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Characterization of (quasi-)ultrametric finite spaces in terms of (directed) graphs [a]

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Characterization of (quasi-)ultrametric finite spaces in terms of (directed) graphs [1]

Vladimir Gurvich Michael Vyalyi

Abstract. Given a complete directed graph (digraph) $D = (V, A)$ and a positive real weight function $d : A \to \{d_1, \ldots, d_k\} \subseteq \mathbb{R}_+$ such that $0 < d_1 < \ldots < d_k$, then, for any $i \in [k] = \{1, \ldots, k\}$ let us set $A_i = \{a = (u, w) \in A \mid d(a) \leq d_i\}$ and assume that every subgraph $D_i = (V, A_i), i \in [k]$, in the obtained nested family is transitive: $(u, w) \in A_i$ when $(u, v), (v, w) \in A_i$ for some $v \in V$ and for all $u, w \in V$ and $i \in [k]$. It is not difficult to verify that the considered weighted digraph (D, d) defines a quasi-ultrametric finite space (QUMFS), that is,
$d(u, w) \geq 0$, $d(u, w) = 0$ iff $u = w$, and $d(u, w) \leq \max(d(u, v), d(v, w)) \; \forall \; u, v, w \in V$.
Moreover, each QUMFS is uniquely (up to an isometry) realized in such a way.
This result implies that each QUMFS is realized by a multi-pole flow network.
In the symmetric case, $d(u, w) = d(w, u)$ for all $u, w \in V$, we obtain the following canonical representation of an ultrametric finite space (UMFS). Let $T = (V, E)$ be a rooted tree in which $L \subseteq V$ is the set of leaves and $v_0 \in N = V \setminus L$ is the root. For any leaf $v \in L$, there is a unique path $p(v)$ from v_0 to v. Furthermore, let $d : N \to \{d_1, \ldots, d_k\} \subseteq \mathbb{R}_+$ be a positive weight function $(0 < d_1 < \ldots < d_k)$ whose weights strictly monotone decrease along each path $p(v)$, $v \in L$. Then, for every two distinct leaves $u, w \in L$, let us set $d(u, w) = d(v(u, w))$, where $v(u, w) \in N$ is the lowest common ancestor of u and w, or in other words, the last common vertex of the paths $p(u)$ and $p(w)$; standardly we set $d(u, w) = 0$ iff $u = w$. Again, it is easy to see that $d(u, w) = d(w, u) \geq 0$ and $d(u, w) \leq \max(d(u, v), d(v, w))$ for all $u, v, w \in L$. Thus, $(T = (V, E), d)$ forms an UMFS. Moreover, every UMFS is uniquely (up to an isometry) realized in this way if we additionally assume that $deg(v_0) \geq 2$ and $deg(v) \geq 3$ for any other $v \in N$.
By this result, somewhat surprisingly, an UMFS can be viewed as a k-person positional game of players $\{1, \ldots, k\}$ such that in every play $p(v)$ from v_0 to v the corresponding players move in a monotone strictly decreasing order.

Keywords: distance, ultrametric, spanning tree, minimum cut, maximum flow, Gomory-Hu tree, widest bottleneck path, decomposing n-graphs, positional game

1 Introduction

1.1 Finite (quasi-)ultrametric spaces and their (directed) graphs

Given a finite set V and a mapping $d : V \times V \to \mathbb{R}_+$, let us consider three standard axioms:

- (i) $d(u, w) = 0$ iff $u = w$;
- (ii) $d(u, w) = d(w, u)$ for all $u, w \in V$;
- (iii) $d(u, w) \leq max(d(u, v), d(v, w))$ for all $u, v, w \in V$.

A pair (V, d) satisfying (i, iii) or (i, ii, iii) is called QUMFS or UMFS, respectively, [6]. In both cases $d(u, w)$ is called the *distance* between u and w.

It is easily seen that in an UMFS the equality in (iii) holds whenever $d(u, v) \neq d(v, w)$; in other words, the largest two distances are equal in any triangle $u, v, w \in V$. For this reason, an UMFS is alternatively called an *isosceles* space; see, for example, [24, 25].

Example 1. *However, the QUMFSs are not necessarily isosceles; let us set, for example,* $d(u, v) = 1, d(v, w) = 3, d(u, w) = 2$ *and* $d(v, u) = d(w, v) = d(w, u) = 10$.

An UMFS (respectively, QUMFS) is conveniently represented by a positively weighted complete graph $G = (V, E), d : E \to \mathbb{R}_+$ (respectively, digraph $D = (V, A), d : A \to \mathbb{R}_+$), where $|V| = n$, $|E| = n(n-1)/2$, $|A| = n(n-1)$, and $|im\ d| = k$. We set $im\ d = \{d_1, \ldots, d_k\}$ and assume that $0 < d_1 < \ldots d_k < \infty$.

A *pseudo*-(Q)UMFS is defined by relaxation of these inequalities to $0 \leq d_1 < \ldots d_k \leq \infty$. (For brevity, we write "a (Q)UMFS" to refer to both an UMFS or QUMFS simultaneously.) Respectively, "iff" should be replaced by "if" in (i).

Remark 1. *Obviously, the ultrametric inequality (iii) is respected by any change of the values of* $d_1, \ldots, d_k$ *provided their order is preserved. In particular, all statements that hold for the (Q)UMFSs are automatically extended to pseudo-(Q)UMFSs too, whenever the values* $d_1 = 0$ *and* $d_k = \infty$ *are allowed. E.g., in Example 1 we could set* $d(u, v) = 0$ *rather than* 1 *and/or* $d(v, u) = d(w, v) = d(w, u) = \infty$ *rather than* 10 *and get a pseudo-QUMFS.*

For simplicity, we will restrict ourselves to the (Q)UMFSs (unless a pseudo-(Q)UMFS is mentioned explicitly, like in Sections 4.1 and 4.2) but keep in mind that all statements hold for the pseudo-(Q)UMFSs as well.

1.2 Main and related results

As it was announced in the title and explained in abstract, in this paper, we will characterize (Q)UMFSs in terms of their (directed) graphs.

In case of UMFSs our characterization is closely related to the recent observations of Demaine, Landau, and Weimann [5] that extend the fundamental results of Gomory and Hu [10, 20, 21]; see the next Section for more details.

On the other hand, our characterization of the UMFSs can be viewed as a restriction of the one-to-one correspondence between the positional game structures and Π- and Δ-free k-graphs, studied by the first author in [11, 12, 13, 14, 17]. Interestingly, the UMFSs correspond to the structures in which players $\{1, \dots, k\}$ move in a strictly monotone decreasing order in every play $p(v)$ from the initial position v_0 to a terminal one $v \in L$; see Section 2 for details.

Let us recall that the semilattices were also characterized as special positional game structures by Libkin and the first author in [30].

Here, we should also mention considerable contribution of A. Lemin and V. Lemin to characterization of ultrametric spaces [24, 25, 26, 27, 28, 29]. In their works, mostly, general (infinite) spaces are studied in an algebraic language. Yet, according to A. Lemin, in the late 90s "Israel Gelfand set a problem to describe all *finite* ultrametric spaces up to isometry using *graph theory language*". In this paper we suggest a solution.

The QUMFSs are considered in the last two sections. In Section 3 they are characterized as nested families of transitive digraphs. It is also shown that the number of pairwise distinct distances in an n-element QUMFS is at most $\frac{1}{2}(n-1)(n+2)$ and this bound is precise.

In Section 4 we consider realization of (Q)UMFSs by networks. In 4.1 and 4.2 we recall two classical problems (maximum flow and maximum bottleneck directed path) that result in QUMFSs. It is easy to show that each QUMFS can be realized by a bottleneck network. For the symmetric case (and UMFSs) this observation was mentioned by Leclerc in [23].

Even earlier, in [10, 20, 21], it was shown that each multi-pole maximum flow network defines a QUMFS and that every UMFS can be realized by a symmetric flow network.

We construct a QUMFS (D^0, d^0) which is not a *flow QUMFS*, that is, it cannot be realized by the set of *all* vertices of a flow network. In fact, (D^0, d^0) is a unique minimal such QUMFS. Moreover, given a QUMFS (D, d) we provide a polynomial algorithm recognizing whether it is a flow QUMFS and providing a corresponding flow network in case of the positive answer. In contrast, every QUMFS can be realized by a *subset* of vertices, or in other words, by a multi-pole flow network. All these results are derived in Section 4.2 from the characterization of a QUMFS (as a nested family of transitive digraphs) obtained in Section 3.

