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Polynomially Computable Bounds for the Probability of the Union of Events

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Abstract. We consider the problem of finding upper and lower bounds for the probability of the union of events when the probabilities of the single events and the probabilities of the intersections of up to m events are given.
It is known that the best possible bounds can be obtained by solving linear programming problems with a number of variables that is exponential in the number of events. Due to their size and structure, these large linear programs are known to be very hard to solve. In the literature simpler, polynomially sized aggregations are considered and numerous closed form or polynomially computable bounds are derived from those.
We present here a new approach which introduces additional constraints to the dual linear programming problems in such a way that those become polynomially solvable. By using different sets of additional constraints, we introduce three new classes of polynomially computable upper and lower bounds. We show that they dominate almost all efficiently computable bounds known in the literature. Furthermore, **by characterizing the vertices of two new classes of polyhedra**, we can show that in two cases our bounds coincide with classical bounds, proving new extremal properties for those well-known bounds. Finally, we provide extensive numerical results comparing the average tightness of the various bounds on large number of instances.

Keywords: Probability bounds, probability of the union, submodular functions, polyhedral combinatorics

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1 Introduction

Starting from the work of Boole, a fundamental problem that has been identified in the theory of probability is that of approximating, or bounding, the probability of a specified event in the algebra determined by a family $A_1, \ldots, A_n$ of events when the probability of some other events in the algebra are known. In this work we address a special problem of this type that has several applications and has been thoroughly investigated in the literature: bounding the probability of the union of $A_1, \ldots, A_n$, when we know the probabilities of the single events A_i and those of all their intersections up to a certain order, called the *order of the bound.*

1.1 Notation and Linear Programming formulation

Let $A_1, ..., A_n$ be arbitrary events in a probability space Ω. The *Boolean probability bounding problem* consists of finding the smallest and largest possible values for the probability of the union $\Pr(A_1 \cup A_2 \cup ... \cup A_n)$ of these events, subject to constraints on the probabilities of their intersections of the form

$$\Pr\left(\bigcap_{i\in I} A_i\right) \;=\; p_I \quad \text{for all} \quad I \subseteq V,\ |I| \leq m, \tag{1.1}$$

where $V = \{1, \ldots, n\}$ is the set of indices, $p = (p_I \mid 1 \leq |I| \leq m)$ are prescribed probabilities, and we set $\bigcap_{i\in\emptyset} A_i = \Omega$ and $p_\emptyset = 1$ by definition. The integer m is usually referred to as the *degree* of these bounds.

Following Hailperin [28] this problem can be formulated as a Linear Programming (LP) problem, involving 2^n variables. For every subset $S \subseteq V$ of indices we define the event

$$C_S = \left(\bigcap_{i\in S} A_i\right) \cap \left(\bigcap_{i\notin S} A_i^c\right),$$

where $A_i^c = \Omega \setminus A_i$, and we associate with this event a decision variable $x_S = \Pr(C_S)$. Note that the equality

$$\bigcap_{i\in I} A_i = \bigcup_{I\subseteq S\subseteq V} C_S$$

holds for all subsets $I \subseteq V$. Since the events C_S are pairwise disjoint we then have

$$\Pr\left(\bigcap_{i\in I} A_i\right) = \sum_{I\subseteq S\subseteq V} \Pr(C_S) = \sum_{I\subseteq S\subseteq V} x_S.$$

With this notation the Boolean probability bounding problem can be stated, following Hailperin [28], as the Linear Programming problem:

$$\begin{array}{rrcll} \max \quad (\min) & \sum_{\emptyset \neq S \subseteq V} x_S & & & \\ \text{s.t.} & \sum_{I \subseteq S \subseteq V} x_S & = & p_I & \forall\ I \subseteq V, \quad |I| \leq m \\ & x_S & \geq & 0 & \forall S \subseteq V \end{array} \tag{1.2}$$

As observed in [28], the above *max* and *min* values, denoted by U_m^* and L_m^* respectively, provide the best possible upper and lower bounds for $\Pr(\bigcup_{i \in V} A_i)$ based on the input $(p_I \mid I \subseteq V,\ |I| \leq m)$. Indeed, from the LP optimum we can easily construct examples of probability spaces Ω and events A_i, $i = 1, ..., n$ which satisfy the equalities in (1.1), and for which the $\Pr(\bigcup_{i \in V} A_i)$ values attain these extremes.

As an illustration consider the case where $n = 3$, $m = 2$, $p_I = 0.5$ for $|I| = 1$, and $p_I = 0.25$ for $|I| = 2$.

$$\begin{array}{lcccccccccl} Max\ (Min) & & & \multicolumn{8}{l}{x_1 + x_2 + x_3 + x_{12} + x_{13} + x_{23} + x_{123}} \\ st & & & & & & & & & & \\ & x_\emptyset & +x_1 & +x_2 & +x_3 & +x_{12} & +x_{13} & +x_{23} & +x_{123} & = & 1 \\ & & x_1 & & & +x_{12} & +x_{13} & & +x_{123} & = & 0.5 \\ & & & x_2 & & +x_{12} & & +x_{23} & +x_{123} & = & 0.5 \\ & & & & x_3 & & +x_{13} & +x_{23} & +x_{123} & = & 0.5 \\ & & & & & x_{12} & & & +x_{123} & = & 0.25 \\ & & & & & & x_{13} & & +x_{123} & = & 0.25 \\ & & & & & & & x_{23} & +x_{123} & = & 0.25 \\ & & \multicolumn{9}{l}{x_\emptyset, x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123} \geq 0.} \end{array}$$

In this example the optimal value of the minimization problem is 0.75, achieved for $x_\emptyset = x_{12} = x_{13} = x_{23} = 0.25$ and $x_1 = x_2 = x_3 = x_{123} = 0$. Thus, if $\Omega = \{\omega_\emptyset, \omega_{12}, \omega_{13}, \omega_{23}\}$ with $\Pr(\omega_\alpha) = x_\alpha = 0.25$, then events $A_1 = \{\omega_{12}, \omega_{13}\}$, $A_2 = \{\omega_{12}, \omega_{23}\}$ and $A_3 = \{\omega_{13}, \omega_{23}\}$ satisfy the input probability equalities, and we have $\Pr(A_1 \cup A_2 \cup A_3) = 0.75$.

