal Process

Let $\{Z(t) : t \ge 0\}$ denote a nonlattice alternating renewal process having the two states 0 and 1 where the definitions are arranged so that the sample paths are cadlag. The notation used here is that of Section 2.2, and in addition $\{U_1, U_2, \ldots\}$ and $\{D_1, D_2, \ldots\}$ form mutually stochastically independent renewal processes in which the distribution of U_1 is denoted by F and that of D_1 is denoted by G. We assume $Z(0) = 1$ w. p. 1; everything that follows is also valid, making the appropriate changes, for $Z(0) = 0$ w. p. 1. Denoting by $A(t)$ the probability that the process is in state 1 at time t, we obtain, as in Section 2.2, $EJ(t, Z) = 2EJ_{\downarrow}(t, Z) - (1 - A(t))$. Therefore, it suffices to determine $EJ_{\downarrow}(t, Z)$.

Theorem 5. For the alternating renewal process described above, in which in addition F and G have densities on $[0, t]$, the expected number of jumps in $[0, t]$ is given by $F(t) + M_H(t) + F * M_H(t)$.

Proof. For $h > 0$ sufficiently small and $x \in [0, t-h]$,

$$\left\{Z(x)=1, Z(x+h)=0\right\} = \left\{x < U_1 \le x+h\right\} \cup \bigcup_{n=1}^{\infty} \left\{x < S_n + U_{n+1} \le x+h\right\}.$$

Then, letting $H = F{*}G$ and M_H denote the renewal function for H,

$$P\left\{Z(x)=1, Z(x+h)=0\right\} = P\left\{x<U_1 \le x+h\right\} + \sum_{n=1}^{\infty} P\left\{x < S_n + U_{n+1} \le x+h\right\}$$

$$= F(x+h) - F(x) + \sum_{n=1}^{\infty}\left[H_n * F(x+h) - H_n * F(x)\right]$$

$$= F(x+h) - F(x) + F * \sum_{n=1}^{\infty}\left[H_n(x+h) - H_n(x)\right]$$

$$= F(x+h) - F(x) + F * \left[M_H(x+h) - M_H(x)\right].$$

.

Divide by h and let $h \to 0^+$ to obtain

$$L_{\downarrow}(x) = \lim_{h\to 0^+} \frac{1}{h} P\left\{Z(x+h)=0, Z(x)=1\right\} = f(x) + F * m_H(x)$$

where f is the density of F and m_H is the renewal density for $H = F{*}G$. It follows from Corollary 5 that

$$EN_{\downarrow}(t) = \int_0^t \left[f(x) + F * m_H(x) \right] dx = F(t) + F * M_H(t) \tag{4.4}$$

and so finally

$$EN(t) = 2\left[F(t) + F * M_H(t) \right] + A(t) - 1 = F(t) + M_H(t) + F * M_H(t)\,, \tag{4.5}$$

noting that $A(t) = 1 - F(t) + [1 - F]*M_H(t)$ [1]. Certainly this result could have been obtained more simply by the standard renewal argument, for example. Its inclusion here gives perhaps the simplest illustration of the formalism developed above.

4.2 Expected Number of Failures in the Separate Maintenance Model

The separate maintenance model in the mathematical theory of reliability has been considered by several authors [3], [4], [16], and others, and computational procedures for the availability of a system of separately maintained components have been developed [17]. The following offers an approach to computing the expected number of failures in this model.

Suppose a complex system is composed of C components, or subassemblies, that are each separately maintained. That is, there is a two-state process $Z_i(t)$ that describes the functioning and non-functioning times of component i for $i = 1, \ldots, C$, and the processes $Z_1(t), \ldots, Z_C(t)$ are mutually stochastically independent. If φ is the (monotone, coherent) structure function [1] of the system, then $Z(t) = \varphi(Z_1(t), \ldots, Z_C(t))$ is a two-state process that describes the functioning and non-functioning times of the whole system. The system availability (Section 2.2), $P\{Z(t) = 1\} = EZ(t)$ is readily computed from $EZ(t) = \varphi(EZ_1(t), \ldots, EZ_C(t))$ [3]. The expected number of failures of the system may be computed using the methods of theorems 1 and 2.

Define

$$L_{\downarrow}(x) = \lim_{h \to 0^+} \frac{1}{h} P\left\{ Z(x) = 1, Z(x+h) = 0 \right\}$$

$$= \lim_{h \to 0^+} \frac{1}{h} P\left\{ \varphi\left(Z_1(x), \ldots, Z_C(x)\right) = 1, \varphi\left(Z_1(x+h), \ldots, Z_C(x+h)\right) = 0 \right\}.$$

Then the expected number of system failures is given by $\int_0^t L_{\downarrow}(x)\, dx$ (Corollary 5).

