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Abstract. For any non-negative integers a and b, we consider the following game $WYT(a,b)$. Given two piles that consist of x and y matches, two players alternate turns; a single move consists of a player choosing x' matches from one pile and y' from the other, such that

$$0 \leq x' \leq x,\ 0 \leq y' \leq y,\ 0 < x' + y', \text{ and } [\min(x', y') < b \text{ or } |x' - y'| < a].$$

The player who takes the last match is the winner in the normal version of the game and (s)he is the loser in its misere version.
It is easy to verify that the cases $(a = 0, b = 1), (a = b = 1)$, and $(b = 1, \forall\, a)$ correspond to the two-pile NIM, the Wythoff, and Fraenkel games, respectively. The concept of the minimum excludant mex is known to be instrumental in solving the last two games. We generalize this concept by introducing a function mex_b (such that $mex = mex_1$) to solve the normal and misere versions of the game $WYT(a,b)$.

Keywords: combinatorial games, impartial games, NIM, Wythoff game, Fraenkel game, minimal excludant, normal and misere versions, Sprague-Grundy function

1 The game $WYT(a, b)$ and its special cases

The game $WYT(a, b)$ was defined in the abstract. By this definition, a player can take any number of matches from one pile and at most $b - 1$ from the other (that is, $\min(x', y') < b$), or he can take two amounts that differ by at most $a - 1$ (that is, $|x' - y'| < a$), yet, in both cases he is not allowed to pass his turn (that is, $x' + y' > 0$).

If $a = 0, b = 1$, we get the standard (and trivial) NIM with two piles. Indeed, the second option, $|x' - y'| < a$, becomes impossible and, hence, either $x' = 0$ or $y' = 0$, but not both.

If $a = b = 1$ then a player can take either

- (i) any positive number of matches from one pile and none from the other (that is, $x' + y' > 0$ and $\min(x', y') = 0$), or
- (ii) the same positive number of matches from each pile (that is, $x' = y' > 0$).

Thus, $WYT(1, 1)$ coincide with the classical game introduced in 1907 by Wythoff [13].

In [5, 6] Fraenkel generalized this game, replacing the equality $x' = y'$ in (ii) by a weaker constraint $|x' - y'| < a$. The resulting game is $WYT(a, 1)$. In this paper, we also replace the equality $\min(x', y') = 0$ in (i) with a weaker constraint $\min(x', y') < b$, getting $WYT(a, b)$.

Remark 1 *We could generalize even further by replacing the inequality* $\min(x', y') < b$ *with* $((x' < b \; \forall \; y')$ *or* $(y' < c \; \forall \; x'))$. *Yet, in Section 7 we will see that the resulting game* $WYT(a, b, c)$ *is trivial unless* $b = c$. *In that section we will also consider two more trivial cases,* $a = 0$ *and* $b = 0$, *but in Sections 1-6 we assume that* $a > 0$ *and* $b > 0$.

The positions of $WYT(a, b)$ are pairs (x, y), where x and y denote the numbers of matches in the two piles. By default, we will assume that $x \leq y$. Furthermore, (x, y) is called a *P-position* if the player who enters it (the Previous player) can win. Otherwise, (x, y) is called an *N-position*, since in this case the player who leaves it (the Next player) can win.

Clearly, each move from a P-position leads to an N-position and for every N-position there is a move to a P-position. To solve a game, it is sufficient to find all its N- or P-positions.

Due to the symmetry of $WYT(a, b)$, a pair (x, y) is a P-position if and only if (y, x) is.

Obviously, there is a unique terminal position $(0, 0)$, since $b > 0$. By definition, $(0, 0)$ is a P-position in the normal version of $WYT(a, b)$ and an N-position in its misere version.

In this paper, both the normal and misere versions are recursively solved, namely, we obtain a recursive formula for the P-positions.