The optimal value of the above maximization problem is 1, achieved for $x_1 = x_2 = x_3 = x_{123} = 0.25$ and $x_\emptyset = x_{12} = x_{13} = x_{23} = 0$. Again, defining $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_{123}\}$ with $\Pr(\omega_i) = \Pr(\omega_{123}) = 0.25$ for $i = 1, 2, 3$, and events $A_i = \{\omega_i, \omega_{123}\}$ for $i = 1, 2, 3$, we get equalities with the input probabilities, and we have $\Pr(A_1 \cup A_2 \cup A_3) = 1$.

1.2 Review of the literature

The Boolean probability bounding problem is a particular instance of the optimization version of the *probabilistic satisfiability* (PSAT) *problem* [26]. The decision version of PSAT consists in determining whether, given the probabilities that m logical sentences defined on

n logical variables are true, such probability assignment is consistent. The optimization version of PSAT in concerned with determining bounds on the probability that an additional sentence is true. PSAT is known to be NP-hard [26].

Both versions of PSAT were first proposed as "general problem in the theory of probability" by George Boole [4] in 1854 in his "An Investigation of The Laws of Thought", as well as in subsequent publications [5, 6]. George Boole suggested algebraic methods to solve it approximately or exactly.

Bonferroni [2] has introduced the inequalities that now carry his name to bound the probability of the union of events by means of the algebraic sum of their moments with alternating signs.

Hailperin [28] formulated Boole's general problem as a Linear Program and showed that Boole's method is equivalent to Fourier's elimination. The linear program for PSAT has an exponential number of variables, and column generation techniques have been implemented to solve large instances by Zemel [63], for a particular reliability problem, and by Georgakopoulos, Kavvadias and Papadimitriou [26], and Jaumard et Al. [32] in the general case.

None of these methods works, however, in polynomial time, since the column generation is shown to be algorithmically equivalent with quadratic binary optimization, an NP-hard optimization problem [32].

Even the LP (1.2) that represents the special problem of bounding the probability of the union of all events seems to be very hard to solve exactly because of the large number of variables. Indeed, this LP involves 2^n variables, but only $1 + \sum_{i=1}^{m} \binom{n}{i}$ constraints. Thus, for fixed m the number of nonzero elements in an optimum solution to (1.2) is bounded by a polynomial in n. This allows, in principle, for the possibility of finding a polynomial-time algorithm for problem (1.2), at least for small values of m. Deza and Laurent [14] pointed out that column generation, or equivalently, the separation problem for the dual, is equivalent computationally to the separation problem for the cut polytope, an analogous result to [32], which linked these problems to the separation over the so called Boolean quadric polytope. Both connections clearly show that the feasibility problem is NP-hard, even for $m = 2$. However, if we assume that the input probability vector $p = (p_I \mid 1 \leq |I| \leq m)$ does indeed correspond to a set of n events, as in (1.1), then only a restricted set of quadratic binary optimization problems or separating hyperplanes for the cut polytope may occur during optimization. Hence, the above arguments do not immediately imply the hardness of the optimization problem (1.2). As far as we know, in spite of the several studies dedicated to this problem [14, 32, 33, 59], the complexity status of this problem, for feasible input, seems to be still open even for $m = 2$. Nevertheless, guaranteeing feasibility of the input parameters may not be easy in many practical applications.

In view of the practical difficulty of solving problem (1.2) exactly, several approaches have been used to obtain closed form bounds or at least polynomially computable bounds for its solution, as described below.

Kounias and Marin [39] utilized Hailperin's linear model in their work on the Boolean probability bounding scheme and generated bounds of order two.

Dawson and Sankoff [12] proposed a sharp lower bound for the probability that at least one out of n events occurs, using the first two binomial moments of the occurrences and a linear programming formulation.

Kwerel [35] and Galambos [19] used different technics to prove the same tight upper bound on the probability of the union of events based on the knowledge of the first two binomial moments. Inequalities involving higher binomial moments have been proposed by Galambos and Mucci [21], Platz [45], and Boros and Prékopa [3].

Recently, De Caen [13] presented a lower bound on the probability of the union of events in terms of the individual event probabilities and the pairwise event probabilities, which was later generalized by Kuai, Alajaji, and Takahara [41].

Bukszár and Prékopa [8] found bounds of order three by means of graph structures called cherry trees or t-cherry trees to bound the union of events and showed that their bounds can be regarded as generalizations of the upper bound introduced by Hunter [30] by means of maximum weight spanning tree. The same spanning tree bound has been obtained independently by Worseley [62], that also describes several applications of the Boolean probability bounding problem. In a later paper Bukszár and Szántai [9] employ the more general concept of hypercherry tree to derive bounds that are shown to be improvements of Tomescu's lower and upper bounds [58]. The t-cherry trees were generalized in [7] using chordal graph structure (t-cherry trees are special chordal graphs). This generalization was indpendently found by Dohmen [15], where it was further generalized to the so called chordal-sieve bounds (which in fact demand parameters p_I for $|I| \geq 3$). Recently, Dohmen [16] introduced a new family of lower bounds based on chordal sieves.

