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MAXIMAL CONDORCET DOMAINS

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RRR 4-2011, MARCH 2, 2011

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Rutcor Research Report
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Maximal Condorcet domains

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Abstract. A Condorcet domain is a subset of the set of linear orders on a finite set of candidates (alternatives to vote), such that if voters preferences are linear orders belonging to such a subset, then the simple majority rule does not yield cycles. It is well-known that the set of linear orders $\mathcal{LO}$ is the Bruhat lattice. We prove that a maximal Condorcet domain is a distributive sublattice in the Bruhat lattice. We give an explicit lattice formula for the simple majority rule. We introduce the notion of symmetric Condorcet domains and give a description of symmetric Condorcet domains of maximal size.

Keywords: Bruhat order, distributive lattice, binary plane tree, simple majority rule

1 Introduction

A Condorcet domain is a subset of the set of linear orders on a finite set of candidates (alternatives to vote), such that if voters preferences are linear orders belonging to such a subset, then the simple majority rule does not yield cycles. We use the abbreviation CD for a Condorcet domain. A CD is *maximal* if it is not possible to add a linear order outside of the CD such that the expanded set is a CD.

In this paper we study normal CDs on the set $[n] = \{1, ..., n\}$. A CD is *normal* if it contains the natural linear order $\alpha = (1 < 2 < ... < n)$ and the inverse to it, the 'anti-natural' linear order $\omega = (n < n-1 < ... < 1)$.

One of the first interesting examples of normal CDs was constructed by Black [3]. Namely he shown that the set of single-peaked linear orders forms a normal CD. Other examples of normal CDs were constructed and studied in [1, 4, 9]; [11] gives a state of art of the theory. In [7] and then in [1, 4, 6], it was revealed connections between normal CDs and the lattice structure on the set $\mathcal{LO}$, so-called the Bruhat lattice. Specifically, Chameni-Nembua in [4] shown that if a CD is a sublattice of the Bruhat lattice then it is a distributive sublattice.

In the paper we prove that any maximal normal CD is a sublattice (hence, a distributive sublattice) of the Bruhat lattice.

We provide an explicit lattice formula for the simple majority rule when preferences of voters belong to a normal CD. In particular, this formula shows that the linear order, corresponding to the majority rule, belongs to sublattice generated by the preferences of voters. Hence, if preferences of all voters belong to a maximal normal Condorcet domain $\mathcal{D}$ then the majority rule ordering belongs to $\mathcal{D}$ as well.

We apply these results to a study of symmetric CDs. A symmetric CD is a domains such that if it contains a linear order, then it contains the opposite linear order as well. We show that any symmetric CD on the set $[n]$ has at most 2^{n-1} elements. We obtain a complete description of symmetric CDs of maximal size 2^{n-1}. A structure of maximal symmetric CDs of smaller size is an open problem. We show that, for every $2 \leq m < n$, there exists a maximal CD of size 2^m.

In Section 2 we recall the notion of weak Bruhat order on the set $\mathcal{LO}_n$ of linear orders on $[n]$ and that the corresponding poset is the Bruhat lattice. In Section 3 we define a key notion of compatibility of linear orders, for the first time considered by Chameni-Nembua in [4]. We prove that any maximal clique (a maximal collection of pairwise compatible linear orders) is a distributive sublattice of the Bruhat lattice. In Section 4 we give an explicit lattice formula for the simple majority rule when preferences of voters belong to a clique. As a consequence we obtain that a maximal normal CD is a distributive lattice. We also get a bijection between normal CDs and cliques. This bijection was established in [4]. We give an example of non-normalizable CD. Section 5 is devoted to general facts on symmetric CDs. In Section 6 we provide a classification of maximal size symmetric CDs in term of plane binary trees. In the last section we construct a maximal symmetric CDs of size 4 for every $n \geq 3$.

2 Bruhat lattice

Fix a natural number n and denote $[n] = \{1, ..., n\}$. A (strict) *linear order* on $[n]$ is a complete irreflexive transitive binary relation $<$ on $[n]$. One can identify a linear order and a word $x_1x_2...x_n$ of n different symbols from $[n]$; $x_1 < x_2 < ... < x_n$. We denote by $\mathcal{LO}_n$ (or $\mathcal{LO}$ if there is no needs to indicate the ground set) the set of linear orders on $[n]$. Elements of $\mathcal{LO}$ are denoted by Greek letters like σ, τ and so on; we use also the notation $<_\sigma$. For a linear order σ we denote by σ° the opposite linear order, that is $x <_\sigma y$ if and only if $y <_{\sigma^\circ} x$. The set $[n]$ possesses two distinguish linear orders: the identical $1 < 2 < ... < n$, which we denoted α, and the opposite to α, which we denoted ω.

Let $\Omega = \Omega_n = \{(i, j), 1 \leq i < j \leq n\}$. A pair (i, j) from Ω is called an *inversion* of a linear order σ if $j <_\sigma i$, that is if σ inverses the natural relation $i < j$. The set of all inversions of σ is denoted $Inv(\sigma)$; it is a subset of Ω. For example, $Inv(\alpha)$ is empty, whereas $Inv(\omega)$ is the whole Ω. In general, $Inv(\sigma^\circ) = \Omega - Inv(\sigma)$.