Then, it was shown in [18] (see also [19, 15, 16]) that the "flow" and "bottleneck" UMFSs both can be realized as the UMFSs of resistances, by the corresponding choice of the two parameters of a conductivity law; see Section 4.3 for more details. Finally, in Section 4.4, we introduce reducible, universal, and complete families of (Q)UMFSs, give examples, and study simple relations between these families.

1.3 Gomory-Hu's representation of UMFSs

The seminal paper [10] begins with the following construction. Given an UMFS defined by a weighted complete graph $(G = (V, E),\ d : E \to \mathbb{R}_+)$, by Kruskal's greedy algorithm [22], let us choose in (G, d) a lightest spanning tree $T = (V, E'), d' : E' \to \mathbb{R}_+)$, where d' is the restriction of d to $E' \subseteq E$. Then, for any $u, w \in V$ there is a unique path $p(u, w)$ in T between u and w. (This path consists of a single edge (u, w) iff $(u, w) \in E'$.)

Theorem 1. *([10]). For all $u, w \in V$, the equality $d(u, w) = \max(d(e) \mid e \in p(u, w))$ holds.*

Proof. $\leq$ can be easily derived by induction from the ultrametric inequality (iii), while $\geq$ follows immediately from the the fact that T is a *lightest* spanning tree of (G, d). □

Let us remark that such a tree may be not unique in (G, d). Nevertheless, all spanning trees in graph $G = (V, E)$ have $|V| = n$ vertices and $|E'| = n-1$ edges; moreover, all lightest such trees have the same weight distribution, uniquely defined by the greedy algorithm [22]; see Section 2.2.3 for more details. The following two corollaries are obvious.

Corollary 1. *([10]). For any weighted tree $T = (V, E'), d' : E' \to \mathbb{R}_+)$ an UMFS is defined by formula of Theorem 1. Conversely, any UMFS can be obtained from a lightest spanning tree of its weighted graph by the construction of Theorem 1.*

Proof. Given a weighted tree $(T = (V, E'), d')$, let us verify the ultrametric inequality (iii).

For any two vertices u, w of a tree there is a unique path $p(u, w)$ between them.

For any three vertices u, v, w of a tree there is a unique vertex o that belongs to all three paths $p(u, v), p(v, w), p(w, u)$. These three paths and $p(o, u), p(o, v), p(o, w)$ consist of the same edges but each edge appears twice and once in the first and last triplet, respectively. Let e be a heaviest one among all these edges. Without any loss of generality, let us assume that e is in $p(o, v)$. Then, by construction, $d(u, v) = d(v, w) \geq d(u, w)$ and (iii) holds.

All other claims of the Corollary, as well as the next Corollary, are straightforward. □

Corollary 2. *([10]). In an UMFS, the distances take at most $n-1$ distinct values, that is, $|im\ d| \leq n-1$, where $n = |V|$.* □

1.4 Cartesian binary trees of UMFSs

Recently, Demaine, Landau, and Weimann [5] applied Theorem 1 to assign a binary Cartesian tree to an UMFS as follows: Delete from $(T = (V, E'), d' : E' \to \mathbb{R}_+)$ a heaviest edge, repeat the same for each of the obtained two weighted subtrees, etc., until only vertices of T remain. Obviously, this procedure will result in a binary rooted tree $T' = (V', E'', v_0)$ whose leaves $L \subseteq V'$ (respectively, all other vertices $N = V' \setminus L$) are in one-to-one correspondence with V (respectively, with E'). It is easily seen that for any two $u, w \in L = V$ the distance $d(u, w)$ is equal to the weight $d(v(u, w))$ of the lowest common ancestor $v(u, w)$ of u and w in T'. Furthermore, these weights monotone decrease (perhaps, non-strictly) along each path $p(v)$ from the root v_0 to a leaf $v \in L$.

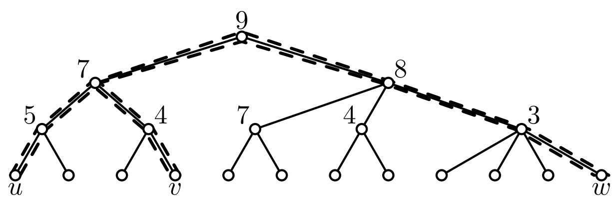

Figure 1: $d(u, v) = 7$, $d(u, w) = d(v, w) = 9$

2 Canonical representation of finite ultrametric spaces

2.1 Ultrametric spaces defined by labeled rooted trees

Let $T = (V, E)$ be a finite rooted tree in which $L \subseteq V$ is the set of leaves and $v_0 \in N = V \setminus L$ is the root. For any leaf $v \in L$, there is a unique path $p(v)$ from v_0 to v. Furthermore, let $d : N \to \{d_1, \ldots, d_k\} \subseteq \mathbb{R}_+$ be a weight function such that $0 < d_1 < \ldots < d_k$.

Then, for each two distinct leaves $u, w \in L$ let us set $d(u, w) = d(v(u, w))$, where $v(u, w) \in N$ is the lowest common ancestor of u and w, or in other words, the last common vertex of paths $p(u)$ and $p(w)$. By definition, $d(u, w) = d(w, u) \geq 0$ and standardly, we set $d(u, w) = 0$ if (and only if) $u = w$. Finally, let us assume for convenience that

- (a) Each vertex $v \in N$ has at least two immediate successors; or in other words, $deg(v_0) \geq 2,\ deg(u) \geq 3\ \forall\ u \in N \setminus \{v_0\}$ (and $deg(v) = 1\ \forall\ v \in L$, by definition).

Proposition 1. *The ultrametric inequality,* $d(v', v'') \leq \max(d(v', v), d(v, v''))\ \forall\ v, v', v'' \in L$ *holds iff the weights (non-strictly) monotone decrease along each path* $p(v),\ v \in L$.

Proof. If all three vertices coincide then all three distances equal 0. If two vertices coincide then distance between them is 0, while two other distances are equal and non-negative. Obviously, the ultrametric inequality holds in both cases. Let us assume that $v, v', v'' \in L$ are pairwise distinct and $u' = u(v, v'), u'' = u(v, v''), u = u(v', v'') \in N$ be the lowest common ancestors of the corresponding three pairs of leaves. Obviously, for any tree, at least two of them coincide, say, $u' = u''$. It is also clear that in this case $u' = u'' \geq u$, that is, $u' = u''$ is an ancestor of u. Hence, the ultrametric inequality holds for $v, v', v'' \in L$ iff $d(u') = d(u'') \geq d(u)$. Moreover, it holds for any three leaves of L iff $d(u') \geq d(u)$ whenever u' is an ancestor of u for any $u, u' \in N$. □

Let us note that Proposition 1 is applicable to the Cartesian binary trees.

2.2 Canonical weighted tree of an UMFS and its applications

2.2.1 Main construction

Let us consider a (pseudo-)UMFS (G, d) given by a complete graph $G = (V, E)$ and a positive weight function $d : E \to \{d_1, \ldots, d_k\} \subseteq \mathbb{R}_+$ satisfying the ultrametric inequality; without loss of generality, we assume that $0 < d_1 < \ldots < d_k$ (recall Remark 1, yet).

For each $i \in [k]$, let us set $D_i = \{d_1, \ldots, d_i\}$, $E_i = \{e \in E \mid d(e) \in D_i\}$, and $G_i = (V, E_i)$; in other words, G_i is the subgraph of G formed by the edges whose weights are at most d_i.

Proposition 2. *For every* $i \in [k] = \{1, \ldots, k\}$*, the subgraph* G_i *is transitive, that is,* $(u, w) \in E_i$ *whenever* $(u, v), (v, w) \in E_i$ *for all* $u, v, w \in V$*; in other words,* G_i *is the union of pairwise vertex-disjoint cliques.*

Proof. Transitivity is just a reformulation of the ultrametric inequality (iii). Furthermore, a connected graph is a clique iff with every two incident edges it contains the whole triangle. □

In particular, G_k is the total clique, while G_{k-1} is the union of $m \geq 2$ pairwise vertex-disjoint cliques each of which defines an ultrametric strict subspace (G', d') of (G, d).
By construction, the largest weight d_k appears in the latter but not in the former, that is, $d_k \in im(d) \setminus im(d')$; in other words, the d_k-edges form a complete m-partite graph.