Let us note that all of these bounds, for $m = 3$ or for $m \geq 3$, offer new and interesting mathematical insight for Boolean bounding problems, however computationally these bounds may not be easy to obtain. Finding the "best" chordal graph or even the "best" cherry tree may be hard optimization problems on their own. Since in our paper we focus on bounds that can be computed in polynomial time in the size of the input, we will not attempt to compare our results to these bounds (also, we focus on the practically most important $m \leq 3$ cases). The only exception is the t-cherry tree bound, for which [8] provides an implementable heuristic procedure, and thus we included this bound in our computational study, despite the fact that a "best" t-cherry tree bound may not be polynomially computable.

Prékopa [46, 47] also provided the best lower and upper bounds for the so-called aggregated version of the Boolean probability bounding problem, in which the individual and joint probabilities are replaced by the first m binomial moments of the random variable that counts the number of events that occur.

Ṕrekopa and Gao [55] show how to obtain several of the previously known bounds by using partial aggregation of the constraints in (1.2).

1.3 Our contribution

In this paper we propose a new approach to obtain bounds for (1.2) based on the the seemingly new idea of *polyhedral tightening* of its dual. Such a tightening does not result in a polynomially sized instance, but, for some types of tightenings, it provides a polynomial time solvable linear programming problem.

In the following section we describe three classes of polynomial time computable bounds (lower bounds on L_m^* and upper bounds on U_m^*) and show that, to the best of our knowledge, these dominate all known polynomially computable bounds from the literature.

By characterizing the vertices of two new classes of polyhedra that might be of independent interest, we obtain a polyhedral interpretation of the upper bound of Kounias [36] and of the Hunter-Worsley spanning tree upper bound $\mathrm{UB}_2(HW)$ ([30, 62]), showing that they coincide with some of the bounds obtained by a certain tightening the dual of (1.2). As a consequence it follows that the Hunter-Worsley bound is the best currently known polynomially computable upper bound on U_2^*.

Furthermore, we show that $\mathrm{UB}_2(HW) = U_2^{**}$, where U_m^{**} is the largest possible value of the probability of the union of the events A_i, $i = 1, ..., n$, under the constraints

$$\begin{aligned} \Pr(A_i) &= p_{\{i\}} \quad \text{for all } i \in V, \text{ and} \\ \Pr\left(\bigcap_{i\in I} A_i\right) &\geq p_I \quad \text{for all } I \subseteq V,\ 2 \leq |I| \leq m. \end{aligned} \tag{1.3}$$

This problem is clearly a relaxation of (1.2), and its dual, which coincides with one of the tightenings that we propose, can be solved in polynomial time for any fixed m.

In Section 3 we recall all the known dominance results among bounds, and we prove new ones among both known and new bounds. The complete (strict) domination and incomparability diagram is reported in the final section.

We also report in Section 4 the results of ample numerical experiences comparing the different bounds. These comparisons give some information on the actual strength of the bounds, and show that the proposed new bounds are not only "dominating" the ones from the literature, but also provide in most cases substantial numerical improvements.

2 Tightening the dual

We observe that problem (1.2) can be slightly simplified by deleting variable $x_\emptyset$ and constraint $\sum_{S\subseteq V} x_S = 1$. Indeed, the global minimum of the objective function $\sum_{\emptyset\neq S\subseteq V} x_S$ is less than or equal to 1 by construction, while if its maximum is found to be larger than 1, then taking into account constraint $\sum\limits_{S\subseteq V} x_S = 1$, the global maximum of the original problem turns out to be 1.

Thus we can rewrite problem (1.2) equivalently as

$$\begin{array}{llrcl} \max\ (\min) & & \sum_{\emptyset \neq S \subseteq V} x_S & & \\ \text{s.t.} & & \sum_{I \subseteq S \subseteq V} x_S & = & p_I \quad \forall\, I \subseteq V, \quad 1 \leq |I| \leq m \\ & & x_S & \geq & 0 \quad \forall \emptyset \neq S \subseteq V. \end{array} \tag{2.1}$$

The dual of this LP is

$$\begin{array}{llrcl} \min\ (\max) & & \sum_{\substack{I \subseteq V \\ 1 \leq |I| \leq m}} p_I w_I & & \\ \text{s.t.} & & w(S) & \geq & 1\ \ (\leq 1) \quad \forall\, S \subseteq V, \quad S \neq \emptyset, \end{array} \tag{2.2}$$

where $w = (w_I \mid 1 \leq |I| \leq m) \in \mathbb{R}^M$, $M = \sum_{i=1}^{m} \binom{n}{i}$, and

$$w(S) = \sum_{\emptyset \neq I \subseteq S} w_I. \tag{2.3}$$

Note that the value of any feasible solution of problem (2.2) provides an upper (lower) bound for problem (2.1). Furthermore the same holds for the optimal solution of any *tightening* of (2.2), i.e., for any problem obtained from (2.2) by adding constraints.

These simple observations will be used in the sequel to obtain polynomial-time computable upper and lower bounds for the Boolean probability bounding problem, to describe new interpretations and alternative ways for computing known bounds, and to establish dominance relations between the new and the known bounds.