To illustrate the application of this result, we consider an C-unit parallel system [1] for which the structure function is

$$\varphi\left(Z_1, \ldots, Z_C\right) = 1 - \prod_{i=1}^{C} \left(1 - Z_i\right). \tag{4.6}$$

Denote by $N_{\downarrow}(t, Z_i)$ the number of down-jumps of Z_i in $[0, t)$; this is the number of failures of unit i in $[0, t)$. In this parallel system, a failure of component i at a given time causes a system failure at that time only if all the other units are in the down state at that time. Let $\{S_{ni} : n = 1, 2, \ldots\}$

denote the sequence of failure times of unit i. Then the number of unit i failures that cause a system failure over the interval [0, t) is given by

$$N_i^*(t) = \sum_{n=1}^{\infty}\left[\prod_{\substack{j=1\\ j\neq i}}^{C}\left(1 - Z_j\left(S_{ni}\right)\right)\right] I\left\{S_{ni} \le t\right\} = \int_0^t \left[\prod_{\substack{j=1\\ j\neq i}}^{C}\left(1 - Z_j\left(u\right)\right)\right] dN_{\downarrow}\left(u, Z_i\right). \qquad (4.7)$$

In the separate maintenance model, the processes Z_1, …, Z_C are mutually stochastically independent, so that

$$EN_i^*(t) = \int_0^t \left[\prod_{\substack{j=1\\ j\neq i}}^{C}\left(1 - A_j\left(u\right)\right)\right] dEN_{\downarrow}\left(u, Z_i\right), \qquad (4.8)$$

where $A_j(u) = P\{Z_j(u) = 1\}$ is the availability for unit j (j = 1, …, C). From (4.4), $EN_{\downarrow}(t, Z_i) = \int_0^t \left[f_i(x) + F_i * m_{H_i}(x)\right] dx = F_i(t) + F_i * M_{H_i}(t)$ where the subscript i indicates the primitives of the alternating renewal process for unit i, i = 1, …, C. With $N_{\downarrow}(t, Z)$ denoting the number of system failures in [0, t], we have

$$EN_{\downarrow}\left(t, Z\right) = \sum_{i=1}^{C} EN_{\downarrow}\left(t, Z_i\right) = \sum_{i=1}^{C}\int_0^t \left[\prod_{\substack{j=1\\ j\neq i}}^{C}\left(1 - A_j\left(u\right)\right)\right]\left[f_i(u) + F_i * m_{H_i}(u)\right] du . \qquad (4.9)$$

Finally, noting that the system availability is $A\left(t\right) = 1 - \prod_{i=1}^{C}\left[1 - A_i\left(t\right)\right]$, (4.9) reduces to

$$EN_{\downarrow}\left(t, Z\right) = \int_0^t \left[1 - A\left(u\right)\right]\frac{A_i\left(u\right)}{1 - A_i\left(u\right)}\left[f_i(u) + F_i * m_{H_i}(u)\right] du . \qquad (4.10)$$

A similar procedure can be followed for other monotone coherent structure functions.

4.3 Regenerative Events

As a further illustration, we develop an expression for the expected number of times a regenerative event occurs in the time interval [0, t). This material is due to N. A. Marlow [13].

Let (Ω, $\mathcal{F}$, P) be a probability space and $\mathcal{R} = \{R(x) : x > 0\}$ be a standard regenerative event in the sense of Kingman [10], with $p(x) = P(R(x))$ for $x > 0$. A standard regenerative event is one for which $R(x) \in \mathcal{F}$ for all $x > 0$, $\lim_{x \to 0^+} p(x) = 1$, and

$$P\left(\bigcap_{i=1}^{n} R\left(x_i\right)\right) = \prod_{i=1}^{n} p\left(x_i - x_{i-1}\right)$$

for all $0 = x_0 < x_1 < \cdots < x_n$. We define $p(0) = 1$. Kingman [10] shows that p is absolutely continuous with respect to Lebesgue measure on $[0, t]$ for $t > 0$ and that the limit

$$\lambda = \lim_{h\to 0^+} \frac{1-p(h)}{h} \tag{4.11}$$

exists. We will confine ourselves to the stable case $\lambda < \infty$ ([10], p. 187).

Define $Z(x, \omega)$ to be the indicator process $Z(x, \omega) = \{I_{R(x)}(\omega) : x > 0\}$ and arrange matters so that the sample paths of Z are cadlag.