The mapping Inv gives an embedding of the set of linear orders $\mathcal{LO}$ into the set of subsets of Ω. To describe the image of this mapping we need some notions.

Definition. A subset S of Ω is *transitive* if $(i, j) \in S$ and $(j, k) \in S$ implies $(i, k) \in S$. A set $S \subset \Omega$ is *co-transitive* if $(i, k) \in S$ implies that, for every j between i and j, either $(i, j) \in S$ or $(j, k) \in S$.

Note that the union of co-transitive sets is co-transitive, and the intersection of transitive sets is transitive.

Lemma 1. *For a subset S of Ω the following assertions are equivalent:*
(i) *S is transitive and co-transitive (for short, TcoT);*
(ii) *$S = Inv(\sigma)$ for some linear order σ.*

The implication (ii) $\Rightarrow$ (i) is obvious. The inverse implication is proved by an explicit construction of the corresponding linear order σ. Namely, we set $i <_\sigma j$ if either $i < j$ and $(i, j) \notin S$ or $j < i$ and $(j, i) \in S$. $\square$

Thus, the set $\mathcal{LO}$ of linear orders on $[n]$ can be identified with the set of TcoT-subsets in Ω. Because of this, one can compare linear orders via the inclusion of their inversion sets.

Definition. For linear orders σ and τ, $\sigma \ll \tau$ if $Inv(\sigma) \subseteq Inv(\tau)$. The relation $\ll$ is the *weak Bruhat order.*

This is one of equivalent definitions of the weak Bruhat order, see, for example, [2]. The set $\mathcal{LO}$ endowed with the weak Bruhat order is a poset $(\mathcal{LO}, \ll)$. The linear order α is the minimal element of the poset whereas ω is the maximal one. The poset $(\mathcal{LO}, \ll)$ is a lattice indeed, see, for example [2, 8]. We show this by the use of the following lemma.

Lemma 2. *Let S be a co-transitive subset in Ω. Then the transitive closure of S is co-transitive as well.*

Proof. Let R be the transitive closure of S. By definition, a pair (i, i') belongs to R if there exists a chain $i = i_0 < i_1 < \cdots < i_p = i'$ such that every neighbor pair $(i_s, i_s + 1)$ belongs to S.

Suppose now that $(i,k) \in R$ and $i < j < k$. We have to prove that either $(i,j) \in R$ or $(j,k) \in R$.

Let $i_0, \ldots, i_p$ be a chain as above which connects i and k. If j is one of the nodes of this chain then (i,j) and (j,k) belong to R. If not then j lies inside some link of the chain, $i_s < j < i_{s+1}$. The pair (i_s, i_{s+1}) belongs to S. Due to the co-transitivity of S, we have either $(i_s, j) \in S$ or $(j, i_{s+1}) \in S$. In the first case we obtain the chain $i = i_0 < \cdots < j_s < j$ connecting i with j and giving $(i,j) \in R$. In the second case we have a chain from j to k and $(j,k) \in R$. $\square$

For linear orders σ and τ, denote $S = Inv(\sigma)$ and $T = Inv(\tau)$. Let R be the transitive closure of $S \cup T$. Since $S \cup T$ is co-transitive, the set R is transitive and co-transitive due to Lemma 2. Therefore R is the inversion set for some linear order ρ. Obviously, $\sigma \ll \rho$ and $\tau \ll \rho$. Now, suppose that ρ' is a linear order such that $\sigma, \tau \ll \rho'$. Let $R' = Inv(\rho')$. Due to $\sigma \ll \rho'$ we have $S \subseteq R'$; similarly $T \subseteq R'$, hence $S \cup T \subseteq R'$. Since R' is a transitive set, R' contains the transitive closure of $S \cup T$ whence $\rho \ll \rho'$.

This proves the existence of the join $\sigma \vee \tau$ in the poset $(\mathcal{LO}, \ll)$. The existence of the meet follows by the formula $\sigma \wedge \tau = (\sigma^\circ \vee \tau^\circ)^\circ$.

Thus, the poset $(\mathcal{LO}, \ll)$ is a lattice called the *Bruhat lattice*. The operation $\sigma \mapsto \sigma^\circ$ is an ortho-complementation of the lattice, that is an antitone involution such that $\sigma \vee \sigma^\circ = \omega$ for every σ.

3 Chameni-Nembua relation

In general case, the set $Inv(\sigma \vee \tau)$ is strictly larger than $Inv(\sigma) \cup Inv(\tau)$. Similarly the set $Inv(\sigma \wedge \tau)$ can be strictly smaller than $Inv(\sigma) \cap Inv(\tau)$.

Example 1. Let $n = 3$, $\sigma = 213$, $\tau = 132$. σ has one inversion $(1,2)$, τ also has one inversion $(2,3)$. However the set $\{(1,2),(2,3)\}$ is not transitive since it does not contain the pair $(1,3)$. If we add this pair, we obtain $\sigma \vee \tau = \omega$. Similarly, $Inv(\sigma^\circ) \cap Inv(\tau^\circ)$ consists of the single pair $(1,3)$ and is not co-transitive.