2 The solution of Fraenkel's games

Let us start with $b = 1$. In this case the game $WYT(a, 1) = WYT(a)$ was solved by Fraenkel; see [5] and [6] for the standard and misere versions, respectively. We will postpone discussion of the misere version until Section 6, where it will be considered in the more general setting of $WYT(a,b)$. As for the standard version of $WYT(a)$, the set of its P-positions $\{(x_n, y_n) \mid n = 0, 1, \ldots\}$ was characterized in [5] by the following recursion:

$$x_n = mex\{x_i, y_i \mid 0 \le i < n\}, \quad y_n = x_n + an, \quad n \ge 0, \tag{1}$$

where the *minimum excludant* function $mex(S)$ is defined for any subset $S \subset \mathbb{Z}_+$ of the non-negative integers as the minimum $z \in \mathbb{Z}_+$ such that $z \notin S$; in particular, $mex(\emptyset) = 0$.

The first ten P-positions of the games $WYT(1)$ and $WYT(2)$ are given in Table 1.

n	x_n	y_n
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23

n	x_n	y_n
0	0	0
1	1	3
2	2	6
3	4	10
4	5	13
5	7	17
6	8	20
7	9	23
8	11	27
9	12	30

Table 1: $(a = b = 1)$ and $(a = 2, b = 1)$

Moreover, Fraenkel solved the recursion and got the following explicit formula for (x_n, y_n).

Let $\alpha = \alpha(a) = \frac{1}{2}(2-a+\sqrt{a^2+4})$ be the (unique) positive root of the quadratic equation $\frac{1}{z} + \frac{1}{z+a} = 1$. In particular, we have $\alpha(1) = \frac{1}{2}(1+\sqrt{5})$, which is the *golden section (or ratio)*, and $\alpha(2) = \sqrt{2}$. Furthermore,

$$x_n = \lfloor \alpha n \rfloor, \quad y_n = x_n + an \equiv \lfloor n(\alpha + a) \rfloor; \quad n \ge 0. \tag{2}$$

As mentioned in [5], the explicit formula (2) solves the game $WYT(a)$ in linear time, in contrast to recursion (1) providing only an exponential algorithm.

3 The recursive solution of $WYT(a,b)$ based on mex_b

The function mex can be generalized as follows. Given an integer $b \ge 1$ and a finite set $S \subset \mathbb{Z}_+$ of m non-negative integers $0 \le s_0 < \cdots < s_{m-1}$, we define $mex_b(S) = s_i + b$, where s_i is the smallest number in S such that $s_{i+1} - s_i > b$. By the definition, we assume that $s_m = \infty$ and $mex_b\{\emptyset\} = 0$; moreover, $s_0 = 0$ will always hold in the sequel.

It is easily seen that the function mex_b is well-defined and that $mex_1 = mex$.

We will show that the recursion (1) can be naturally extended to the game $WYT(a,b)$.

Theorem 1 *The set of P-positions $\{(x_n, y_n) \mid n = 0, 1, \ldots\}$ of the game $WYT(a, b)$ is determined by the same recursive formula (1), in which mex is replaced with mex_b, yet:*

$$x_n = mex_b\{x_i, y_i \mid 0 \leq i < n\}, \;\; y_n = x_n + an; \;\; n \geq 0. \tag{3}$$

The first ten P-positions of the games $WYT(1, 2)$ and $WYT(2, 3)$ are given in Table 2.

We will postpone the proof of Theorem 1 till Section 5. Now let us derive from it the following useful property of P-positions $(x_n, y_n) = (x_n(a, b), y_n(a, b))$ showing how do they depend on the parameters a and b.

Corollary 1 *For all non-negative integer a, b and k, n we have*

$$x_n(ka, kb) = kx_n(a, b) \;\; \text{and} \;\; y_n(ka, kb) = ky_n(a, b). \tag{4}$$

Proof: It is obvious that for any integer $k > 0$, (3) holds for some a, b and x_n, y_n for all $n \geq 0$ if and only if it holds for $a' = ka, b' = kb$ and $x'_n = kx_n, y'_n = ky_n$ for all $n \geq 0$. It is also clear that (4) holds when $k = 0$; in this case $x_n = y_n = 0$ and (3) holds too. □

4 The Bouton - von Neumann algorithm for $WYT(a, b)$

An algorithm finding all P- and N-positions was suggested in 1901 by Bouton in [2] for the normal and misere versions of NIM with k piles. Then in [11], it was further generalized for the games modeled by arbitrary acyclic digraphs.