Each graph (G', d') can be similarly decomposed in its turn and we can proceed until every considered clique becomes a single vertex of V. Obviously, the above procedure results in a rooted tree $T = (V', E')$ whose set of leaves $L \subseteq V'$ is in one-to-one correspondence with V and every other vertex $u \in N = V' \setminus L$ is assigned to an intermediate clique of G and labeled by some d_i; in particular, the root $v_0 \in N$ is labeled by d_k.

By construction, T satisfies the assumption (a) as well as the following property:

- (b) Labels d_i strictly monotone decrease along each path $p(v)$ from v_0 to $v \in L$.

Thus, Propositions 1 and 2 imply the following canonical representation of the UMFSs.

Theorem 2. *The above construction is a one-to-one correspondence between the UMFSs and labeled trees satisfying (a) and (b).* □

An example illustrating this theorem is given in Figure 3.

2.2.2 Canonical and Cartesian trees

Clearly, the above canonical representation of an UMFS (G, d) is closely related to its Cartesian binary trees. Yet, the former is unique, while the latter may be numerous and not satisfy (b).

It is also clear that for any $i \in [k]$ the set of vertices labeled by d_i form a forest in a Cartesian tree. Furthermore, contracting the subtrees of this forest for each $i \in [k]$ we obtain a tree that satisfies (b) and still defines the same UMFS (G, d). By Theorem 2, it must be the canonical tree of (G, d). Thus, for all Cartesian trees, the above contraction results in the same (canonical) tree. This statement is obvious for the Cartesian trees corresponding to a fixed minimum weight spanning tree of (G, d). However, there might be many such spanning trees and, hence, Theorem 2 is essential, in general.

The following inequalities and equalities result from Theorem 2 immediately:

Corollary 3. *Let T be the canonical tree of an UMFS (G, d), then*

$$k \leq |N_T| \leq |L_T| - 1 = n - 1.$$

Moreover, the second inequality holds with equality iff tree T is binary, while the first one iff every label d_i, $i \in [k]$, appears in T only once. □

2.2.3 Enumerating all minimum weight spanning trees of an UMFS and counting the corresponding unique weight distribution

The canonical tree $T = (V', E')$ is instrumental for efficient enumeration of all minimum weight spanning trees in $(G = (V, E), d)$. Let us recall that G_{k-1} is the union of at least two pairwise vertex-disjoint cliques $C_1, \ldots, C_m \subseteq V$ of G such that every edge between two distinct cliques is of the largest weight d_k, while each edge within a clique is of a strictly lesser weight; in other words, the d_k-edges form the complete m-bipartite graph with parts $C_1, \ldots, C_m$.

Let us choose in G any $m - 1$ edges that would form a spanning tree in the factor-graph obtained from G by contracting each of the m cliques to a vertex. It is clear that all $m - 1$ chosen edges are of weight d_k and that every spanning tree on V must contain such a selection. Then, let us repeat the same procedure for each of the cliques $C_1, \ldots, C_m$, etc., until every obtained clique becomes a vertex. Obviously, this procedure results in a minimum weight spanning tree; moreover, each one can be obtained in this way.

Combining these arguments with the Broder and Mayr [4] algorithm for counting the minimum weight spanning trees in graphs, we obtain a very efficient enumeration procedure. For any given k, it outputs the kth minimum weight spanning tree, with respect to the lexicographic order, in time poly$(\log k, |V|)$.

Also, the above procedure makes it obvious that all minimum weight spanning trees of a given UMFS $(G = (V, E), d)$ have a unique weight distribution, which is explicitly determined by the canonical tree $T = (V', E')$ as follows.

Let $S(v)$ denote the set of all immediate successors of a vertex $v \in V'$ and $s(v) = |S(v)|$. Clearly, $s(v_0) = deg_T(v_0)$ for the root and $s(v) = deg_T(v) - 1$ for any other vertex $v \in V'$; in particular, $s(v) = 0$ for the leaves $v \in L(T)$. Then, it is also easy to see that for each $i \in [k] = \{1, \ldots, k\}$ the corresponding weight d_i appears $\sum_{v \in N \mid d(v) = d_i}(s(v) - 1)$ times.

2.3 Canonical trees of UMFSs and positional game structures

We will show that the above representation of an UMFS (G, d) by its canonical labeled tree is a special case of the following one-to-one correspondence between complete edge-colored graphs and positional game structures studied in [11, 12, 13, 14, 17].

2.3.1 Complementary connected k-graphs

Let us label the edges of G by colors $i \in [k] = \{1, \ldots, k\}$ rather than by weights d_i.

A k-graph $\mathcal{G} = (V; E_1, \ldots, E_k)$ is a complete graph on vertices $V = \{v_1, \ldots, v_n\}$ whose $\binom{n}{2}$ edges are partitioned into k subsets (colored by k colors) some of which might be empty.

A k-graph $\mathcal{G}$ is called *complementary connected* (CC) if $k \geq 2, n \geq 2$, and the complement $\overline{G}_i$ to each of k chromatic components $G_i = (V, E_i)$ of $\mathcal{G}$ is connected on V; in other words, for each $u, w \in V$ and $i \in [k]$ there is an i-free path between u and w in $\mathcal{G}$.

By convention, we assume that $\mathcal{G}$ is *not* CC when $n = 1$. It is also easily seen that there is no CC k-graph with $n = 2$. Yet, CC k-graphs exist for any $k \geq 2$ and $n \geq 3$.

The following two examples Π and Δ in Figure 2 will play an important role:

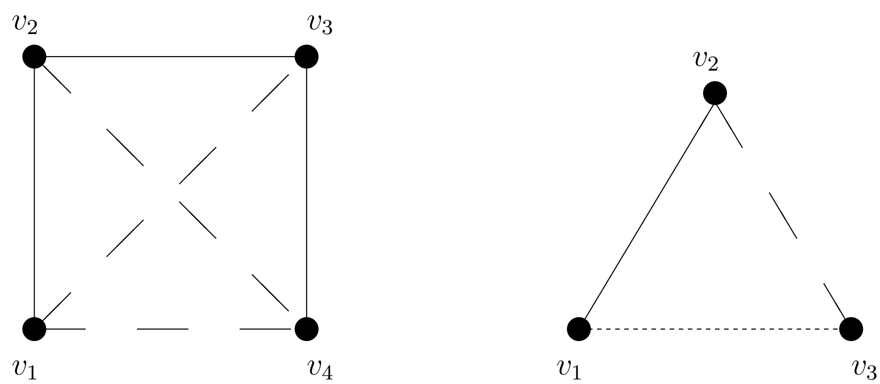

Figure 2: k-graphs Π and Δ.

Π is defined for any $k \geq 2$ by $V = \{v_1, v_2, v_3, v_4\}$;
$E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$, $E_2 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$, and $E_i = \emptyset$ whenever $i > 2$;

Δ is defined for any $k \geq 3$ by $V = \{v_1, v_2, v_3\}$,
$E_1 = \{(v_1, v_2)\}$, $E_2 = \{(v_2, v_3)\}$, $E_3 = \{(v_3, v_1)\}$, and $E_i = \emptyset$ whenever $i > 3$.

In other words, Δ is a three-colored triangle, while Π has two non-empty chromatic components each of which is isomorphic to P_4. It is easy to verify that Π and Δ are CC but their proper subgraphs are already not CC, for every $k \geq 2$; in other words, Π and Δ are minimal CC k-graphs. It was shown in [11] that there are no others.

Theorem 3. *([11]; see also [13, 14]) Every CC k-graph contains a* Π *or* Δ *as a subgraph.* □

2.3.2 Canonical decomposition of Π- and Δ-free k-graphs

Given a Π- and Δ-free k-graph $\mathcal{G} = (V; E_1, \ldots, E_k)$, by Theorem 3, there is an $i \in [k]$ such that the complement $\overline{G}_i$ to the chromatic components $G_i = (V, E_i)$ is not connected on V. It is easy to show that such an $i \in [k]$ is unique.