Nevertheless, there are cases when the set $Inv(\sigma) \cup Inv(\tau)$ is transitive. In such a case we denote by $\sigma \cup \tau$ the join $\sigma \vee \tau$. Similarly, in the case of co-transitivity of the set $Inv(\sigma) \cap Inv(\tau)$, we denote by $\sigma \cap \tau$ the meet $\sigma \wedge \tau$.

Definition. Linear orders σ and τ are *compatible* if the set $Inv(\sigma) \cup Inv(\tau)$ is transitive and the set $Inv(\sigma) \cap Inv(\tau)$ is co-transitive. The same terminology is applied to TcoT-sets as well.

For instance, if $\sigma \ll \tau$ then σ and τ are compatible. In particular, α and ω are compatible with any linear order. Linear orders σ and σ° are compatible for any σ. Linear orders σ and τ are compatible if and only if σ° and τ° are compatible.

Lemma 3. *Let three linear orders ρ, σ, and τ be pairwise compatible. Then ρ is compatible with $\sigma \cup \tau$ and $\sigma \cap \tau$ and $\rho \cap (\sigma \cup \tau) = (\rho \cap \sigma) \cup (\rho \cap \tau)$.*

Proof. We prove the compatibility of ρ and $\sigma \cup \tau$; the compatibility ρ and $\sigma \cap \tau$ is proved similarly. Let R, S, and T be the inversion sets for ρ, σ and τ. We have to check that the sets $R \cup (S \cup T)$ and $R \cap (S \cup T)$ are transitive and co-transitive.

The co-transitivity of $R \cup S \cup T$ is obvious. The transitivity follows from a simple observation. If two pairs (i, j) and (j, k) belong to $R \cup S \cup T$ then they belong to at most two of the sets R, S or T, for instance, to R and T. But the union of R and T is transitive, hence $(i, k) \in R \cup T \subseteq R \cup S \cup T$.

The transitivity of the intersection $R \cap (S \cup T)$ follows from transitivity of R and $S \cup T$. The set $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$ is co-transitive as the union of two co-transitive sets $R \cap S$ and $R \cap T$. □

A *clique* is a subset in $\mathcal{LO}$ which consists of pairwise compatible linear orders.

Because of Lemma 3, we can expand any clique by adding joins and meets of elements of the clique. In particular, if a clique is maximal (by inclusion) then it is a lattice, moreover, a distributive lattice. This proves the following

Theorem 1. *Let $\mathcal{C}$ be a maximal clique. Then $\mathcal{C}$ is a distributive sublattice of the Bruhat lattice $(\mathcal{LO}, \ll)$.*

Example 2. Let us list all maximal cliques in the case $n = 3$. It was shown in Example 1 that there are two pairs of non-compatible linear orders: 213 and 132, and 231 and 312. All other pairs are compatible. Therefore a maximal clique must contain exactly one element from the first pair and one from the second pair. We obtain four maximal cliques:

1) 123, 132, 312, and 321. The alternative 2 is never the worst. This is the *peak* domain $\mathcal{D}_3(\cap)$.

2) 123, 213, 231, and 321. Here 2 is never the best; it is the *pit* domain $\mathcal{D}_3(\cup)$.

3) 123, 213, 312, and 321. Here 1 and 2 appear a tight group separated from the alternative 3. We denote this domain as $\mathcal{D}(\rightarrow)$.

4) 123, 132, 231, and 321. Here 1 is separated from 2 and 3, this is the domain $\mathcal{D}(\leftarrow)$.

For what follows, it is useful to know properties of the compatibility relation under restrictions to subsets.

Let $\varphi : [m] \rightarrow [n]$ be a strictly increasing (hence, injective) mapping. Then the restriction mapping

$$\varphi^* : \mathcal{LO}_n \rightarrow \mathcal{LO}_m$$

is compatible with the Bruhat posets structures. Indeed, if σ is a linear order on $[n]$ then $Inv(\varphi^*(\sigma)) = \varphi^*(Inv(\sigma))$. Further, φ^* commutes with the ortho-complementations. Moreover, φ^* sends α_n in α_m and ω_n in ω_m. In general case, φ^* does not commute with joins and meets.

Example 3. Let $\sigma = 213$, $\tau = 132$; then $\sigma \vee \tau = 321$. Under the restriction to the subset $\{1, 3\}$, σ is sent into 13, τ is sent into 13, and their join 13 differs from the restriction of 321 to $\{1, 3\}$ which is equal to 31. Similarly for the meet of 231 and 312.

However, there are two cases when the restriction commutes with $\vee$ and $\wedge$. The first case is when φ is the natural inclusion of $[n-1]$ into $[n]$. The second case is when σ and τ are compatible.

Proposition 1. *Let σ and τ be compatible linear orders on $[n]$. Then, for any strictly increasing mapping $\varphi : [m] \to [n]$, the linear orders $\varphi^*(\sigma)$ and $\varphi^*(\tau)$ are compatible. Moreover, $\varphi^*(\sigma \cup \tau) = \varphi^*(\sigma) \cup \varphi^*(\tau)$ and $\varphi^*(\sigma \cap \tau) = \varphi^*(\sigma) \cap \varphi^*(\tau)$.*

The proof follows from a simple remark: if $S \subseteq \Omega_n$ is a transitive (or co-transitive) set then $\varphi^*(S)$ is a transitive (respectively, co-transitive) set in Ω_m.