Let $\mathrm{UB}_m(\mathcal{F})$ ($\mathrm{LB}_m(\mathcal{F})$) denote the optimal value of the following tightening of the dual problem (2.2):

$$\begin{array}{llrcl} \min\ (\max) & & \sum_{\substack{I \subseteq V \\ 1 \leq |I| \leq m}} p_I w_I & & \\ \text{s.t.} & & w(S) & \geq & 1 \quad (\leq 1) \quad \forall\, S \subseteq V, \quad S \neq \emptyset \\ & & w & \in & \mathcal{F} \end{array} \tag{2.4}$$

where $\mathcal{F}$ is a polyhedral set of vectors $w = (w_I \mid 1 \leq |I| \leq m) \in \mathbb{R}^M$, that is described by a finite set of linear inequalities of the form $a^T w \leq b$, where $a = (a_I \mid 1 \leq |I| \leq m) \in \mathbb{R}^M$, and $b \in \mathbb{R}$. Note that we can also view vectors $w \in \mathcal{F}$ as set functions over $\mathcal{P}(V)$ defined by (2.3).

Theorem 2.1 *If membership in $\mathcal{F}$ can be checked in polynomial time, and if every vector in $\mathcal{F}$ defines a set function that can be minimized (maximized) over $\mathcal{P}(V)$ in polynomial time, then $\mathrm{UB}_m(\mathcal{F})$ ($\mathrm{LB}_m(\mathcal{F})$) can be computed in polynomial time.*

Proof A vector w belongs to the feasible region of (2.4) whenever $w \in \mathcal{F}$, and $\min_{\emptyset \neq S \subseteq V} w(S) \geq 1$ ($\max_{\emptyset \neq S \subseteq V} w(S) \leq 1$). Since membership in $\mathcal{F}$ can be checked in polynomial time and functions in $\mathcal{F}$ can be minimized (maximized) in polynomial time on $\mathcal{P}(V)$, we deduce that the separation problem for (2.4) can also be solved in polynomial time. The proof then follows from the equivalence between optimization and separation for Linear Programming described, e.g., in [27]. □

2.1 Submodular Bounds

We now provide a first example of a family $\mathcal{F}$ satisfying the assumptions of Theorem 2.1 for the case of $\mathrm{UB}_m(\mathcal{F})$, i.e., for the case of minimization in problem (2.4).

A real function f defined on the family $\mathcal{P}(V)$ of all subsets of a finite set V is *submodular* if $f(S \cap T) + f(S \cup T) \leq f(S) + f(T)$ for all $S, T \in \mathcal{P}(V)$.

Let

$$\mathcal{F}_{sub} = \{w = (w_I \mid 1 \leq |I| \leq m) \in \mathbb{R}^M : w(S) \text{ is submodular}\},$$

and note that the submodularity condition on $w(S)$ can be enforced with the (exponentially many) linear inequalities $w(S \cap T) + w(S \cup T) \leq w(S) + w(T)$, with $S, T \subseteq V$.

Since submodular functions can be minimized in polynomial time on $\mathcal{P}(V)$ (see [31, 43] and references therein), if submodularity of $w(S)$ can also be checked in polynomial time, then $\mathrm{UB}_m(\mathcal{F})$ can be computed in polynomial time by Theorem 2.1. Note that w may be viewed as the set of coefficients of a (multilinear-polynomial) pseudo-Boolean function of degree m, and in these terms submodularity is equivalent with the nonpositivity of the second order partial derivatives (see, e.g., [44]). Hence, when $m = 2$ it follows that $w(S)$ is submodular if and only if $w_I \leq 0$ for all $I \subseteq V$ with $|I| = 2$. Furthermore, when $m = 3$ Billionnet and Minoux [1] have shown that submodularity of $w(S)$ is equivalent to the conditions $w_I + \sum_{J \supset I} (w_J)^+ \leq 0$ for all $I \subseteq V$ with $|I| = 2$, where, for $a \in \mathbb{R}$, $(a)^+ = \max\{0, a\}$ denotes the positive part of a. We have thus obtained the following:

Theorem 2.2 *If $m \leq 3$ and $\mathcal{F} = \mathcal{F}_{sub}$, then $\mathrm{UB}_m(\mathcal{F}_{sub})$ can be computed in polynomial time.* □

The proof of the previous Theorem cannot be straightforwardly extended to the case $m > 3$, since the problem of checking submodularity of a (multilinear-polynomial) pseudo-Boolean function of degree 4 is NP-complete [24]. We believe that computing $UB_m(\mathcal{F}_{sub})$ is NP-hard when $m > 3$, but we do not yet have a proof for this.

An analogous argument shows that functions in the class $\mathcal{F}_{sup}$ of supermodular functions can be recognized in polynomial time for $m \leq 3$. Furthermore, since supermodular functions can be maximized in polynomial time, the lower bound $\mathrm{LB}_m(\mathcal{F}_{sup})$ can also be computed in polynomial time. However these bounds turn out to be very loose in practice (see Section 4), so we do not further analyze this class of bounds.

2.2 Nonpositive Bounds

Another family of functions satisfying the assumptions of Theorem 2.1 is

$$\mathcal{F}_N = \{w = (w_I \mid 1 \leq |I| \leq m) \in \mathbb{R}^M : w_I \leq 0 \text{ for all } I \text{ with } 1 < |I| \leq m\}.$$

Indeed, every function in $\mathcal{F}_N$ is submodular (see, e.g., [1, 44]) so that $\mathcal{F}_N \subseteq \mathcal{F}_{sub}$ (with equality holding when $m = 2$). Thus, the functions in $\mathcal{F}_N$ can be minimized in polynomial time on $\mathcal{P}(V)$. Furthermore, membership in $\mathcal{F}_N$ can be checked in polynomial time (for fixed m) by checking the linear inequalities $w_I \leq 0$. From Theorem 2.1 we then derive the following result.