Lemma 1. *Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs on the common vertex-set V such that both complementary graphs $\overline{G}_1 = (V, \overline{E}_1)$ and $\overline{G}_2 = (V, \overline{E}_2)$ are not connected. Then $E_1 \cap E_2 \neq \emptyset$.*

Proof. Let $V_i \subset V$ be a connected component of $\overline{G}_i$, then all edges between V_i and $V \setminus V_i$ belong to E_i, for both $i = 1$ and $i = 2$. Then $E_1 \cap E_2 \neq \emptyset$, since $V_i \neq \emptyset$ and $V_i \neq V$ for both $i = 1$ and $i = 2$. □

So, let $G_i = (V, E_i)$ be a unique not CC component of $\mathcal{G}$. Let us decompose its complement into connected components and consider the corresponding induced k-graphs (note that there are at least two of them). Each such k-graph $\mathcal{G}'$ is still Π- and Δ-free. Hence, there exists a unique $j \in [k]$ (note that $j \neq i$, since $\overline{G_i}$ was decomposed into connected components) such that ... etc. Thus, we get a decomposition rooted tree $T = T(\mathcal{G}) = (V', E')$ whose leaves $L \subseteq V'$ are in one-to-one correspondence with V, while all other vertices $N = V' \setminus L$ are labeled by the colors of $[k]$.

By construction, property (a) holds for $T(\mathcal{G})$, yet, (b) should be weaken as follows:

- (b') The labels are distinct for every two adjacent vertices of N.

The labeled rooted tree $T(\mathcal{G})$ was interpreted in [11, 12, 13, 14] as a positional game structure in which $[k]$ is the set of players. Then, condition (a) means that there is no position with a unique (forced) possible move, while (b') means that no player has two successive moves.

2.3.3 UMFSs as positional game structures

Given a complete labeled graph (G, d) that defines an UMFS, it is enough to replace each label d_i by the color i for every $i \in [k]$ to obtain a k-graph $\mathcal{G}$.

Theorem 4. *A k-graph $\mathcal{G} = (V; E_1, \ldots, E_k)$ can be realized by an UMFS (G, d) iff $\mathcal{G}$ is Δ-free and has no $m \geq 2$ triangles colored $(i_i, i_2, i_2), (i_2, i_3, i_3), \ldots (i_{m-1}, i_m, i_m), (i_m, i_1, i_1)$.*

Proof. The existence of a three-colored triangle Δ is in contradiction with the ultrametric inequality, while a two-colored triangle $(i_\ell, i_\ell, i_{\ell+1})$ may exist but then $d_{i_\ell} > d_{i_{\ell+1}}$, by the same inequality. Hence, the distances $d_1, \ldots, d_k$ corresponding to the colors $1, \ldots, k$ can be ordered iff $\mathcal{G}$ contains no cycle of m triangles mentioned in the theorem. □

Remark 2. *Let us notice that k-graph Π contains two triangles colored $(i_i, i_2, i_2), (i_2, i_1, i_1)$. Hence, the right-hand side of Theorem 4 implies that the k-graph $\mathcal{G}$ is Π- and Δ-free.*

Then, according to the previous subsection, $\mathcal{G}$ can be represented by a unique tree $T(\mathcal{G})$. It remains to note that the labeling becomes special, since property (b) is stronger than (b'). One can interpret an UMFS as a positional game structure in every play (a path from v_0 to a leave $v \in L$) of which each player makes at most one move.

2.3.4 CIS property of UMFSs

Given a k-graph $\mathcal{G} = (V; E_1, \ldots, E_k)$, let $S_i \subseteq V$ be an inclusion-maximal independent set of $G_i = (V, E_i)$ and let $S = \cap_{i=1}^k S_i$. It is easily seen that $|S| \leq 1$. Indeed, if $v, v' \in S$ then edge (v, v') would have no color in $\mathcal{G}$. We say that $\mathcal{G}$ is a *CIS* k-graph, or that it has the *CIS property*, if $S \neq \emptyset$ for every selection $\{S_1, \ldots, S_k\}$.

Theorem 5. *([11]; see also [14]). Every Π- and Δ-free k-graph has the CIS property.* □

Remark 3. *A new proof was recently obtained in [17]. It was also conjectured in [11] that every non-CIS k-graph contains a Δ; this conjecture is still open; see [1] for more details.*

It was shown in [2] that Π- and Δ are the only (locally) minimal non-CIS k-graphs.

Applying Theorem 5 to the UMFSs we obtain the following statement.

Corollary 4. *Given an UMFS (G, d) with $im(d) = \{d_1, \ldots, d_k\}$, for each $i \in [k]$, let $S_i \subseteq V$ be an inclusion-maximal vertex set in which no two vertices are at distance d_i.*
Then, $S = \cap_{i=1}^{k} S_i \neq \emptyset$, that is, every such k sets contain a unique common vertex. □

3 Representing QUMFSs by nested transitive digraphs

3.1 Transitive directed graphs

A directed graph (digraph) $D = (V, A)$ is called *transitive* if

$$\text{for any } \ u, w \in V \text{ we have: } \ (u, w) \in A \ \text{ whenever } \ (u, v), (v, w) \in A \ \text{ for a } \ v \in V. \tag{1}$$

Proposition 3. *Let $D = (V, A)$ be a transitive digraph.*

- *If C is a directed cycle then $(u, w) \in A$ for any two vertices u and w of C, or in other words, the vertices of C induce a complete subdigraph in D.*

Furthermore, let $D' = (V', A')$ and $D'' = (V'', A'')$ be two complete subdigraphs of D.

- *If $V' \cap V'' \neq \emptyset$ then a complete digraph is induced by $V' \cup V''$;*
- *If $V' \cap V'' = \emptyset$ and there is an arc $(v', v'') \in A$ such that $v' \in V'$, $v'' \in V''$ then $(w', w'') \in A$ for all $w' \in V'$, $w'' \in V''$.*

Proof. All these three statements result immediately from the transitivity of D. □

The above three claims completely clarify the structure of a transitive digraph $D = (V, A)$.

It is uniquely defined by a partition $V = V_1 \cup \ldots \cup V_m$ and *acyclic* digraph $D' = (V', A')$, where $V' = \{v_1, \ldots, v_m\}$. A complete subdigraph is induced in D by each V_i and $(w_i, w_j) \in A$ whenever $w_i \in V_i, w_j \in V_j$, and $(v_i, v_j) \in A'$; here $i, j \in [m] = \{1, \ldots, m\}$ and $i \neq j$.

In other words, the vertices of a transitive digraph are partitioned into several pairwise disjoint classes (of equivalent vertices) on which a partial order is defined.

Let us notice that in the symmetric (non-directed) case [$(u, w) \in A$ whenever $(w, u) \in A$] the partial order becomes trivial. As we already know, a transitive graph is just the union of m cliques; respectively, its complement is a complete m-partite graph.

3.2 Characterizing QUMFSs

Given a complete digraph $D = (V, A)$ and a positive weighting $d : A \to \{d_1, \ldots, d_k\} \subseteq \mathbb{R}_+$ such that $0 < d_1 < \ldots < d_k$, then, for any $i \in [k] = \{1, \ldots, k\}$ let us set

$$A_i = \{(u, w) \in A \mid d(u, w) \leq d_i\}$$

Theorem 6. *The weighted digraph $(D = (V, A), d)$ defines a QUMFS iff all subdigraph $D_i = (V, A_i)$, $i \in [k]$, of the corresponding nested family are transitive.*

Proof. Only if part: Assume that $D_i = (V, A_i)$ is not transitive for some $i \in [k]$, that is, $d(u, v) \leq d_i$ and $d(v, w) \leq d_i$, while $d(u, w) > d_i$; then $max(d(u, v), d(v, w)) < d(u, w)$.
If part: Conversely, let us assume that $max(d(u, v), d(v, w)) < d(u, w)$ for some $u, v, w \in V$. Then $d(u, v) \leq d_i$ and $d(v, w)) \leq d_i$, while $d(u, w) > d_i$, where $d_i = max(d(u, v), d(v, w))$. Hence, transitivity fails for D_i. □

Corollary 5. *Every QUMFS is uniquely (up to an isometry) realized by Theorem 6*

Proof. Given a QUMFS, let consider the complete digraph on its elements and introduce the weights equal to the corresponding distances. □

Example 2. *Let us consider three vertices $V = \{u, v, w\}$ and define the distances as follows:* $d_1 = d(u, w) = 1, d_2 = d(u, v) = d(w, v) = 2, d_3 = d(w, u) = 3, d_4 = d(v, u) = d(v, w) = 4.$

It is easy to verify the ultrametric inequality and that the next four nested arc-sets $A_1 = \{(u, w)\}, A_2 = \{(u, w), (u, v), (w, v)\}, A_3 = \{(u, w), (u, v), (w, v), (w, u), A_4 = A$ *form transitive digraphs $D_i = (V, A_i),\ i = 1, 2, 3, 4.$*

3.3 Upper bound $k \leq \frac{1}{2}(n-1)(n+2)$ is precise for QUMFSs

First, let us recall that $k \leq n - 1$ for an n-element UMFS and that this bound is precise. From Theorem 6, we will derive a much larger, but also precise, upper bound for QUMFSs.