The reverse to this remark is partly true. Namely, if the restriction of S to every triple ijk is transitive (co-transitive) then S is transitive (co-transitive). In particular, *if the restriction of linear orders σ and τ to every triple are compatible, then σ and τ are compatible.*

4 Cliques and Condorcet domains

Recall that a Condorcet domain (CD) is a subset $\mathcal{D} \subseteq \mathcal{LO}$ such that the simple majority rule does not yield cycles. Let us say this more precisely. A *$\mathcal{D}$-opinion* is a mapping $\nu : \mathcal{D} \to \mathbb{N} = \{0, 1, 2, ...\}$. Intuitively, this means that $\nu(\sigma)$ voters have σ as their preference on the set $[n]$ of alternatives. The number $|\nu| = \sum_{\sigma \in \mathcal{D}} \nu(\sigma)$ is equal to total number of voters. Let $sm(\nu)$ denote the binary relation (the 'social preference') on $[n]$ constructed by the simple majority rule: $i \; sm(\nu) \; j \Leftrightarrow$ the number of voters which prefer i to j is strictly larger than the number of voters having opposite preference. If (obviously, asymmetric) relation $sm(\nu)$ is acyclic for every $\mathcal{D}$-opinion ν, the set $\mathcal{D}$ is called a *Condorcet domain* (or an *acyclic set of linear orders*).

It is well known (see, for example, [11]) that in this definition one can consider only $\mathcal{D}$-opinions with odd number of voters. In this case the relation $sm(\nu)$ is complete (and is a linear order provided that $\mathcal{D}$ is a CD). We give an explicit polynomial expression for this order.

Let $\mathcal{C}$ be a subset of linear orders and ν be a $\mathcal{C}$-opinion. We say that a subset $A \subseteq \mathcal{C}$ is *ν-admissible* if $\nu(A) := \sum_{\sigma \in A} \nu(\sigma) > |\nu|/2$.

Theorem 2. *Let $\mathcal{C}$ be a clique and let $|\nu|$ be odd. Then*

$$sm(\nu) = \bigvee_A \Big(\bigwedge_{\sigma \in A} \sigma \Big),$$

where A runs over ν-admissible subsets of $\mathcal{C}$.

Proof. It suffices to check the that the left and the right hand sides of the formula coincide under the restriction to arbitrary two-element subset in $[n]$. Let $\varphi : [2] \to [n]$ be a strictly increasing mapping. By definition, $\varphi^*(sm(\nu)) = sm(\varphi^*(\nu))$ (where $\varphi^*(\nu)$ is the corresponding recalculated opinion). Due to Proposition 1, $\varphi^*(\vee_A(\wedge A)) = \vee_A(\wedge \varphi^*(A))$. Thus, we have to check the equality in the case $n = 2$. In this case there are only two linear orders: $\alpha = 12$ and $\omega = 21$. And an opinion ν is given by two numbers: $\nu(\alpha)$ and $\nu(\omega)$, $|\nu| = \nu(\alpha) + \nu(\omega)$. Exactly one of these numbers is larger than $|\nu|/2$.

Suppose that $\nu(\alpha) > |\nu|/2$. Then $sm(\nu) = \alpha$. On the other hand, any ν-admissible set A contains α, so that $\wedge A = \alpha$ and the join of all $\wedge A$ is equal to α as well.

Suppose now that $\nu(\omega) > |\nu|/2$. Then $sm(\nu) = \omega$. In this case the set $A_0 = \{\omega\}$ is admissible, and $\wedge A_0 = \omega$. Hence the join (over all admissible A) is equal to ω as well. □

Corollary. *Let $\mathcal{C}$ be a maximal clique. Then, for every $\mathcal{C}$-opinion ν (with odd $|\nu|$), the social preference $sm(\nu)$ belongs to $\mathcal{C}$.*

Indeed, due to Theorem 1, $\mathcal{C}$ is a sublattice of the Bruhat lattice. Therefore, due to maximality, the right hand side of the above formula belongs to $\mathcal{C}$.

Remark. The simple majority rule is an anonymous rule. The so-called majority systems $\mathcal{F}$ (see [10]) allows us to construct more general rules $a_{\mathcal{F}}$ for aggregation of linear orders. Such a rule maps the set $\mathcal{LO}^N$ of preference profiles (where N is a set of voters) in the set of tournaments. The same reasons as above show that if the image of a profile $p : N \to \mathcal{LO}$ lies in a maximal clique $\mathcal{C}$ then the aggregated preference $a_{\mathcal{F}}(p)$ i) is a linear order, and ii) belongs to $\mathcal{C}$. The assertion i) was proved in [10].

It also follows from Theorems 1 and 2 that any clique is a CD. Thus we get a half of the proof of the following proposition from [4].

Proposition 2. *Let $\mathcal{C}$ be a subset of $\mathcal{LO}$ containing α and ω. Then $\mathcal{C}$ is a CD if and only if $\mathcal{C}$ is a clique.*

Proof. It remains to show that a normal CD is a clique. It is well-known (see, for example, [11]) that $\mathcal{D}$ is a CD if and only if the restrictions of $\mathcal{D}$ to every triple ijk ($i < j < k$) get into one of the domains $\mathcal{D}_3(\cap)$, $\mathcal{D}_3(\cup)$, $\mathcal{D}_3(\to)$ or $\mathcal{D}_3(\leftarrow)$ from Example 2. Because of the remark at the end of Section 3, the proposition follows. □

Recall that a subset $\mathcal{D} \subset \mathcal{LO}$ is *normal* if it contains α and ω. From Proposition 2 and Theorem 1 we obtain the following corollary, which is a strengthening of an Abello's result [1].