Theorem 6. Let $\{R(x) : x > 0\}$ be a standard, stable regenerative event and $Z(x, \omega) = I_{R(x)}(\omega)$. Then

$$EJ_{\uparrow}(t,Z) = p(t) - 1 + \lambda\int_{[0,t)} p(u)\, du \quad \text{and} \quad EJ_{\downarrow}(t,Z) = \lambda\int_{[0,t)} p(u)\, du\,. \tag{4.12}$$

Proof. For $h > 0$, define $\xi_{\uparrow}(x, h) = P\{Z(x) = 0, Z(x + h) = 1\}$ for $0 \le x \le t - h$ and $\xi_{\uparrow}(x, h) = 0$ for $t - h \le x \le t$. Then $\xi_{\uparrow}(x, h) = p(x + h) - p(x)p(h)$ for $0 \le x \le t - h$ and $\xi_{\uparrow}(x, h) = 0$ for $t - h \le x \le t$. Therefore,

$$\frac{1}{h}\xi_{\uparrow}(x,h) = \frac{1}{h}\left[p(x+h) - p(x)\right] + \frac{1}{h}\left[1 - p(h)\right]p(x), \quad 0 \le x \le t - h\,.$$

By the absolute continuity of p, the limit

$$L_{\uparrow}(x) = \lim_{h\to 0^+}\frac{1}{h}\xi_{\uparrow}(x,h) = p'(x) + \lambda p(x)$$

exists and

$$\int_{[0,t)} L_{\uparrow}(u)\, du = p(t) - 1 + \lambda\int_{[0,t)} p(u)\, du\,.$$

Next,

$$\frac{1}{h}\int_0^{t-h}\xi_{\uparrow}(u,h)\, du = \frac{1}{h}\int_0^{t-h}\left[p(u+h) - p(u)\right] du + \frac{1}{h}\int_0^{t-h}\left[1 - p(h)\right]p(u)\, du\,.$$

Continuity of p on $[0, t]$ implies that

$$\lim_{h\to 0^+}\frac{1}{h}\int_0^{t-h}\xi_{\uparrow}(u,h)\, du = p(t) - 1 + \lambda\int_{[0,t)} p(u)\, du < \infty\,.$$

It follows from the up-jump version of Corollary 5 that the first part of the Theorem is established. A similar argument establishes the second part of the Theorem. ■

5 COUNTING THE NUMBER OF JUMPS

If all jumps are of unit size (as, for example, in the two-state stochastic process or a birth-death process), the cumulative jump random variable also counts the number of jumps. In general, there still may be interest in counting the number of jumps even when the jumps are not of unit size. The observation that enables us to do this is that we can count the number of intervals that contain a single jump and this will equal the number of jumps.

Let $J_\downarrow(t, X, \omega)$ denote the number of down-jumps of a cadlag+ function $X(\cdot, \omega)$. Divide the interval into two equal parts $[0, t/2)$ and $[t/2, t)$. From the definition, it is clear that the cumulative down-jump random variable over $[t/2, t)$ is equal to $J_\downarrow(t, X, \omega) - J_\downarrow(t/2, X, \omega)$ and may reasonably be denoted by $J_\downarrow(t/2, t; X, \omega)$. Continue bisection in this manner and define

$$B_\downarrow^k(t, X, \omega) = \sum_{j=0}^{k-1} I\left\{ J_\downarrow\left(\frac{jt}{2^k}, \frac{(j+1)t}{2^k}; X, \omega \right) \geq 1 \right\}, \quad k = 1, 2, \ldots. \tag{6.1}$$

$B_k(t, X, \omega)$ counts the number of intervals in the k^{th} bisection in which there is at least one jump of X. Then $J_\downarrow(t, X, \omega) = \min\{k : B_\downarrow^k(t, X, \omega) = B_\downarrow^{k+1}(t, X, \omega)\}$. That is, if the number of intervals containing a jump does not change when the next bisection step is performed, the number of down-jumps of X is equal to the number of intervals containing a down-jump in the current bisection step. The same method may be applied to up-jumps.

6 CONCLUSION

We have augmented and generalized the work of Marlow [13] by developing an integral representation, inspired by the Friedrich mollifier method, for the cumulative jump random variables associated with a stochastic process having cadlag sample paths and at most a finite number of discontinuities in any finite interval. The analysis proceeds on a sample path basis, so the question of what kinds of stochastic processes this may apply to is settled by examining the sufficient conditions for theorems 2 and 4. For example, the analysis does apply to a regular birth-and-death process because such a process has step-function sample paths and $E\left|X(x) - X(x^-)\right| \leq 1$. It is of considerable interest to find expressions for higher moments of the cumulative jump random variables.