The algorithm works recursively as follows. First, let us find all terminal (that is, of out-degree 0) positions and denote the resulting set P_0. Then, let N_0 be the set of all positions from which P_0 can be reached by a single move. Let us remove all these positions, $P_0 \cup N_0$ (as well as all arcs incident to them) from the digraph and repeat the whole procedure, thus getting in a similar fashion P_1 and N_1, etc. Obviously, $(P_0 \cup P_1 \cup \ldots)$ is the set of all P-positions.

In Figure 1, the above algorithm is illustrated for the game $WYT(1, 2)$, whose positions (x, y) are represented by the squares of the planar grid. The P- and N-positions are denoted by the circles and lines, respectively. The only terminal position is $(x_0, y_0) = (0, 0)$. Since $a = 1$ and $b = 2$, the corresponding set N_0 consists of two columns $\{(x, y) \mid x \leq 1\}$, two rows $\{(x, y) \mid y \leq 1\}$, and the main $\{(x, y) \mid x = y \geq 1\}$, excluding the position $(0, 0)$ itself. Obviously, from these and only from these positions $(0, 0)$ can reached by a single move, in accordance with the rules of the game. After eliminating $P_0 \cup N_0$, we obtain two new terminal positions, $P_1 = \{(2, 3), (3, 2)\}$. Then N_1 is constructed in a similar way. Positions $(2, 3)$ and $(3, 2)$ can be reached by a single move from the set N_1 that can be represented as the union of the following subsets: three vertical rays $\{(x, y) \mid 2 \leq x \leq 4 \leq y\}$, three horizontal rays $\{(x, y) \mid 2 \leq y \leq 4 \leq x\}$, two diagonal rays $\{(x, y) \mid 4 \leq y = x + 1\}$, $\{(x, y) \mid 4 \leq x = y + 1\}$, and one extra position $(x, y) = (3, 3)$. After eliminating $P_1 \cup N_1$ we obtain two new terminal positions, $P_2 = \{(5, 7), (7, 5)\}$, etc.; see Figure 1 and Table 2 as an illustration.

To prove Theorem 1, we have to describe this recursion for $WYT(a, b)$ with more details.