Corollary. *Let $\mathcal{D}$ be a maximal normal Condorcet domain. Then it is a distributive sublattice of the Bruhat lattice.*

We say that a Condorcet domain $\mathcal{D}$ is *normalizable* if there exists a pair of opposite linear orders σ and σ° such that the set $\mathcal{D} \cup \{\sigma, \sigma^\circ\}$ is a CD as well. Without loss of generality, one can assume that $\sigma = \alpha$.

The class of normalizable Condorcet domains does not exhaust all Condorcet domains. Namely, the following example shows the existence of a non-normalizable CD.

Example 4. Let the domain $\mathcal{D}$ in $\mathcal{LO}_4$ consists of four linear orders $\alpha = 1234$, $\beta = 2314$, $\gamma = 2413$ G $\delta = 2143$. It can be directly proved that $\mathcal{D}$ is a CD. We assert that this domain is not normalized. In other words, for any pair σ, σ° of opposite linear orders, the domain $\mathcal{D} \cup \{\sigma, \sigma^\circ\}$ is not a CD.

For proving we consider the restriction to the subset $\{1, 3, 4\}$, that is we delete the alternative 2. We obtain four orders 134, 314, 413 and 143. These orders form a maximal CD on the set $\{1, 3, 4\}$. From this observation, one can see that, after deleting 2, the pair σ, σ° has to be the pair $314, 413$. Wlog we assume that the restriction of σ is 314. Thus, we have to examine four possibility for σ: 2314, 3214, 3124 and 3142.

1) $\sigma = 2314$ or 3214. Delete the alternative 3 (and denote the resriction by $'$). Then $\sigma'^\circ = 412$, $\alpha' = 124$, $\delta' = 241$ form a cyclic triple. Therefore $\mathcal{D} \cup \{\sigma^\circ\}$ is not a CD.

2) $\sigma = 3124$ or 3142. Deleting the alternative 4 we obtain the following cyclic triple $\sigma' = 312$, $\alpha' = 123$, $\beta' = 231$. Therefore $\mathcal{D} \cup \{\sigma\}$ is not CD.

5 Symmetric Condorcet domains

A large class of the so-called tiling type (or peak-pit) CDs was considered in [5] (see also [1, 9, 6]). A tiling type CD is a normal CD and it does not contains other pairs of opposite linear orders, except α and ω. Here we present a kind of a complementary class of normal CDs.

Definition. A Condorcet domain $\mathcal{D}$ is called *symmetric* if, for every linear order $\sigma \in \mathcal{D}$, the the opposite linear order σ° belongs to the domain $\mathcal{D}$ as well.

In other words, $\mathcal{D}^\circ = \mathcal{D}$. Domains $\mathcal{D}_3(\rightarrow)$ and $\mathcal{D}_3(\leftarrow)$ are examples of symmetric CDs. Of course, a symmetric CD is normalizable. In what follows it is convenient to consider normal symmetric CDs. We do not loss generality under this assumption.

Suppose that $\mathcal{D}$ is a symmetric CD, that is a symmetric clique (by virtue of Theorem 1). If a linear order σ is compatible with every elements of $\mathcal{D}$ then its opposite order σ° is also compatible with every element of $\mathcal{D}$ (and with σ). Therefore any symmetric CD can be extended to a maximal and symmetric CD.

How to construct symmetric CDs? Suppose we have a partition $\Omega = S_1 \coprod ... \coprod S_t$ ($\coprod$ denotes the disjoint union). Denote $\mathcal{B}(S_1, ..., S_t)$ the Boolean algebra of subsets of Ω generated by $S_1, ..., S_t$. It consists of $S(I) = \cup_{i \in [t]} S_i$, where I runs over subsets of $[t]$.

Lemma 3. *In these notations suppose that the sets S_i, $i = 1, ..., t$ are co-transitive. Then any element $S(I)$ of the Boolean algebra $\mathcal{B}(S_1, ..., S_t)$ is a TcoT-set.*

Indeed, any set $S(I)$ is co-transitive as the union of co-transitive sets S_i and is transitive as the complement (in Ω) to the co-transitive set $S([t] - I)$.

In particular, the Boolean algebra $\mathcal{B}(S_1, ..., S_t)$ consists of pairwise compatible sets and, due to Proposition 2, defines a CD, which is denoted by the same symbol. Since $\mathcal{B}(S_1, ..., S_t)$ is stable with respect to $^\circ$, this CD is symmetric. It can be non-maximal if there exists a finer partition. But any maximal symmetric CD $\mathcal{D}$ has such a form. Indeed, due to Theorem 1 and Proposition 2, $\mathcal{D}$ is a distributive sublattice of the Bruhat lattice. By virtue of symmetry, $\mathcal{D}$ is ortho-complemented. Therefore $\mathcal{D}$ is a Boolean lattice. Let $\sigma_1, ..., \sigma_t$ be atoms of the Boolean lattice, and $S_i = Inv(\sigma_i)$, $i = 1, ..., t$, be the corresponding inversion sets. Obviously, S_i do not intersect and cover Ω.