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APPENDIX

In this Appendix, we show how Marlow's original computation [13] generalizes to yield the number of jumps, regardless of size, of a *cadlag* step function of bounded variation.

Suppose $0 = u_0 < u_1 < \cdots < u_n < t$ and $f(x) = \sum_{i=1}^{n} \alpha_i I_{[u_{i-1}, u_i)}(x)$ for $x \in [0, t]$. Then

$$\iint_{0 \le x \le y \le t} \lambda^2 e^{-\lambda(y-x)} \left[f(y) - f(x) \right]^{\vee} dy\, dx = \sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} \int_x^t \lambda^2 e^{-\lambda(y-x)} \left[f(y) - \alpha_i \right]^{\vee} dy\, dx$$

because the integral over $[u_n, t]$ is zero by virtue of f being a step function. Then this becomes

$$\sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} \left[\int_x^{u_i} + \sum_{j=i+1}^{n} \int_{u_{j-1}}^{u_j} + \int_{u_n}^{t} \right] \lambda^2 e^{-\lambda(y-x)} \left[f(y) - \alpha_i \right]^{\vee} dy\, dx .$$

The integral from x to u_i is zero because $f(y) = \alpha_i$ for all $y \in [u_{i-1}, u_i)$. What remains is equal to

$$\sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} \sum_{j=i+1}^{n} \int_{u_{j-1}}^{u_j} \lambda^2 e^{-\lambda(y-x)} \left[f(y) - \alpha_i \right]^{\vee} dy\, dx + \sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} \int_{u_n}^{t} \lambda^2 e^{-\lambda(y-x)} \left[f(y) - \alpha_i \right]^{\vee} dy\, dx .$$

The first term is

$$\lambda^2 \sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} e^{\lambda x} \sum_{j=i+1}^{n} \left[\alpha_j - \alpha_i \right]^{\vee} \int_{u_{j-1}}^{u_j} e^{-\lambda y}\, dy\, dx = \lambda \sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} e^{\lambda x} \sum_{j=i+1}^{n} \left[\alpha_j - \alpha_i \right]^{\vee} \left[e^{-\lambda u_{j-1}} - e^{-\lambda u_j} \right] dx$$

$$= \lambda \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left[\alpha_j - \alpha_i \right]^{\vee} \left[e^{-\lambda u_{j-1}} - e^{-\lambda u_j} \right] \int_{u_{i-1}}^{u_i} e^{\lambda x}\, dx$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left[\alpha_j - \alpha_i \right]^{\vee} \left[e^{-\lambda u_{j-1}} - e^{-\lambda u_j} \right] \left[e^{\lambda u_i} - e^{\lambda u_{i-1}} \right]$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left[\alpha_j - \alpha_i \right]^{\vee} \left[e^{-\lambda(u_{j-1} - u_i)} - e^{-\lambda(u_j - u_i)} - e^{-\lambda(u_{j-1} - u_{i-1})} + e^{-\lambda(u_j - u_{i-1})} \right].$$

As $\lambda \to +\infty$, the only term in the inner sum that survives is the $j = i + 1$ term, and this is equal to

$$\sum_{i=1}^{n} \left(\alpha_{i+1} - \alpha_i \right)^{\vee} .$$

The second term is

$$\lambda^2 \sum_{i=1}^{n} [\alpha_n - \alpha_i]^\vee \int_{u_{i-1}}^{u_i} e^{\lambda x} \int_{u_n}^{t} e^{-\lambda y}\, dy\, dx = \lambda \sum_{i=1}^{n} [\alpha_n - \alpha_i]^\vee \left[e^{-\lambda u_n} - e^{-\lambda t} \right] \int_{u_{i-1}}^{u_i} e^{\lambda x}\, dx$$

$$= \sum_{i=1}^{n} [\alpha_n - \alpha_i]^\vee \left[e^{-\lambda u_n} - e^{-\lambda t} \right] \left[e^{\lambda u_i} - e^{\lambda u_{i-1}} \right]$$

$$= \sum_{i=1}^{n} [\alpha_n - \alpha_i]^\vee \left[e^{-\lambda(u_n - u_i)} - e^{-\lambda(t-u_i)} - e^{-\lambda(u_n - u_{i-1})} + e^{-\lambda(t-u_{i-1})} \right]$$

and this term goes to zero as $\lambda \to +\infty$. Finally, we obtain

$$\lim_{\lambda \to +\infty} \iint_{0 \le x \le y \le t} \lambda^2 e^{-\lambda(y-x)} [f(y) - f(x)]^\vee \, dy\, dx = \sum_{i=1}^{n} (\alpha_{i+1} - \alpha_i)^\vee ,$$

as desired.