Theorem 7. *The number k of pairwise distinct distances in a QUMFS on n elements is at most $\binom{n}{2} + n - 1 = \frac{1}{2}(n-1)(n+2)$.*

Proof. Let QUMFS (D, d) be standardly given by a complete digraph $D = (V, A)$ and weighting $d : A \to \{d_1, \ldots, d_k\}$ such that $d_1 < \ldots < d_k$. Each arc (u, w) belongs to $n - 2$ triangles $(u, v), (v, w), (u, w)$, where $v \in V \setminus \{u, w\}$. If $d(u, w) = d_k$ then value d_k must appear at least $n - 2$ times more among $\{d(u, v), d(v, w) \mid v \in V \setminus \{u, w\}\} \subseteq D_k = \{d_1, \ldots, d_k\}$, by the ultrametric inequality. Furthermore, if $d(u_0, w_0) = d_k$ and $d(u_1, w_1) = d_{k-1}$ then the values d_k and d_{k-1} must appear at least $(n - 2) + (n - 3) = 2n - 5$ times more among

$$D_0 \cup D_1 = \{d(u_0, v_0), d(v_0, w_0) \mid v_0 \in V \setminus \{u_0, w_0\}\} \cup \{d(u_1, v_1), d(v_1, w_1) \mid v_1 \in V \setminus \{u_1, w_1\}\} \subseteq D_k.$$

Indeed, it is enough to notice that $|D_0| = |D_1| = n - 2$ and $|D_0 \cap D_1| = 1$ whenever $u_0 = u_1$ or $w_0 = w_1$; otherwise $D_0 \cap D_1 = \emptyset$ and, hence, $|D_0 \cup D_1| \geq 2n - 5$, by Theorem 6.

In general, for any $j \in \{1, \ldots, n - 2\}$, if $d(u_i, w_i) = d_{k-i}$ for all $i \in I_j = \{0, \ldots, j\}$ then the values $\{d_{k-i} \mid i \in I_j\}$ must appear at least

$$\sum_{i=0}^{j}(n - 2 - i) = (j + 1)(n - 2 - j/2)$$

times more among

$$\bigcup_{i=0}^{j} D_i = \bigcup_{i=0}^{j}\{d(u_i, v_i), d(v_i, w_i) \mid v_i \in V \setminus \{u_i, w_i\}\}.$$

Indeed, it is enough to notice that D_j and $\cup_{i=1}^{j-1} D_i$ have at most j elements in common, which happens when $u_0 = u_1 = \ldots = u_j$ or $w_0 = w_1 = \ldots = w_j$.

In particular, substituting $j = n-2$, we conclude that there are at least $\frac{1}{2}(n-1)(n-2)$ repetitions among the distances $d(u, w)$ for $u, w \in V$. Hence, for the number k of pairwise distinct distances we obtain $k \leq n(n-1) - \frac{1}{2}(n-1)(n-2) = \frac{1}{2}(n-1)(n+2)$. □

It is also not difficult to demonstrate that this bound is precise.

Example 3. *Given $V = \{v_1, \ldots, v_n\}$, first, let us consider the $\binom{n}{2}$ ordered pairs $v_i, v_j \in V$ such that $i < j$ in the lexicographic order and assign the distinct values $1, \ldots, \binom{n}{2}$ to the $\binom{n}{2}$ corresponding distances $d(v_1, v_2), \ldots, d(v_1, v_n), d(v_2, v_3), \ldots, d(v_2, v_n), \ldots, d(v_{n-1}, v_n)$.*

Then, let us consider the remaining $\binom{n}{2}$ pairs $v_i, v_j \in V$, for which $i > j$, in the inverse lexicographic order and set: $d(v_n, v_{n-1}) = \binom{n}{2}+1, d(v_n, v_{n-2}) = d(v_{n-1}, v_{n-2}) = \binom{n}{2}+2, \ldots,$ $d(v_n, v_1) = \ldots = d(v_2, v_1) = \binom{n}{2} + n - 1 = \frac{1}{2}(n-1)(n+2)$.

Thus, $k = \frac{1}{2}(n-1)(n+2)$ and it is easy to verify that the ultrametric inequality holds.

Moreover, it was shown by Frank and Frisch in [9] that the equality $k = \frac{1}{2}(n-1)(n+2)$ may hold already for the flow QUMFSs; see Section 4.2 for the definitions.

4 Realizing (Q)UMFSs by networks

4.1 Bottleneck QUMFSs

Let (D, c) be a network defined by a strongly connected digraph $(D = (V, A))$ and strictly positive weight function, $c : A \to \{c_1, \ldots, c_k\} \in \mathbb{R}_+$, where $0 < c_1 < \ldots < c_k$.

We will interpret $c(u, w)$ as a *width* of the arc $(u, w) \in A$, that is, the largest size of on object that can pass (u, w). Then, obviously, the width of a directed path (dipath) $p(u, w)$ from u to w is the minimum of the widths of its arcs $C(p(u, w)) = \min\{c(e) \mid e \in p(u, w)\}$.

Let us define the width $C(u, w)$ as the largest size of on object that can pass from u to w for all $u, w \in V$ (not necessarily $(u, w) \in A$). Obviously, $C(u, w) = \max\{C(p(u, w) \mid p(u, w)\}$ is the width of a max min (or widest bottleneck) dipath from u to w.

By definition, function $C : V \times V \to \mathbb{R}_+$ takes only the (strictly positive) values $c_1, \ldots, c_k$; it cannot be equal to 0, since D is a strongly connected digraph; see Remark 4 below.

Lemma 2. *The inequality $C(u, w) \geq \min(C(u, v), C(v, w))$ holds for all $u, v, w \in V$.*

Proof. If an object can pass from u to v and from v to w then it can pass from u to w. □

Let $d(u, w) = C^{-1}(u, w)$ be the inverse width and $d(v, v) = 0$ for all $u, v, w \in V$.

Proposition 4.
- *Mapping d is a QUMFS for any network (D, c).*
- *Furthermore, d is an UMFS whenever (D, c) is symmetric, that is, $(u, w) \in A$ iff $(w, u) \in A$ and $c(u, w) = c(u, w)$ for all $u, w \in V$.*
- *Moreover, all (Q)UMFSs can be realized in this way.*

Proof. It is easily seen that the inequality of Lemma 2 for C is equivalent with the ultrametric inequality for d and the first statement follows. The second one is obvious.

To realize a given QUMFS d by a network (D, c) it is enough to define $D = (V, E)$ as the complete digraph on V and set $c(u, w) = d^{-1}(u, w)$ for all $u, w \in V$. Obviously, the obtained network (D, c) is symmetric whenever d is an UMFS. □

For the UMFSs, the above two statements were mentioned by Leclerc in [23].

Remark 4. *According to Remark 1, we can easily adjust the above definitions and statements for the case of pseudo-(Q)UMFSs: It is sufficient to allow for functions d, c and C the values 0 and ∞ (assuming standardly that they are mutually inverse, $0^{-1} = \infty$ and $\infty^{-1} = 0$). Also we should include all, not only strongly connected, digraphs $D = (V, A)$ into consideration and set $d(u, w) = \infty$ (and $C(u, w) = 0$) whenever there is no dipath from u to w.*

4.2 Flow QUMFSs

4.2.1 Ultrametric inequality for inverse capacities

Given a network (D, c), let us now interpret $c(u, w)$ as a *capacity* of the arc $(u, w) \in A$, that is, the largest amount of a material that can be transported along (u, w) per a unit time.

Then, obviously, the capacity of a dipath $p(u, w)$ from u to w is again the minimum of the capacities of its arcs, $C(p(u, w)) = \min\{c(e) \mid e \in p(u, w)\}$. Furthermore, for all $u, w \in V$ (not necessarily $(u, w) \in A$), let us define the capacity $C(u, w)$ as the largest amount of a material that can be transported in a unit time from u to w, assuming that all other vertices are transient and the conservation law holds for each of them.

Function $C : V \times V \to \mathbb{R}_+$ can take only the (strictly positive) values $c_1, \dots, c_k$ and their sums. (It cannot be equal to 0, since D is a strongly connected digraph; see Remark 4, yet).