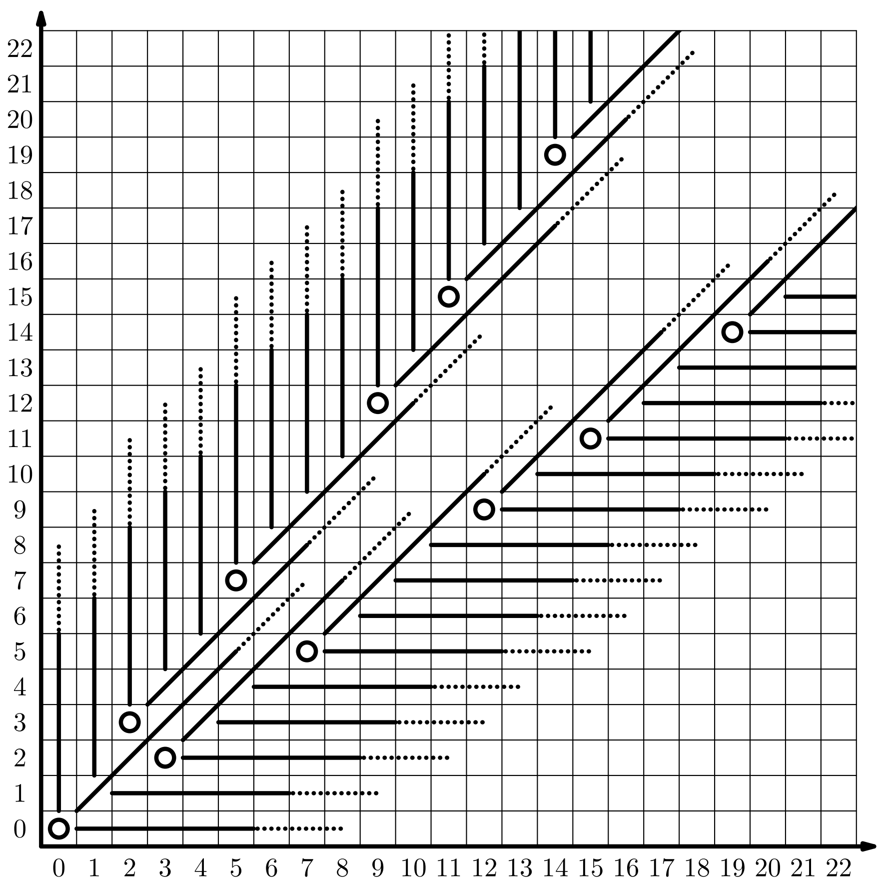

Figure 1: Bouton - von Neumann algorithm for $a = 1$ and $b = 2$. The circles and lines (horizontal, vertical, and diagonal) denote the P- and N-positions, respectively.

Lemma 1 *Every position (x^0, y^0) of the game $WYT(a,b)$ can be reached from the*

- *b vertical rays: $\{(x,y) \mid x^0 \leq x < x_0 + b, y \geq y_0\} \setminus \{(x_0, y_0)\}$,*
- *b horizontal rays: $\{(x,y) \mid y^0 \leq y < y_0 + b, x \geq x_0\} \setminus \{(x_0, y_0)\}$, and*
- *$2a-1$ diagonal rays: $\{(x,y) \mid |(y-x) - (y^0 - x^0)| < a,\ x+y > x^0 + y^0\}$.*

Proof: These three statements represent just a reformulation of the rules of $WYT(a,b)$. □

Let us remark that the considered three sets are not pairwise disjoint. For example, the intersection of the first two is the $b \times b$ square without "the lower left corner" (x_0, y_0).

In the beginning, there is a unique terminal position $z_0 = (x_0, y_0) = (0,0)$ and $P_0 = \{z_0\}$; respectively, N_0 consists of b horizontal, b vertical, and $2a-1$ diagonal rays, in accordance with Lemma 1. Yet, after Step 0 the picture is slightly changed.

Lemma 2 *For each step $n > 0$ the set $P_n = \{z_n, z'_n\} = \{(x_n, y_n), (x_n, y_n)\}$ consists of two symmetric positions, while N_n contains exactly $2a$ diagonal rays: a for z_n and a for z'_n. As for the numbers $k(n) = k_v(n) = k_h(n)$ of the vertical and horizontal rays in N_n, we get bounds $b < k(n) \leq 2b$.*

Figure 1 can serve as an illustration of these claims.

Proof: The symmetry of the coordinates x and y is implied by the rules of $WYT(a,b)$. Step 0 eliminates $P_0 \cup N_0$ and, in particular, the main diagonal $\{(x,y) \mid x = y\}$. By this, after each step $n \geq 1$, the set of the remaining positions Q_n is partitioned into two disjoint subsets

$Z_n = \{(x,y) \in Q_n \mid x < y\}$ and $Z'_n = \{(x,y) \in Q_n \mid x > y\}$,

each of which, has a unique terminal position, $z_n = (x_n, y_n)$ and $z'_n = (y_n, x_n)$, respectively.