Proposition 3. *The size of symmetric CD does not exceed* 2^{n-1}.

Proof. We can assume that symmetric CD $\mathcal{D}$ is maximal. Therefore it has the form $\mathcal{B}(S_1, ..., S_t)$ for some partition of Ω by TcoT-sets, and its size is equal to 2^t. It remains to remark that every (non-empty) co-transitive set $S \subseteq \Omega$ contains a 'short arrow', that is a pair of the form $(i, i+1)$. Therefore every set S_r contains a 'short arrow'. Since there are $n-1$ 'short arrows', we obtain that $t \leq n-1$. □

6 Structure of symmetric Condorcet domains of maximal size

From Proposition 3, we know that the size of a symmetric CD is less or equal to 2^{n-1}. Therefore any symmetric CD of size 2^{n-1} is maximal. Here we give a complete description of such CDs. This result uses the following simple construction (a particular case of more general block construction from [9]).

Let $1 < m < n$, and suppose we have two normal CDs, $\mathcal{D}$ on the set $[m]$ and $\mathcal{D}'$ on the set $[m'] = [n-m]$. Denote by $\mathcal{D} * \mathcal{D}'$ the set of linear orders on $[n]$ of the form $\sigma\sigma'$ or $\sigma'\sigma$, where $\sigma \in \mathcal{D}$, $\sigma' \in \mathcal{D}'$. Here we consider σ and σ' as words in the alphabet $1, \dots, m$ and $m+1, \dots, n$ respectively. We regard the set $[m']$ as the set of the last $n-m$ symbols in $[n]$.

Lemma 4. $\mathcal{D} * \mathcal{D}'$ *is a CD on* $[n]$.

Intuitively, it is quite clear. Suppose we have a $\mathcal{D} * \mathcal{D}'$-opinion. At first we decide whether the first m alternatives stay above or below than the last m' alternatives. Then, due to Theorems 1 and 2, we rank the first m alternatives ($\mathcal{D}$ is a CD) and similarly for the last m' alternatives ($\mathcal{D}'$ is a CD).

One can argue more formally. We have to check that every two linear orders from $\mathcal{D} * \mathcal{D}'$ are compatible. Here four cases are possible. For instance, $\sigma\sigma'$ and $\tau\tau'$, or $\sigma\sigma'$ and $\tau'\tau$. In any case, it is useful to consider the inversion sets for $\sigma\sigma'$ and $\sigma'\sigma$. Let $S = Inv(\sigma)$, $S' = Inv(\sigma')$. Then $Inv(\sigma\sigma') = S \cup S'$ whereas $Inv(\sigma'\sigma) = S \cup S' \cup P$, where $P = \{(i,j), i \leq m, j \geq m+1\}$. Consider, for example, the join of $\sigma\sigma'$ and $\tau'\tau$ (setting $T = Inv(\tau)$ and $T' = Inv(\tau')$). Let us prove that $(S \cup S') \cup (T \cup T' \cup P)$ is transitive (the co-transitivity is obvious). The set $S \cup T$ is transitive due to compatibility of σ and τ; $S' \cup T'$ is transitive due to compatibility of σ' and τ'. The union of these transitive sets is transitive because the end of a first arrow cannot be the origin of a second one. Adding P does not violate the transitivity. Similarly with the other cases.

Lemma 5. *Let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be symmetric CDs. Then* $\mathcal{D} * \mathcal{D}'$ *is a symmetric CD.*

This follows from $(\sigma\sigma')^\circ = \sigma'^\circ\sigma^\circ$.

Proposition 5. *Let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be maximal normal CDs. Then* $\mathcal{D} * \mathcal{D}'$ *is a maximal normal CD.*

Proof. Suppose that a linear order ρ is compatible with all elements from $\mathcal{D} * \mathcal{D}'$. Then it is compatible with the linear order $\omega_m \omega_{m'}$ whose inversion set is equal to $\Omega_m \cup \Omega_{m'}$. In particular, $Inv(\rho) \cup (\Omega_m \cup \Omega_{m'})$ must be transitive. Further, due to compatibility of ρ and $\alpha_{m'}\alpha_m$, the intersection $Inv(\rho) \cap P$ must be co-transitive. We assert that it is possible only in two cases: either $Inv(\rho)$ contains P or $Inv(\rho)$ does not intersect P.

Namely, suppose that $R = Inv(\rho)$ contains a pair of the form (i, j), where $i \leq m$, $j \geq m + 1$. Since the set $R \cap P$ is co-transitive, it contains the short arrow $(m, m + 1)$. The set Ω_m contains any arrow (i, m) $(i \leq m)$, the set $\Omega_{m'}$ contains any arrow $(m + 1, j)$ $(j \geq m + 1)$. Due to the transitivity, $R \cup \Omega_m \cup \Omega_{m'}$ contains all arrow of the form (i, j), that is $R \supset P$.