Theorem 2.3 *For any fixed m, $UB_m(\mathcal{F}_N)$ can be computed in polynomial time.*

Note that $UB_m(\mathcal{F}_N)$ is only an upper bound for the optimal value of problem (2.1), which gives the tightest possible upper bound for the probability of a union of events when the probabilities of the single events and those of all their intersections up to order m are fixed. However, we will now show that $UB_m(\mathcal{F}_N)$ is the tightest possible upper bound for the related problem of bounding the probability of a union of events when the probabilities of the single events are fixed and those of their intersections *are not smaller than certain prescribed thresholds*. Thus, the latter problem can be solved in polynomial time by Theorem 2.3.

Theorem 2.4 *Given probabilities $p = (p_I \mid I \subseteq V,\ 1 \leq |I| \leq m)$, let us consider events A_i, $i = 1, ..., n$, such that*

$$\Pr(A_i) \ = \ p_{\{i\}} \text{ for } i \in V, \quad \text{and} \quad \Pr\left(\bigcap_{i\in I} A_i\right) \ \geq \ p_I \text{ for } I \subseteq V,\ 1 < |I| \leq m.$$

Then, the maximal value of $\Pr(A_1 \cup \cdots \cup A_n)$ subject to the above constraints is equal to $UB_m(\mathcal{F}_N)$, and hence it is computable in polynomial time for any n and fixed m.

Proof We can formulate the above problem analogously to problem (2.1) as the following linear programming problem:

$$\begin{array}{llll} \max & \sum\limits_{\emptyset\neq S\subseteq V} x_S & & \\ \text{s.t.} & \sum\limits_{I\subseteq S\subseteq V} x_S & = & p_I \quad \forall\, I \subseteq V, \quad |I| = 1 \\ & \sum\limits_{I\subseteq S\subseteq V} x_S & \geq & p_I \quad \forall\, I \subseteq V, \quad 1 < |I| \leq m \\ & x_S & \geq & 0 \quad\ \forall\, S \subseteq V, \quad S \neq \emptyset. \end{array} \tag{2.5}$$

The dual of the above problem is

$$\begin{array}{lrcl} \min & \sum_{\substack{I \subseteq V \\ 1 \le |I| \le m}} p_I w_I & & \\ \text{s.t.} & w(S) & \geq & 1 \quad \forall\, S \subseteq V, \quad S \neq \emptyset \\ & w_I & \leq & 0 \quad \forall\, I \subseteq V, \quad 1 < |I| \le m, \end{array}$$

which coincides with the tightened dual (2.4) in case of $\mathcal{F} = \mathcal{F}_N$. Hence the optimal solution of (2.5) is equal to $UB_m(\mathcal{F}_N)$ by linear programming duality, and this bound can thus be computed in polynomial time by Theorem 2.3. □

2.3 The Case $m = 2$

While theoretically interesting, the polynomiality results obtained above are based on the ellipsoid algorithm and do not lead immediately to practically efficient methods to compute the bounds $UB_m(\mathcal{F}_N)$ and $UB_m(\mathcal{F}_{sub})$ in the general case. However, for the case of bounds of order two (i.e., when $m = 2$), we now show that these two bounds coincide with the well-known Hunter-Worsley bound [30, 62], which can be efficiently computed by solving a minimum spanning tree problem in a graph with $|V|$ nodes.

For $m = 2$ we use the notation $w = (w^1, w^2)$, with $w^1 \in \mathbb{R}^n$, $w^2 \in \mathbb{R}^{\binom{n}{2}}$, $w^1_i = w_{\{i\}}$, and $w^2_{ij} = w_{\{i,j\}}$ for all $i < j$. Thus, for $S \subseteq V$ we have $w(S) = w^1(S) + w^2(S)$. For simplicity we also let $p_i = p_{\{i\}}$ and $p_{ij} = p_{\{i,j\}}$ for all $i < j$.

When $m = 2$, we have $\mathcal{F}_{sub} = \mathcal{F}_N = \{(w^1, w^2) \in \mathbb{R}^{n+\binom{n}{2}} : w^2 \le 0\}$, so that $UB_2(\mathcal{F}_N)= UB_2(\mathcal{F}_{sub})$ is the optimum value of the problem

$$\begin{array}{lrcl} \min & \sum_{i=1}^{n} p_i w^1_i + \sum_{1 \le i < j \le n} p_{ij} w^2_{ij} & & \\ \text{s.t.} & w^1(S) + w^2(S) & \geq & 1 \quad \forall\, S \subseteq V, \quad |S| \neq \emptyset \\ & w^2 & \leq 0. & \end{array} \tag{2.6}$$

We now proceed to prove that all the vertices of the feasible region R of (2.6) have the form $(\mathbf{1}, -y_T)$, where $\mathbf{1} \in \mathbb{R}^n$ denotes the vector of all ones, and $y_T \in \mathbb{R}^{\binom{n}{2}}$ denotes the incidence vector of a spanning tree T of the complete graph on V.

We say that a family $\mathcal{F} \subseteq \mathcal{P}(V)$ is a *crossing family* if $S \cap T \in \mathcal{F}$ and $S \cup T \in \mathcal{F}$ hold for all subsets $S, T \in \mathcal{F}$ such that $S \cap T \neq \emptyset$.