By Lemma 1, z_n and z'_n can be reached from exactly b vertical (and b horizontal) rays each. Obviously, the corresponding two sets of rays can overlap but cannot coincide. Hence, $b < k(n) \leq 2b$. Also, z_n and z'_n can be reached from exactly $2a-1$ diagonal rays each, but only a from these $2a-1$ rays belong to N_n, while $a-1$ were eliminated with N_{n-1} in the previous step. Furthermore, the obtained two sets, which consist of the remaining a diagonal rays each, are disjoint, since they belong to Z_n and Z'_n, respectively. □

5 Two proofs of Theorem 1

Let us recall that the possible moves (x', y'), in a position (x,y) of $WYT(a,b)$, are defined by the inequalities: $0 \leq x' \leq x, 0 \leq y' \leq y, 0 < x' + y'$, and

(i) $\min(x', y') < b$ or (ii) $|x' - y'| < a$.

Respectively, we will distinguish moves of the types (i) and (ii).

n	x_n	y_n
0	0	0
1	2	3
2	5	7
3	9	12
4	11	15
5	14	19
6	17	23
7	21	28
8	25	33
9	27	36

n	x_n	y_n
0	0	0
1	3	5
2	8	12
3	11	17
4	15	23
5	20	30
6	26	38
7	29	43
8	33	49
9	36	54

Table 2: $(a = 1,\ b = 2)$ and $(a = 2,\ b = 3)$.

Proof 1: It results easily from the Bouton - von Neumann algorithm. The pairs $P_n = \{z_n, z'_n\} = \{(x_n, y_n), (y_n, x_n)\}$, where $x_n \leq y_n$, were recursively defined in Section 4 for $n = 0, 1, \dots$ We have to show that (x_n, y_n) satisfy the algebraic recursion (3).

By Lemma 2, each set N_n contains a diagonal rays with $x \leq y$ from which z_n can be reached by a single move of type (ii) (and other a diagonal rays with $x \geq y$ assigned to z'_n). These observations immediately imply the equality $y_n = x_n + an$ of (3).

Furthermore, each set N_n contains two sets of vertical rays (which may overlap): b rays from which z_n can be reached by a single move of type (i) and b rays assigned to z'_n. (The similar statement holds for the horizontal rays, as well.) These observations imply the equality $x_n = mex_b\{x_i, y_i \mid 0 \leq i < n\}$ completing the proof of the recursion (3). $\square$

In the above proof we demonstrated that the Bouton - von Neumann recursive algorithm results in the recursive formula (3) for the P-positions. Below we provide another proof based on the inverse approach, which is standard for the impartial games. However, for $WYT(a, b)$ this second proof is longer than the first one, because of a detailed case analysis.

Proof 2: Let $z_n = (x_n, y_n)$ be defined by the recursion (3); in particular, $x_n \leq y_n$; furthermore, $P_n = \{z_n, z'_n\} = \{(x_n, y_n), (y_n, x_n)\}$ for $n = 0, 1, \dots$ We have to show that $P = \cup_{n=0}^{\infty} P_n$ is the set of all P-position of the game $WYT(a, b)$. Then, equivalently, the complementary set $N = \overline{P}$ will consist of all N-positions of the game. We will verify the following two standard properties:

- (j) each move from a P-position results in an N-position;
- (jj) from each N-position a P-position can be reached by a single move.

Claim (j) means that there is no legal move from one P-position to another. Let us consider two pairs (x_n, y_n) and (x_m, y_m) satisfying (3); in particular, $x_n \leq y_n$ and $x_m \leq y_m$.

Without loss of generality, we can assume than $z_n = (x_n, y_n)$ is the first P-position. Yet, the second one can be either (x_m, y_m) or (y_m, x_m). These two positions coincide if and only if $m = 0$, in which case $(x_0, y_0) = (y_0, x_0) = (0, 0)$.

Claim (j) obviously holds when $n = m$. Indeed, there is no move from a position (x, y) to (x, y) or (y, x), since the players are not allowed to pass or add matches to a pile.

The case $n \leq m$ is easy too, for the same reason. Indeed, by (3), both x_k and y_k are the strictly monotone increasing functions of k and the players cannot add matches to a pile.