Lemma 3. *([10]). The inequality $C(u, w) \geq \min(C(u, v), C(v, w))$ holds for all $u, v, w \in V$.*

This is an exact copy of Lemma 2. However, the proof cannot be just copied. Indeed, assuming that $C(u, v) \geq x$ and $C(v, w) \geq x$ we have to show that $C(u, w) \geq x$. It would suffice just to sum up two x-flows that realize $C(u, v)$ and $C(v, w)$. Yet, by this operation, the capacity of an edge can be exceeded. However, the result can be easily derived from the maximum flow and minimum cut theorem [7].

For the symmetric networks and UMFSs the proof was given by Gomory and Hu [10].

The same arguments work for the digraphs and QUMFSs as follows.

Proof. By the Ford-Fulkerson theorem, $C(u, w)$ is equal to the capacity of a critical (directed) cut (U, V) of u from w, where $V = U \cup W$, $U \cap W = \emptyset$, $u \in U$ and $w \in W$. Obviously, the same (U, V) cuts v from w (respectively, u from v) whenever $v \in U$ (respectively, $v \in W$). It is easily seen that in both cases the inequality follows. □

Again, let $d(u, w) = C^{-1}(u, w)$ be the inverse capacity and $d(v, v) = 0$ for all $u, v, w \in V$.

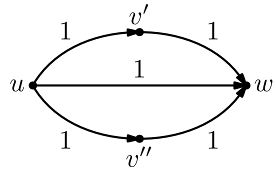

Figure 3

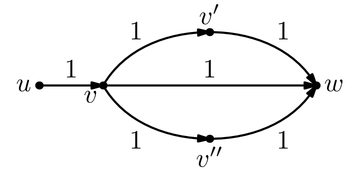

Figure 4

Proposition 5. *([10]).*

- *Mapping d defines a QUMFS for every network (D, c).*
- *Furthermore, d is an UMFS whenever (D, c) is symmetric.*
- *Moreover, any UMFS (but not any QUMFS) can be realized in this way.*

Proof. As in Proposition 3, the first statement immediately follows from Lemma 3, while the second one is obvious. Finally, the last one results from Corollary 1 as follows.

Given an UMFS (G, d), where $G = (V, E)$ is the complete graph on V, let us construct a lightest spanning tree $T = (V, E')$ in (G, d) and set $c(e) = d^{-1}(e)$ for all $e \in E'$. It is easily seen that the obtained symmetric flow network (T, c) defines the original UMFS (G, d). $\square$

Remark 5. *Here, we should repeat Remark 4, word to word.*

4.2.2 Not every QUMFS is a flow QUMFS

A (Q)UMFS generated by a bottleneck or flow network is naturally called a bottleneck or flow (Q)UMFS, respectively. By Proposition 3, every (Q)UMFS is a bottleneck (Q)UMFS. Furthermore, by Proposition 5, every UMFS is a flow UMFS. However, it is not difficult to construct a non-flow QUMFS. It appears even simpler to start with a pseudo-QUMFS.

Example 4. *Let us consider pseudo-QUMFS (D^0, d^0) defined by the following (not complete) digraph $D^0 = (V^0, A^0)$, given in Figure 3, and unit weight function $d^0 : A^0 \to \{1\}$:*

$$V^0 = \{u, v', v'', w\},\ A^0 = \{(u, v'), (u, v''), (v', w), (v'', w), (u, w)\},\ d^0(a) \equiv 1\ \forall\ a \in A. \quad (2)$$

Obviously, digraph D^0 is transitive and, by convention, $d(a, b) = \infty$ (that is, $C(a, b) = 0$) whenever $(a, b) \notin A$. Yet, (D^0, d^0) cannot be realized by a flow network. Indeed, to have $d(u, v') = d(u, v'') = d(v', w) = d(v'', w) = 1$ we must set $c(u, v') = c(u, v'') = c(v', w) = c(v'', w) = 1$. But then, $C(u, w) \geq 2$ and $d(u, w) \leq 0.5$, while $d^0(u, w) = 1$; a contradiction.

Remark 6. *Let us notice, however, that we obtain a flow pseudo-QUMFS just replacing $d^0(u, w) = 1$ by any $d^0(u, w) \in [0; 0.5]$ and keeping all other distances as in Example 4.*

Example 5. *To get a non-flow QUMFS, let us extend D^0 to a complete digraph on V^0 and introduce large enough $d(a, b)$, say $d(a, b) = 10$, for all $a, b \in V^0$ such that $(a, b) \notin A^0$.*

4.2.3 Yet, every QUMFS can be realized by a multi-pole flow network

First, let us show that the pseudo-QUMFS (D^0, d^0) of Example 4 can be easily realized by a flow network on a slightly larger vertex-set.

To do so, let us replace u by a new arc (u, v), as in Figure 4, and consider the network $(D = (V, A), c)$ in which

$V = \{u, v, v', v'', w\}$, $A = \{(u, v), (v, v'), (v, v''), (v', w), (v'', w), (v, w)\}$, and $c(a) = 1 \;\forall\, a \in A$.

It is not difficult to verify that

$C(u, v') = C(u, v'') = C(v', w) = C(v'', w) = C(u, w) = 1$ and $C(a, b) = 0$ for all other $a, b \in V^0 = \{u, v, v', v'', w\}$. Hence, $d(u, v') = d(u, v'') = d(v', w) = d(v'', w) = d(u, w) = 1$, as requested. Let us notice that $C(v, w) = 3$ and $d(v, w) = 1/3$ but it does not matter.

A still larger multi-pole network is needed to realize the similar QUMFS in which ∞ is replaced, say, by 10. We leave this analysis to the reader. Instead, let us demonstrate how the procedure works in general implying that every QUMFS is a subspace of a flow QUMFS.

Theorem 8. *Each pseudo-QUMFS given by a weighted digraph $(D = (V, A), d)$ can be realized by a multi-pole flow network $(D' = (V', A'), V \subseteq V', c)$, where $V \subseteq V'$ is a set of poles and $d(u, w) = C^{-1}(u, w)$ for all $u, w \in V$.*

For the beginning, let us consider the case $k = 1$, that is, $0 \le d_1 \le \infty$.

Let us replace in the digraph $D = (V, A_1)$ each vertex $v \in V$ by an arc (v, v^1), every arc $(v, u) \in A_1$ by an arc (v^1, u), and set $c(e) = d_1^{-1}(e)$ for every obtained arc e. Since A_1 is transitive, in the obtained weighted digraph, we get $C(u, w) = d_1^{-1}$ for all $(u, w) \in A_1$ and $C(u, w) = 0$ for all other pairs $u, w \in V$.

Proof of Theorem 8. In general, from $i = k$ to $i = 0$ do: For each vertex $v \in V$ introduce a new vertex v^i and arc (v, v^i), then, replace every arc $(v, u) \in A_i$ by (v^i, u), and set $c(e) = d_i^{-1} - d_{i+1}^{-1}$ for all new arcs. Standardly, we assume that $0 \le d_1 < \ldots < d_k \le \infty$ and set $d_0 = 0, d_{k+1} = \infty$. Then, by transitivity of A_i for all $i \in [k]$, we obtain

$$C(u, w) = \sum_{i \mid (u,w) \in A_i} d_i^{-1} - d_{i+1}^{-1} = d_{i(u,w)}^{-1} = d^{-1}(u, w) \tag{3}$$

for every ordered pair $u, w \in V$, where $i(u, w) = \max(i \mid (u, w) \in A_i)$. □

In particular, these arguments work for a complete digraph D, that is, for a QUMFS.

4.2.4 Recognizing flow QUMFSs and realizing them by flow networks; $k = 1$

Given a QUMFS (D, d) in which $D = (V, A)$ is a complete digraph and $d : A \to \{d_1, \ldots, d_k\}$ is a weight function such that $0 < d_1 < \ldots < d_k < \infty$, assume for the beginning that $k = 1$. In this case, QUMFS (D, d) is defined by d_1 and the corresponding (typically, not complete) digraph $D_1 = (V, A_1)$. Then, the results of subsections 4.2.2 and 4.2.3 provide the following characterization of the flow QUMFSs.

Proposition 6. *A QUMFS (D, d) (of $k = 1$) is realized by a flow network iff digraph $D_1 = (V, A_1)$ does not contain $D^0 = (V^0, A^0)$ from Example 4 as an induced subdigraph.*

Proof. The "only if" part was already shown in Example 4; let us prove the inverse statement.