Lemma 2.5 *Let $f : \mathcal{P}(V) \to \mathbb{R}$ be a submodular function such that $f(S) \ge \alpha$ for some real $\alpha \in \mathbb{R}$ and for all nonempty subsets $S \subseteq V$. Then, the family $\mathcal{T} = \{S \subseteq V \mid S \neq \emptyset,\ f(S) = \alpha\}$ is a crossing family.*

Proof Let $S, T \in \mathcal{T}$ with $S \cap T \neq \emptyset$. Then, $2\alpha \leq f(S\cap T) + f(S \cup T) \leq f(S) + f(T) = 2\alpha$. Hence, $f(S \cap T) + f(S \cup T) = 2\alpha$. Since $f(S \cup T) \geq \alpha$ and $f(S \cap T) \geq \alpha$, we obtain $f(S \cap T) = f(S \cup T) = \alpha$. □

Lemma 2.6 *For every vertex (w^1, w^2) of the feasible region R of problem (2.6) we have $w^1 = \mathbf{1}$.*

Proof Note that for every $w = (w^1, w^2)$ with $w^2 \leq 0$ the set function $w(S) = w^1(S) + w^2(S)$ is submodular. We call a set $S \in \mathcal{P}^*(V) = \mathcal{P}(V) \setminus \{\emptyset\}$ *w-tight* if $w(S) = 1$. By Lemma 2.5 the family $\mathcal{T}(w)$ of w-tight sets is a crossing family. Thus, for every $i \in V$ the family $\mathcal{T}_i(w) = \{S \in \mathcal{T}(w) : i \in S\}$ is a sublattice of $\mathcal{P}(V)$, and hence it has a unique least element $L_i(w) = \bigcap_{S \in \mathcal{T}_i(w)} S \neq \emptyset$.

We will prove by contradiction that $L_i(w) = \{i\}$ for all $i \in V$ and for every vertex $w = (w^1, w^2)$ of the feasible region R of problem (2.6). This claim implies the statement of the lemma, since the tightness of subsets of size 1 means that we have $w_i^1 = 1$ for all $i \in V$.

To see the above claim, let us first observe that if for some $w \in R$ we have $\mathcal{T}(w) = \emptyset$, then for some small $\epsilon > 0$ we have both $(1+\epsilon)w \in R$ and $(1-\epsilon)w \in R$, implying that w cannot be a vertex of R. Consequently, we can assume that $\mathcal{T}(w) \neq \emptyset$. Let us assume then that the set $\bar{S}_i = L_i(w) \setminus \{i\}$ is nonempty for some $i \in V$. Then, since $w(\bar{S}_i) \geq 1$ and $w_i^1 > 1$, there is an index $j \in \bar{S}_i$ such that $w_{ij}^2 < 0$. Furthermore, every subset $S \in \mathcal{P}^*(V)$ that contains i and does not contain j is not w-tight. Take $0 < \epsilon < \min\{w(S) - 1 \mid i \in S,\ j \notin S\}$, and define $\bar{w}_i^1 = w_i^1 - \epsilon$, $\bar{w}_{ij}^2 = w_{ij}^2 + \epsilon$ and $\bar{w}_k^1 = w_k$, $\bar{w}_{kh}^2 = w_{kh}$ otherwise. For any $S \in \mathcal{P}^*(V)$ we have that if $i \notin S$, or if both $i, j \in S$, then $\bar{w}(S) = w(S)$. Furthermore, if $i \in S$ and $j \notin S$, then $\bar{w}(S) = w(S) - \epsilon > 1$. Hence, $\bar{w} \neq w$ is another feasible solution to (2.6) such that $\mathcal{T}(w) = \mathcal{T}(\bar{w})$, that is $\bar{w}$ satisfies the same subsystem of equalities as w. This contradicts the fact that w is a basic feasible solution of (2.6). □

Theorem 2.7 *The vertices of the feasible region R of (2.6) are the points $(\mathbf{1}, -y_T)$, where y_T denotes the incidence vector of a spanning tree T of the complete graph on V.*

Proof Let $(\overline{w}^1, \overline{w}^2)$ be a vertex of R. By Lemma 2.6 we have $\overline{w}^1 = \mathbf{1}$. Thus $\overline{w}^2$ must be a vertex of the polyhedron described by the system of inequalities

$$\begin{aligned} w^2(S) &\geq 1 - |S| \quad \forall\, S \subseteq V,\ |S| > 1 \\ w^2 &\leq 0. \end{aligned}$$

By setting $y = -w^2$, this is equivalent to the system

$$\begin{aligned} y(S) &\leq |S| - 1 \quad \forall\, S \subseteq V,\ |S| > 1 \\ y &\geq 0, \end{aligned}$$

which is exactly the spanning tree polytope of the complete graph on V (see e.g., [17, 57]). □

Thus, for $m = 2$, the upper bounds $\mathrm{UB}_2(\mathcal{F}_N)$ and $\mathrm{UB}_2(\mathcal{F}_{sub})$ coincide with the Hunter-Worsley upper bound [30, 62]:

Corollary 2.8 *The following equalities hold:*

$$\mathrm{UB}_2(\mathcal{F}_N) \;=\; \mathrm{UB}_2(\mathcal{F}_{sub}) \;=\; \mathrm{UB}_2(HW) \;=\; \sum_{i\in V} p_i - \max_{T\in\mathcal{T}} \sum_{(i,j)\in T} p_{ij}, \tag{2.7}$$

where $\mathcal{T}$ is the set of all spanning trees of the complete graph on V. □

As a consequence the Hunter-Worsley bound (2.7) is the *exact solution* of problem (2.5), which provides the tightest upper bound for the probability of a union of events when the probabilities of the single events are fixed and those of their intersections are not smaller than certain prescribed thresholds.

Let us add that [54] showed that the Hunter-Worsley bound corresponds to a dual feasible basis of the linear programming formulation (2.1) in case of $m = 2$. For any spanning tree T of the complete graph on V, the value $\sum_{i\in V} p_i - \sum_{(i,j)\in T} p_{ij}$ was shown by [59] to be the optimum of the linear programming problem (2.6) with the additional restriction that $w^2_{ij} = 0$ for all $(i,j) \notin T$. Recently [61] noted that bounds corresponding to feasible solutions of (2.6) satisfying $w^1 = \mathbf{1}$ and $w^2_{ij,} \in \{0, -1\}$ are sharp. Theorem 2.7 and Corollary 2.8 can be viewed as common generalization of the above cited results.