Thus, we can assume that $n > m$ and $z_n = (x_n, y_n)$. Then, we have to consider two cases: (t) $z_m = (x_m, y_m)$, (tt) $z_m = (y_m, x_m)$, and show that in each case the move from z_n to z_m is illegal in $WYT(a, b)$. Let us recall that there are two types of moves: (i) and (ii).

In case (t) both are forbidden by (3). Indeed, $y_n = x_n + an$ and $y_m = x_m + am$; by subtraction we get $y' = y_n - y_m = x_n - x_m + a(n - m) = x' + a(n - m)$ and, hence, $y' - x' = a(n - m) \geq a$, showing that there is no move of type (ii) from z_n to z_m.

Furthermore, there is no move of type (i) from z_n to z_m either. Indeed, by (3), we have $x_n = mex_b\{x_i, y_i \mid 0 \leq i < n\}$. Hence, $y_n - y_m \geq x_n - x_m \geq x_n - x_{n-1} \geq b$.

Let us notice that in case (tt) there is no move of type (i) from $z_n = (x_n, y_n)$ to $z_m = (y_m, x_m)$, for the same reason: x_i and y_i simultaneously appear in (3) as arguments of mex_b, implying that $y_n - x_m \geq x_n - y_m \geq b$.

Finally, there is no move of type (ii) either. Indeed, by (3),

$$y' = y_n - x_m = x_n - y_m + a(n - m) = x' + a(n - m).$$

If $x_n < y_m$ then $x' < 0$ and the move from $z_n = (x_n, y_n)$ to $z_m = (y_m, x_m)$ is illegal, since it makes larger the number of matches in the first pile. If $x_n \geq y_m$, the move is still illegal, since $y' - x' = a(n - m) \geq a$, in contradiction with (ii).

Now, let us consider claim (jj). Obviously, it can be reformulated as follows:

For each position $z = (x, y)$ either $z \in P$ or P can be reached from z by a single move.

Again, due to symmetry, we can restrict ourselves to the case $x \leq y$.

Let us set $n = n(x, y) = \lfloor \frac{y-x}{a} \rfloor$. It is easily seen that if $x \geq x_n$ and $y \geq y_n$ then either $z = (x, y) = (x_n, y_n) = z_n \in P$ or there is a move of type (ii) from z to z_n in $WYT(a, b)$.

Furthermore, given $z = (x, y)$ and $n \in \mathbb{Z}_+$ such that

$$(0 \leq x - x_n < b \text{ and } 0 \leq y - y_n) \text{ or } (0 \leq x - x_n \text{ and } 0 \leq y - y_n < b),$$

then, either $z = (x, y) = (x_n, y_n) = z_n \in P$ or there is a move of type (i) from z to z_n.

Similarly, there is a move of type (i) from $z = (x, y)$ to $z'_n = (y_n, x_n)$ whenever

$$(0 \leq x - y_n < b \text{ and } 0 \leq y - x_n) \text{ or } (0 \leq x - y_n \text{ and } 0 \leq y - x_n < b).$$

It is not difficult to verify that every position $z = (x, y)$ belongs to at least one of the above cases. Indeed, by construction, b horizontal rays (as well as b vertical rays) are assigned to every position of $P = \{z_n, z'_n \mid n = 0, 1, \ldots\}$, by the definition of N_n in Section 4. On the other hand, the minimum number of rows (respectively, columns) between two neighbor positions of P is at most b, by (3) and the definition of mex_b. Hence, every column $x = const$ (and row $y = const$) appear among the rays of N_n for some n; see Figure 1 as an example.