Thus, either P does not intersect R or $P \subseteq R$. In the first case ρ has the form $\tau\tau'$, where τ is a linear order on $[m]$ and τ' is a linear order on $[m']$. In the second case ρ has the form $\tau'\tau$. Since τ is compatible with every element of $\mathcal{D}$, $\tau \in \mathcal{D}$ due to maximality of $\mathcal{D}$. Similarly $\tau' \in \mathcal{D}'$. $\square$

Let us note a particular case of the construction when $m = n - 1$. In this case we take $\mathcal{D}'$ as the set of all linear orders on the set $\{n\}$. This set contains a single element denoted as n (we hope that this misuse of language does not yield a confusion). In this case the set $\mathcal{D} * \mathcal{D}' = \mathcal{D} * n$ consists of linear orders of the form σn or $n\sigma$, where $\sigma \in \mathcal{D}$. If $\mathcal{D}$ is a symmetric (respectively, maximal) CD on $[n - 1]$ then $\mathcal{D} * n$ is a symmetric (respectively, maximal) CD on $[n]$ of the double size.

This construction can be iterated. In particular, one can define by induction a CD for any binary parenthesization of the expression $1 * 2 * ... * n$. For instance, $(1 * (2 * 3)) * (4 * 5)$. Here the symbol i is considered as a unique linear order on the set $\{i\}$. (It is well-known that such a parenthesization is equivalent to a plane binary tree with n leaves. Exercise 6.19 in Stanley's book [12] contains yet 64 interpretations of parenthesizations.)

Example 5. Obviously, $1 * 2 = \mathcal{LO}_2$. The domain $(1 * 2) * 3$ is exactly the Condorcet domain on [3], in which the alternative 3 is never middle, that is the domain $\mathcal{D}_3(\rightarrow)$ from Example 2. Similarly $1 * (2 * 3)$ is the domain $\mathcal{D}_3(\leftarrow)$.

Example 6. Let us consider in details the case $(...((1 * 2) * 3)...) * n$. Linear orders from this domain are characterized by the following property: every alternative i either is better than any element from the set $[i - 1]$ or is worse than any element from it. N.S. Kukushkin proposed the following interpretation of such orders. Let us consider alternatives as 'reform projects', ordered by their degree of 'radicalism'. Due to increasing radicality, every project is perceived either better than all of the project that preceded it, or worse than all of them. This domain is somewhat similar the Black's domain of single-peaked preferences. As for the single-peaked domain, there is a simple inductive procedure for aggregation of these preferences. At first, we compare the alternatives n and $n - 1$. If n wins in the comparison, it becomes the best in the group sense. In the opposite case it is worst. Next we compare $n - 1$ and $n - 2$, and so on.

Similarly one can aggregate the preferences for any CD, produced by a parenthesization of $1 * 2 * ... * n$.

Let $\mathcal{P}$ be a binary parenthesization of $1 * 2 * ... * n$, and $\mathcal{D}(\mathcal{P})$ be the corresponding CD. We saw that $\mathcal{D}(\mathcal{P})$ is a symmetric CD of size 2^{n-1}, hence is a maximal CD. We assert that the converse is also true.

Theorem 3. *Let $\mathcal{D}$ be a symmetric Condorcet domain of size 2^{n-1}. Then it has the form $\mathcal{D}(\mathcal{P})$ for some parenthesization $\mathcal{P}$ of $1 * 2 * ... * n$.*

We need the following

Lemma 6. *Let P be a co-transitive set in Ω, which contains the 'long arrow' $(1, n)$ and a unique short arrow $(m, m+1)$. Then $P = \{(i, j),\ i \le m < j\}$.*

Proof. First of all, $P \subseteq \{(i, j),\ i \le m < j\}$. Indeed, if P contains an arrow (i, j) with $i, j \le m$ then it contains a short arrow of the same form. This contradicts the uniqueness of the short arrow $(m, m+1)$. Similarly for $i, j > m$.

On the other hand, every arrow (i, j) with $i \le m,\ j \ge m+1$ belongs to P. Indeed, applying the definition of co-transitivity to the arrow $(1, n)$ and the symbol i, we obtain that either $(1, i) \in P$ or $(i, n) \in P$. The first case is impossible, since $P \subseteq \{(i, j),\ i \le m < j\}$. Therefore $(i, n) \in P$. Applying again the definition of co-transitivity to the arrow (i, n) and the symbol j, we obtain $(i, j) \in P$. $\Box$

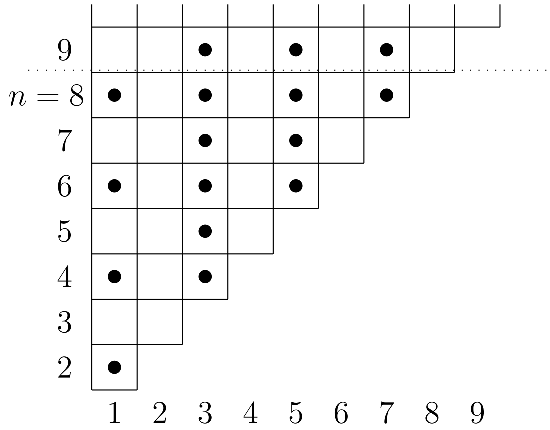
Proof of Theorem 3. It follows from Section 5 that the CD $\mathcal{D}$ is defined by a partition $\Omega = S_1 \coprod ... \coprod S_{n-1}$ consisting of $n-1$ TcoT-sets $S_1, ..., S_{n-1}$. One of them (say, S_1) contains the long arrow $(1, n)$. Moreover, S_1 (as well as all other S_r) contains a unique short arrow $(m, m+1)$. Due to Lemma 6, $S_1 = \{(i, j),\ i \le m < j\}$. This gives us the first decomposition of $[n]$ on two sub-intervals: $[m]$ and $[m+1..n]$. Now we repeat the previous arguments to the interval $[m]$ and decompose it on two sub-intervals. Similarly for the interval $[m+1..n]$, and so on. This process leads to a parenthesization of $1 * 2 * ... * n$ and completes the proof. $\Box$