Using the equalities (2.7) we will show in Section 3 that the Hunter-Worsley bound dominates all the polynomially computable degree two upper bounds that we are aware of.

2.4 Decomposition Bounds

In this section we analyze another family of functions $\mathcal{F}$ that satisfies the assumptions of Theorem 2.1: the family $\mathcal{F}_{dec}$ of decomposable functions defined by

$$\mathcal{F}_{dec} = \left\{ w = (w_I \mid 1 \le |I| \le m) \in \mathbb{R}^M \;\middle|\; \begin{array}{c} w_I = \sum_{i\in I} u_i^{|I|} \quad \forall I \subseteq V,\ 1 \le |I| \le m, \\ \text{for some} \quad u^1, \cdots, u^m \in \mathbb{R}^n \end{array} \right\}.$$

For any m, membership in $\mathcal{F}_{dec}$ can clearly be checked in time polynomial in n. Furthermore, we shall prove that any function in $\mathcal{F}_{dec}$ can be minimized (maximized) in polynomial time on $\mathcal{P}(V)$.

Theorem 2.9 *If $w \in \mathcal{F}_{dec}$, then a global maximizer (minimizer) of $w(S)$ on $\mathcal{P}(V)$ can be computed in $O(n^2 \log n)$ time.*

Proof Let us first note that members of $\mathcal{F}_{dec}$ can be uniquely described by vectors $u^k \in \mathbb{R}^n$, $k = 1, ..., m$. Furthermore, for a subset $S \subseteq V$ of size $s = |S|$ we have

$$w(S) \;=\; \sum_{\substack{I \subseteq S \\ 1 \leq |I| \leq m}} w_I \;=\; \sum_{k=1}^{\min\{m,s\}} \binom{s-1}{k-1} \sum_{i \in S} u_i^k \;=\; \sum_{k=1}^{m} \binom{s-1}{k-1} \sum_{i \in S} u_i^k \tag{2.8}$$

since $\binom{s-1}{k-1} = 0$ for all $k > s$. Introducing $\beta_i^s = \sum_{k=1}^m \binom{s-1}{k-1} u_i^k$ for all $i \in V$ and $s = 1, ..., n$, we can rewrite the above as

$$w(S) \;=\; \sum_{i \in S} \beta_i^s.$$

Thus, for finding the maximum (minimum) of $w(S)$ among subsets of size $|S| = s$ all we need is to sort the β_i^s values in $O(n \log n)$ time, and choose the s largest (smallest). Doing this for all possible sizes $s = 1, ..., n$ and choosing the best over all subsets, we can find the global maximizer (minimizer) of $w(S)$ over $\mathcal{P}(V)$ in $O(n^2 \log n)$ time. □

Thus Theorem 2.1 immediately implies the following:

Theorem 2.10 *For any m, $\mathrm{LB}_m(\mathcal{F}_{dec})$ and $\mathrm{UB}_m(\mathcal{F}_{dec})$ can be computed in polynomial time.* □

Again, this computational efficiency result is obtained through equivalence between separation and optimization, which is based on the ellipsoid algorithm. However, also in this case when $m = 2$ we can show that $\mathrm{UB}_2(\mathcal{F}_{dec})$ can be computed very efficiently by solving a minimum spanning star problem in a graph with $|V|$ nodes. In fact, we shall show that when $m = 2$, $\mathrm{UB}_2(\mathcal{F}_{dec})$ coincides with the well-known upper bound $\mathrm{UB}_2(Ko)$ introduced by Kounias [36].

When $m = 2$ every $w \in \mathcal{F}_{dec}$ can be identified with (u^1, u^2), where $u^k \in \mathbb{R}^n$ for $k = 1, 2$. For notational convenience, let us introduce $x = u^1$ and $y = -u^2$. Thus, by (2.8) we have for all $S \subseteq V$ that

$$w(S) \;=\; x(S) - (|S| - 1)y(S),$$

and hence for $\mathcal{F} = \mathcal{F}_{dec}$ problem (2.4) becomes

$$\begin{array}{lll} \min \;\; (\max) & \sum_{i \in V} \left(p_i x_i - \sum_{j \neq i} p_{ij} y_i \right) & \\ \text{s.t.} & x(S) - (|S| - 1)y(S) \geq 1 \;\; (\leq 1) & \forall\, \emptyset \neq S \subseteq V \end{array} \tag{2.9}$$

We are going to show now that the feasible region of problem (2.9) in case of minimization has only n vertices and of a simple structure, so that its solution can be described in a very simple way.

Given a positive integer n, we set $V_n = \{1, 2, ..., n\}$, and we denote by $\mathbf{1_n} \in \mathbb{R}^{V_n}$ the vector of all ones and by $e^i \in \mathbb{R}^{V_n}$ the ith unit vector, for $i \in V_n$. Furthermore, we define

$$P_n = \{(u, v) \mid u(S) - (|S| - 1)v(S) \geq 1 \ \ \forall \emptyset \neq S \subseteq V_n\} \tag{2.10}$$

and

$$C_n = \{(u, v) \mid u(S) - (|S| - 1)v(S) \geq 0 \ \ \forall \emptyset \neq S \subseteq V_n\}. \tag{2.11}$$

Theorem 2.11 *For $n \geq 3$ the vertices of P_n are the n points $(\mathbf{1_n}, e^i)$ for $i = 1, \ldots, n$. In other words, we have*

$$P_n = \left\{ \sum_{i \in V_n} \alpha_i(\mathbf{1_n}, e^i) + (u, v) \;\middle|\; \begin{array}{c} \sum_{i \in V_n} \alpha_i = 1 \\ \alpha_i \geq 0 \ \forall i \in V_n \text{ and} \\ (u, v) \in C_n \end{array} \right\}.$$

Let us note that $(\mathbf{1_n}, e^i) \in P_n$ for all $i \in V_n$. The above statement claims that these are vertices of P_n and that there are no other vertices.