Let us show that each position $z = (x, y)$, with $x \le y$, belongs either to P or to a vertical or diagonal ray. Let us order the P-positions by the first coordinate; e. g., for $WYT(1, 2)$ in Figure 1 we obtain

$z_0 = z'_0 < z_1 < z'_1 < z_2 < z'_2 < z_3 < z_4 < z'_3 < z_5 < z'_4 < \dots$, since
$x_0 = y_0 = 0, x_1 = 2, y_1 = 3, x_2 = 5, y_2 = 7, x_3 = 9, x_4 = 11, y_3 = 12, x_5 = 14, y_4 = 15, \dots;$

Let us insert x into this sequence and consider the following four cases:

(I) $x_n \le x < x_{n+1}$, (II) $y_n \le x < y_{n+1}$, (III) $x_n \le x < y_m$, and (IV) $y_m \le x < x_n$,

By (3), both x_n and y_n are strictly monotone increasing functions of n. Furthermore, $y_n = x_n + an > x_n$ whenever $n > 0$. Hence, $m \le n$. Also by (3), $y_{n+1} = x_{n+1} + a(n + 1)$, $y_n = x_n + an$; hence, $y_{n+1} - y_n = x_{n+1} - x_n + a \ge b + a$ implying that case (II) is impossible. In contrast, the other three cases may take place, as the above example shows. In cases (I) and (III), (x, y) is covered by a vertical ray whenever $x \ge y_n$ and by a diagonal ray otherwise. In case (IV), (x, y) is always covered by a vertical ray, since $y_m \le x \le y$. □

6 The misere version of $WYT(a, b)$

The Bouton - von Neumann algorithm can be easily adapted to the misere version of any game modeled by an arbitrary acyclic digraph. Let us just add to this digraph one new position and an arc leading to it from each terminal position of the original game.

In particular, for $WYT(a, b)$ we add one new position $(*, *)$ and one new possible move from $(0, 0)$ to $(*, *)$. Obviously, the misere version of the original game is equivalent to the normal version of the newly obtained game. Hence, we can apply the standard algorithm to the modified game rather than developing a "misere version of the algorithm" for the original game. Thus, for all $b \ge 1$ we obtain the following recursion:

For $a = 1$ we have $(x_0, y_0) = (b + 1, b + 1)$, for $n = 0$, while for $n \ge 1$ we obtain

$$x_n = mex_b\{x_i, y_i \mid 0 \le i < n\}, \quad y_n = x_n + an,$$

as before. From this recursion, it is easy to derive that for the normal and misere versions the sets of P-positions P_N and P_M "almost coincide". More precisely,

$P_N \setminus P_M = \{(0, 0), (b, b + 1), (b + 1, b)\}; \quad P_M \setminus P_N = \{(0, 1), (1, 0), (b + 1, b + 1)\}.$

In [1], games of this type (in which P_M and P_M differ "just slightly") are called *tame*. Thus, games $WYT(a, b)$ are tame for $a = 1$ and arbitrary $b \ge 1$. As it was shown in [10], another example of a tame game is the game Euclid introduced in [3]; see also [7, 14].

For the case $a > 1$ we obtain the following, slightly different, recursion:

$$x_n = mex_b\{x_i, y_i \mid 0 \le i < n\}, \quad y_n = x_n + an + 1, \quad \forall\, n \ge 0.$$

It is well-known that P_N is the set of the zeros of the so-called Sprague-Grundy function [12, 9] and it follows from results of Fergusson [4] that P_M is the set of ones of this function; see also [8]. In particular, sets P_M and P_N are disjoint.

We omit the proofs of these recursions, since they are similar to the proofs in the previous two sections. For $b = 1$ and an arbitrary a these results were obtained by Fraenkel in [6]. He also derived and explicit formula for x_n and y_n, thus, giving a linear time algorithm that solves the misere version of the game $WYT(a)$.

7 Simple cases

(i) $a = b = 0$: In this case there are no moves at all, that is, each position is terminal, i.e., it is losing for the normal and winning for the misere version of $WYT(0, 0)$.

(ii) $a = 0$ and $b \geq 1$: In this case, it is easy to verify that $x_n = y_n = bn$ for all $n \geq 0$ in the normal version and $x_0 = y_0 = (0, 1)$, $x'_0 = y'_0 = (1, 0)$, and $x_n = y_n = bn + 1$ for all $n \geq 1$ in the misere version of $WYT(0, b)$.