By Theorem 6, digraph $D_1 = (V, A_1)$ is transitive and hence, by Proposition 3, its structure is described as follows: D_1 is uniquely defined by a partition $V = V_1 \cup \ldots \cup V_m$ and by an *acyclic* transitive digraph $D' = (V', A')$ such that $V' = \{v_1, \ldots, v_m\}$, a complete subdigraph is induced in D_1 by each V_i; furthermore, $(w_i, w_j) \in A_1$ iff $w_i \in V_i, w_j \in V_j$, and $(v_i, v_j) \in A'$, for any $i, j \in [m] = \{1, \ldots, m\}$ such that $i \neq j$. In other words, A' defines a partial order P over V'; see Section 3.1.

It is easily seen that D_1 contains D^0 as an induced subdigraph if and only if D' does.

Finally, let $D'' = (V', A'')$ denote the so-called *Hasse diagram* of P; in other words, $D'' = (V', A'')$ is a (unique) subdigraph of $D' = (V', A')$ such that $(v_i, v_j) \in A''$ iff v_j is a *cover* of v_i in D', that is, $(v_i, v_j) \in A'$ but $(v_i, v), (v, v_j) \in A'$ holds for no $v \in V'$.

Now, for each arc $(u, w) \in A_1$ let us introduce its capacity $c(u, w)$ as follows:

$$c(u,w) = \begin{cases} (d_1(n_i - 1))^{-1} & \text{if } u, w \in V_i \text{ for some } i \in [m], \text{where } n_i = |V_i|; \\ (d_1 n_i n_j)^{-1} & \text{if } u \in V_i\,,\ w \in V_j\,, \text{and } v_j \text{ is a cover of } v_i; \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

We will show that the total capacity $C(u, w) = d_1^{-1}$ whenever $(u, w) \in A_1$ and $C(u, w) = 0$ otherwise. Let us assume that $u \in V_i, w \in V_j$ and consider the following three cases:

Case 1: $i = j$. In this case $(u, w) \in A_1$ and, hence, there are $n_i - 1$ dipaths from u to w in D_1, one of which consists of 1 arc, while the remaining $n_i - 2$ consist of 2 arcs each.
By (4), each of these $n_i - 1$ dipaths is of capacity $((n_i - 1)d_1)^{-1}$ and, hence, $C(u, w) = d_1^{-1}$.

Case 2: $i \neq j$ and $(u, w) \in A_1$, i.e., v_j is a successor (but not necessarily a cover) of v_i.
Let us suppose that (t) there are (at least) two dipaths from v_i to v_j in D''.
Then, each of them contains at least two arcs, by the definition of the Hasse diagram.
Hence, by transitivity, (tt) D' (or equivalently, D_1) contains D^0 as an induced subdigraph, in contradiction with the main assumption of the theorem.

Remark 7. *In fact, the converse is also true, that is, statements (t) and (tt) are equivalent.*

Thus, (ttt) there is a unique dipath from v_i to v_j in D'. In this case again $C(u, w) = d_1^{-1}$ for every $u \in V_i$ and $w \in V_j$, by (4). Indeed, cut (V_i, V_j) contains $n_i n_j$ arcs of capacity $(n_i n_j d_1)^{-1}$ each and, by (ttt), there are no other dipaths from V_i to V_j. Hence, $C(u, w) \leq d_1^{-1}$. On the other hand, $n_i(n_i - 1)$ arcs of V_i of capacity $(d_1(n_i - 1))^{-1}$ each, $n_j(n_j - 1)$ arcs of V_j of capacity $(d_1(n_j - 1))^{-1}$ each, and $n_i n_j$ arcs from V_i to V_j of capacity $(n_i n_j d_1)^{-1}$ each are, obviously, sufficient to transport d_1^{-1} from u to w.

Case 3: $i \neq j$ and $(u, w) \notin A_1$, that is, v_j is not a successor of v_i in D'. Then, there is no dipath from v_i to v_j in D'', D', or D and hence $C(u, w) = 0$. □

Remark 8. *Standardly, we extend Proposition 6 to pseudo-QUMFSs just assuming that* $0 \le d_1 \le \infty$ *rather than* $0 < d_1 < \infty$*.*

Let us underline that, in Proposition 6, only an *induced* subdigraph D^0 is an obstruction.

Example 6. *If we extend the digraph* $D^0 = (V^0, A^0)$ *from Example 4 by one new arc* (v', v) *and define* $d(a) = 1$ *for all* $a \in A'$*, while* $d(a) = \infty$ *for all* $a \notin A'$*, we will not get a pseudo-QUMFS, since the obtained digraph* $D' = (V, A')$ *is not transitive.*

To get a transitive digraph $D = (V, A)$*, we have to add one more arc* (v', v'')*.*

Then, to get a pseudo-QUMFS (D, d)*, we just extend* d *by setting* $d(v', v'') = 1$*, too.*

By Proposition 6, the obtained pair (D, d) *is a flow pseudo-QUMFS and the capacities of the corresponding flow network are defined by (4) as follows:*

$c(u, v') = c(v', u) = c(v'', w) = 1, c(u, v'') = c(v', v'') = \frac{1}{2}$ *and* $c(x, y) = 0$ *for other* $x, y \in V$*.*

Example 7. *Similarly, if we extend* $D^0 = (V^0, A^0)$ *by two new arcs* (v', u) *and* (v'', u) *and standardly define* $d(a) = 1$ *for all* $a \in A'$*, while* $d(a) = \infty$ *for all* $a \notin A'$*, we will not get a pseudo-QUMFS, since the obtained digraph* $D' = (V, A')$ *is not transitive.*

To get a transitive digraph $D = (V, A)$*, we add two more arcs* (v', v'') *and* (v'', v')*. Then, to get a pseudo-QUMFS* (D, d)*, we just extend* d *by setting* $d(v', v'') = d(v'', v') = 1$*, too. By Proposition 6,* (D, d) *is a flow pseudo-QUMFS. The capacities of the corresponding flow network are defined by (4) as follows:*

$c(x, y) = \frac{1}{2}, c(x, w) = \frac{1}{3}$ *for any distinct* $x, y \in \{u, v', v''\}$*, while* $c = 0$ *for all remaining arcs.*

Example 8. *Finally, let us extend* $D^0 = (V^0, A^0)$ *by two new arcs* (v', u) *and* (w, v'') *and in the obtained digraph* $D' = (V, A')$ *standardly define* $d(a) = 1$ *for all* $a \in A'$ *and* $d(a) = \infty$ *for all* $a \notin A'$*. Again, to get a transitive digraph* $D = (V, A)$*, we add the arc* (v', v'') *and, to get a pseudo-QUMFS* (D, d)*, we extend* d *by setting* $d(v', v'') = 1$*. Then, by Proposition 6,* (D, d) *is a flow pseudo-QUMFS and the capacities of the corresponding flow network are defined by (4) as follows:*

$c(x, y) = \frac{1}{4}$ *for all* $x \in \{u, v'\}, y \in \{v'', w\}$*, while* $c(u, v') = c(v', u) = c(v'', w) = c(w, v'') = 1$*, and* $c = 0$ *for all remaining arcs.*

In three above examples, it is not difficult to compute the effective capacities and verify the equality $C(x, y) = d^{-1}(x, y)$ for every pair of distinct vertices $x, y \in V = \{u, v', v'', w\}$.

4.2.5 Recognizing flow QUMFSs and realizing them by flow networks

Now, let us consider the general case: $k \ge 1$. Given a pseudo-QUMFS $(D = (V, A), d)$ in which $d : A \to \{d_1, \ldots, d_k\}$ and $0 \le d_1 < \ldots < d_k \le \infty$, we wish either to construct on the same digraph a flow network $(D = (V, A), c)$ whose effective capacities $C(u, w)$ are equal to the inverse distances $d^{-1}(u, w)$ for all $u, w \in V$, or to prove that there is no such network.

We will need several iterations. The first one is as follows. Let us consider the (transitive) digraph $D_k = (V, A_k)$ and define the acyclic transitive digraph $D' = (V', A')$ and Hasse diagram $D'' = (V', A'')$ as in the previous subsection, in which we had $k = 1$.

Furthermore, let us assign the capacity $c(a) = d_k^{-1}$ to each $a \in A''$ and compute the effective capacity $C'(v_i, v_j)$ for all $v_i, v_j \in V'$ in the obtained flow network (D'', c).