7 One more example

Here we show that (for any $n \ge 3$) there exists a maximal symmetric CD $\mathcal{Q}_n \subseteq \mathcal{LO}_n$ of size 4. It consists of four linear orders α, ω, σ and σ°, where σ has the form $24...(2k)1(2k \pm 1)...53$ (here $2k \pm 1$ is equal to $2k+1 = n$, if n is odd, and $2k-1 = n-1$ if n is even). In other words, at first even symbols go in increasing order, further the symbol 1 stays, and then odd symbols (up to 3) go in decreasing order. For example, $\sigma = 2461753$ for $n = 7$, whereas $\sigma = 24681753$ for $n = 8$. The opposite order σ° is arranged similarly: at first odd symbols (beginning with 3) go, further 1, and next even symbols go in decreasing order.

Proposition 5. *The domain $\mathcal{Q}_n = \{\alpha, \sigma, \sigma^\circ, \omega\}$ is a maximal Condorcet domain.*

Proof. Let $S = Inv(\sigma)$; this set is depicted by black circles in the figure below.



The picture is slightly differ for even and odd n. Here $n = 8$.

Suppose that the domain $\mathcal{Q}_n$ is not maximal. Then there exists a TcoT-set T which is compatible with S and $\Omega - S$ (and differing from them). Passing to $T \cap S$ or to $T \cap (\Omega - S)$, we can suppose that T is contained in S or in $\Omega - T$. We consider in detail the case when non-empty TcoT-set T is contained in $\Omega - S$ and $T \cup S$ is transitive (the other case is considered similarly); we have to show that T coincides with $\Omega - S$.

We will argue by induction. The induction base, $n = 3$, is obviously true. Therefore we assume $n \geq 4$. We will denote the reduction of the symbol n (or restriction on $[n-1]$) by a prime. By induction the domain $\mathcal{D}' = \{\alpha', \omega', \sigma', (\sigma')^\circ\}$ is maximal.

We claim that T' is non-empty as well. Indeed, if T' is empty then T consists only of arrows of the form (i, n). Moreover, T is co-transitive and, hence, contains the short arrow $(n-1, n)$. Since $T \subseteq \Omega - S$, this is possible only at odd n. (In this case i has to be a even symbol or the symbol 1.) Further, $(1, n-1) \in S$; due to transitivity of $T \cup S$ we obtain that $(1, n) \in T \cup S$. Since $(1, n) \notin S$, we have $(1, n) \in T$. Since T is co-transitive, we obtain $(3, n) \in T$, in contradiction with $(3, n) \in S$.

Thus, T' is non-empty. By the inductive assumption T' coincides with $\Omega' - S$. Therefore $T \cup S$ contains whole $\Omega' = \Omega_{n-1}$. In other words, every arrow (i, j) belongs to $T \cup S$ provided that $j \neq n$. Moreover, the set $T \cup S$ is transitive. Recall we should show that $T = \Omega - S$. We consider the cases of even and odd n separately.

1. *n is odd.* In this case $(1, 3) \in \Omega' \subseteq T \cup S$ and $(3, n) \in S$; due to transitivity of $T \cup S$ we obtain $(1, n) \in T \cup S$. But $(1, n) \notin S$ and hence $(1, n) \in T$. Let now i be a even symbol. It lies between 1 and n; due to co-transitivity of T we obtain that either $(1, i) \in T$ or $(i, n) \in T$. The first is impossible, since then $(1, i) \in \Omega - S$ that is not case. Therefore, for every even i we have $(i, n) \in T$. Together with $(1, n) \in T$ this gives the equality $T = \Omega - S$.

2. *n is even.* In this case $(2, 3) \in \Omega' \subseteq T \cup S$ and $(3, n) \in S$; the transitivity of $T \cup S$ implies $(2, n) \in T \cup S$. But $(2, n) \notin S$, hence $(2, n) \in T$. Again, let i be a even symbol (more than 2). It lies between 2 and n; due to the co-transitivity of T, we have either $(2, i) \in T$ or $(i, n) \in T$. The first is impossible since then $(2, i) \in T \subseteq \Omega - S$, that is not the case. Thus, for every even symbol i we have $(i, n) \in T$, which gives $T = \Omega - S$. $\square$

Now one can easily construct a maximal (and symmetric) CD of the size 2^m for any m, $2 \leq m < n$. For this, one takes a domain of the form $\mathcal{Q}_{n-m+2} * ((...(n-m+3)*...)*n)$. For instance, at $n=5$ the CD $\mathcal{Q}_4 * 5$ has the size 8. It would be interesting to find a structure which describes maximal symmetric CDs of any size.

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