Let us notice that the game is tame if $b = 1$, while for $b > 1$, the sets P_N and P_M are the zeros and ones of the Sprague-Grundy function of $WYT(0, b)$; in particular, $P_N \cap P_M = \emptyset$.

(iii) $b = 0$ and $a \geq 1$: In this case, a position (x, y) is terminal if and only if $x = 0$ or $y = 0$; furthermore, there is a move from each non-terminal position to a terminal one.

Finally, let us consider a more general game $WYT(a, b, c)$, in which the set of possible moves (x', y') in a position (x, y) is defined by the following inequalities:

$0 \leq x' \leq x,\ 0 \leq y' \leq y,\ 0 < x' + y'$, and $[\ |x' - y'| < a$, or $x' < b$, or $y' < c]$.

Obviously, $WYT(a, b, c) = WYT(a, b)$ when $b = c$. Let us consider the case $b \neq c$. Applying again the Bouton - von Neumann algorithm we obtain for the P-positions (x_n, y_n) of the game $WYT(a, b, c)$ the simple explicit formula: $x_n = n,\ y_n = n \min(b, c);\ \forall\ n \geq 0$. Thus, it appears that symmetry of $WYT(a, b)$ with respect to x and y is very essential.

8 Open problems and conjectures

The main open problem is to find a polynomial algorithm solving the game $WYT(a, b)$. Such an algorithm would obviously result from explicit formulae for $x_n(a, b)$ and $y_n(a, b)$. Yet, they are known only for $b = 1, a \geq 0$ (formula (2) by Fraenkel) and for $a = 0, b \geq 1$ (when $x_n = y_n = bn$; see the previous section). Let us also recall that $x_n(ka, kb) = kx_n(a, b)$ and $y_n(ka, kb) = ky_n(a, b)$ for all non-negative integer a, b and k, n, by Corollary 1. In general, we have the recursion (3) that gives only an exponential algorithm for $WYT(a, b)$. By this recursion, $x_n = x_n(a, b)$ is a function of n of a linear order of magnitude.

We conjecture that the limits $L(a, b) = \lim_{n \to \infty} \frac{x_n(a,b)}{n}$ exist (and are algebraic numbers) for all integer $a \geq 0$ and $b \geq 1$.

a	b				
	1	2	3	4	5
0	1.	2.	3.	4.	5.
1	1.618	3.080	4.530	5.978	7.418
2	1.414	3.236	4.296	6.159	7.180
3	1.303	2.613	4.854	5.616	6.895
4	1.236	2.828	3.752	6.472	7.016
5	1.193	2.404	3.798	4.847	8.090

Table 3: The hypothetical and approximate values of the limits $L(a,b) = \lim_{n\to\infty} \frac{x_n(a,b)}{n}$ for $a, b \in [0,5]$; these limits are known to exist for $b = 1$, for $a = b$, and for $(a,b) = (4,2)$.

This conjecture, if true, and recursion (3) would easily result in the following properties:

- (i) $\lim_{n\to\infty} \frac{y_n(a,b)}{n} = L(a,b) + a$, since $y_n(a,b) = x_n(a,b) + an \;\; \forall\, a \geq 0, b \geq 1, n \geq 0$;
- (ii) $b \leq L(a,b) \leq 2b$, since $b \leq [x_{n+1}(a,b) - x_n(a,b)] \leq 2b \;\; \forall\, a \geq 0, b \geq 1, n \geq 0$;
- (iii) $L(ka,kb) = kL(a,b) \;\; \forall\, a \geq 0, b \geq 1, k \geq 1$, by Corollary 1.

As we already know, limits $L(a,b)$ do exist when $b = 1$ or $a = 0$; moreover,

$L(ka,k) = kL(a,1) = \frac{k}{2}(2 - a + \sqrt{a^2+4})$, $L(0,kb) = kL(0,b) = kb$ for all integer $k \geq 1$.

For small a, b the (hypothetical and approximate) values of $L(a,b)$ are given in Table 3.

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