Then, let us recall the original digraph D_k, define $c(u, w)$ for all $(u, w) \in A_k$ by formula (4) (in which $k = 1$), and compute the effective capacities $C(u, w)$ for all $u, w \in V$. Finally, in the QUMFS (D, d), let us compare $C(u, w)$ and $C'(v_i, v_j)$ for all $u \in V_i, w \in V_j$.

If $C(u, w) < C'(v_i, v_j)$ for some $u, w \in V$ then, obviously, (D, d) is not a flow QUMFS.

Otherwise, let us update $d(u, w)$ by setting $d^1(u, w) = (C(u, w) - C'(v_i, v_j))^{-1}$. In particular, $d^1(u, w) = \infty$ if and only if $C(u, w) = C'(v_i, v_j)$. Obviously, this equality holds whenever $(v_i, v_j) \in A''$. Then, let us repeat the whole procedure for the obtained pseudo-QUMFS $(D^1 = (V, A^1), d^1)$, etc., getting $(D^\ell = (V, A^\ell), d^\ell)$ after each iteration $\ell = 0, 1, \dots, L$ (assuming that $d^0 = d_k$, $D^0 = D_k$, and $A^0 = A_k$ for the initial iteration).

Let us note that the distances $d^\ell(u, w)$ are monotone non-decreasing in ℓ and at least one of them becomes ∞ in each step. Hence, the arc-sets A^ℓ are strictly monotone decreasing in ℓ implying that $L < n(n - 1)$ where $n = |V|$. After L iterations we either prove that (D, d) is not a flow pseudo-QUMFS, or realize it by a flow network introducing the cumulative capacities: $c(a) = \sum_{\ell=0}^{L} c^\ell(a)$ for all $a \in A$. Obviously, the obtained algorithm is polynomial for the pseudo-QUMFSs and for QUMFSs, in particular. □

Remark 9. *Let us notice that, unlike the arc-sets A^ℓ, the numbers of pairwise distinct distances may not decrease in ℓ, that is, strict inequalities $k^\ell < k^{\ell+1}$ may hold.*

4.3 Finite ultrametric spaces of resistances

Both bottleneck and flow UMFS can be realized as resistance distances [18]; see also [19, 16].

Given a (non-directed) *connected* graph $G = (V, E)$ in which each edge $e \in E$ is an isotropic conductor with the monomial conductivity law

$$y_e^* = y_e^r / \mu_e^s.$$

Here y_e is the voltage, or potential difference, y_e^* current, and μ_e is the resistance of e, while r and s are two strictly positive real parameters independent of $e \in E$.

In particular, the case $r = 1$ corresponds to Ohm's low in electricity, while $r = 0.5$ is the standard square law of resistance typical for hydraulics or gas dynamics. Parameter s, in contrast to r, is redundant; yet, it will play an important role too.

It is not difficult to see that for any two arbitrary nodes $u, w \in V$ the obtained two-pole circuit (G, u, w) satisfies the same monomial conductivity law $y_{u,w}^* = y_{u,w}^r / \mu_{u,w}^s$, where $y_{u,w}^*$ is the total current and $y_{u,w}$ voltage between u and w, while $\mu_{u,v}$ is the *effective resistance* of (G, u, w). In other words, (G, u, w) can be effectively replaced by a single edge $e = (u, w)$ of resistance $\mu_{u,w}$ with the same r and s. Obviously, $\mu_{u,w} = \mu_{w,u}$, due to symmetry (isotropy) of the conductivity functions; it is also clear that $\mu_{u,w} > 0$ whenever $u \neq w$; finally, by convention, we set $\mu_{u,w} = 0$ for $u = w$.

In [18], it was shown that for arbitrary three nodes u, v, w the following inequality holds

$$\mu^{s/r}_{u,w} \leq \mu^{s/r}_{u,v} + \mu^{s/r}_{v,w}.$$

In [19] it was also shown that it holds with equality if and only if node v belongs to every path between u and w.

Clearly, if $s \geq r$ then we obtain the standard triangle inequality $\mu_{u,w} \leq \mu_{u,v} + \mu_{v,w}$ and the ultrametric inequality $\mu_{u,w} \leq \max(\mu_{u,v}, \mu_{v,w})$ appears when $s/r \to \infty$.

Thus, a circuit can be viewed as a metric space in which the distance between any two nodes u and w is the effective resistance $\mu_{u,w}$. Playing with parameters r and s, one can get several interesting examples. Let $r = r(t)$ and $s = s(t)$ depend on a real parameter t; in other words, these two functions define a curve in the positive quadrant $r \geq 0, s \geq 0$.

It is shown in [18, 16] that for the next four limit transitions, as $t \to \infty$, for all pairs of poles $u, w \in V$, the limits $\mu_{u,w} = \lim_{t\to\infty} \mu_{u,w}(t)$ exist and can be interpreted as follows:

- (i) The effective Ohm resistance between poles u and w, when $s(t) = r(t) \equiv 1$, or more generally, whenever $s(t) \to 1$ and $r(t) \to 1$.

- (ii) The standard length (travel time or cost) of a shortest route between terminals u and w, when $s(t) = r(t) \to \infty$, or more generally, $s(t) \to \infty$ and $s(t)/r(t) \to 1$.

- (iii) The inverse width of a maxmin path between terminals u and w when $s(t) \to \infty$ and $r(t) \equiv 1$, or more generally, $r(t) \leq const$, or even more generally $s(t)/r(t) \to \infty$.

- (iv) The inverse capacity between terminals u and w, when $s(t) \equiv 1$ and $r(t) \to 0$; or more generally, when $s(t) \to 1$, while $r(t) \to 0$.

Obviously, all four example define metric spaces, since in all cases $s(t) \geq r(t)$ for any sufficiently large t. Moreover, for the last two examples the ultrametric inequality holds for any $u, v, w \in V$, because $s(t)/r(t) \to \infty$, as $t \to \infty$, in the cases (iii) and (iv).

These examples allow us to interpret s and r as parameters of a transportation problem.

In particular, s can be viewed as a measure of divisibility of a transported material; $s(t) \to 1$ in examples (i) and (iv), because liquid, gas, or electrical charge are fully divisible; in contrast, $s(t) \to \infty$ for (ii) and (iii), because a car, ship, or individual transported from u to w are indivisible. Furthermore, the ratio s/r can be viewed as a measure of subadditivity of the transportation cost; so $s(t)/r(t) \to 1$ in examples (i) and (ii), because in these cases the cost of transportation along a path is additive, i.e., is the sum of the costs of the edges that form this path; in contrast, $s(t)/r(t) \to \infty$ for (iii) and (iv), because in these cases only edges of the maximum cost (capacity or of the bottleneck width) matter. Other values of parameters s and s/r, between 1 and ∞, correspond to an intermediate divisibility of the transported material and subadditivity of the transportation cost, respectively.

4.4 Reducible, universal, and complete families of (Q)UMFSs

Given two (Q)UMFSs $(D' = (V', A'), d')$ and $(D'' = (V'', A''), d'')$ with a common vertex-set $V \subseteq V' \cap V''$, let us call (D', d') and (D'', d'') V-isometric if $d'(u, w) = d''(u, w)$ for all (ordered) pairs $u, w \in V$.

A family F of (Q)UMFSs will be called:

- *reducible* if for any $(D' = (V', A'), d') \in F$ and $V \subseteq V'$ there is a $(D = (V, A), d) \in F$ such that (D, d) and (D', d') are V-isometric;

- *universal* if for any (Q)UMFS $(D = (V, A), d))$ there is a (Q)UMFS $(D' = (V', A'), d') \in F$ such that $V \subseteq V'$ and restriction of the latter (Q)UMFS to V coincide with the former one;

- *complete* if F contains all (Q)UMFSs.

Obviously, a complete family is universal and a universal one is reducible; moreover, both these containments are strict.

In the last section of [16], it was shown that every k-pole n-vertex network, $k \leq n$, can be replaced by an equivalent k-vertex network when $r = s = 1$; in other words, the corresponding family of the resistance UMFSs is reducible.

In fact, the same arguments work for $r = 1$ and any $s > 0$. In particular, the symmetric bottleneck networks (for which $r = 1, s \to \infty$) generate reducible UMFSs too. Moreover, this family of UMFSs is complete, as well as the family of UMFSs generated by the symmetric flow networks (for which $s = 1, r \to 0$); see Sections 4.1 and 1.3, respectively.

As for the families F_B and F_C of QUMFSs generated by the general (not necessarily symmetric) bottleneck and flow networks, F_B is complete, while F_C is universal but not complete, by the results of Sections 4.1 and 4.2, respectively.

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