We shall prove this theorem by induction on n. Before presenting the proof, let us introduce a number of useful technical lemmas.

Lemma 2.12 *Assume $(u, v) \in C_n$ and $\alpha_i \geq 0$ such that $\sum_{i \in V_n} \alpha_i = 1$. If the vector $(x, y) = \sum_{i \in V_n} \alpha_i(\mathbf{1_n}, e^i) + (u, v)$ is a vertex of P_n, then we must have $(u, v) = (0, 0)$.*

Proof For $1 > \epsilon > 0$ we have $(1 \pm \epsilon)(u, v) \in C_n$, and thus $(x^1, y^1) = \sum_{i \in V_n} \alpha_i(\mathbf{1_n}, e^i) + (1 + \epsilon)(u, v)$ and $(x^2, y^2) = \sum_{i \in V_n} \alpha_i(\mathbf{1_n}, e^i) + (1 - \epsilon)(u, v)$ are different vectors in P_n if $(u, v) \neq (0, 0)$. Since $(x, y) = \frac{1}{2}(x^1, y^1) + \frac{1}{2}(x^2, y^2)$, the vector (x, y) could not be a vertex. Thus, we must have $(u, v) = (0, 0)$. □

Lemma 2.13 *If $(u, v) \in C_{n-1}$, $\alpha \geq 0$ and $\beta \leq 0$, then $(u, \alpha, v, \beta) \in C_n$.*

Proof For a subset $S \subseteq V_n$ introduce $g(S) = u(S \setminus \{n\}) - (|S| - 1)v(S \setminus \{n\})$, and note that we have

$$(u, \alpha)(S) - (|S| - 1)(v, \beta)(S) = \begin{cases} g(S) & \text{if } n \notin S, \\ g(S) + \alpha - (|S| - 1)\beta & \text{if } n \in S. \end{cases}$$

Since $\alpha - (|S| - 1)\beta \geq 0$ and $g(S) = u(S \setminus \{n\}) - (|S| - 1)v(S \setminus \{n\}) \geq 0$ by our assumptions, the claim follows. □

For the subsequent claims let us assume the following condition:

(I) P_{n-1} has the structure as claimed in Theorem 2.11.

Lemma 2.14 *If condition (I) holds and $(x, y) \in P_n$ is a vertex, then we have $y \geq 0$.*

Proof We shall prove that $y_i \geq 0$ for all indices $i \in V_n$. To simplify notation, we shall present this proof only for $i = n$. Let $\widetilde{x} = (x_1, ..., x_{n-1})$, and $\widetilde{y} = (y_1, ..., y_{n-1})$, and denote by $\widetilde{e} \in \mathbb{R}^{V_{n-1}}$ the vector of all ones, and by $\widetilde{e}^i \in \mathbb{R}^{V_{n-1}}$ the ith unit vector, $i \in V_{n-1}$. By definition (2.10) we have $(\widetilde{x}, \widetilde{y}) \in P_{n-1}$. Thus, by condition (I) we have some nonnegative reals $\alpha_i \geq 0$, $i \in V_{n-1}$ satisfying $\sum_{i \in V_{n-1}} \alpha_i = 1$, and a vector $(\widetilde{u}, \widetilde{v}) \in C_{n-1}$ such that

$$(\widetilde{x}, \widetilde{y}) \; = \sum_{i \in V_{n-1}} \alpha_i(\mathbf{1_{n-1}}, \widetilde{e}^i) \; + \; (\widetilde{u}, \widetilde{v}). \tag{2.12}$$

Since $\sum_{i \in V_{n-1}} \alpha_i = 1$, the following equality is implied by (2.12):

$$(x, y) \; = \sum_{i \in V_{n-1}} \alpha_i(\mathbf{1_{n-1}}, 1, \widetilde{e}^i, 0) \; + \; (\widetilde{u}, x_n - 1, \widetilde{v}, y_n). \tag{2.13}$$

Furthermore, if $y_n < 0$, then Lemma 2.13 is applicable, and we get that $(\widetilde{u}, x_n - 1, \widetilde{v}, y_n) \in C_n$, which is a nonzero vector since $y_n \neq 0$. Thus, by Lemma 2.12, the vector (x, y) cannot be a vertex, contradicting our choice of (x, y). This contradiction shows that we must have $y_n \geq 0$, as claimed. □

Lemma 2.15 *If condition (I) holds, $(x, y) \in P_n$ is a vertex, and $y_i = 0$ for some $i \in V_n$, then $x = \mathbf{1_n}$ and $y = e^j$ for some $j \neq i$.*

Proof Without any loss of generality, we can assume that $i = n$. Let us then consider equations (2.12) and (2.13) as in the proof of Lemma 2.14. Since $x_n \geq 1$ and $y_n = 0$, Lemma 2.13 is applicable and $(\widetilde{u}, x_n - 1, \widetilde{v}, 0) \in C_n$ is implied. By Lemma 2.12 we must have $(\widetilde{u}, \widetilde{v}) = (0, 0)$ and $x_n = 1$. We also must have $\alpha_i \in \{0, 1\}$ for all $i \in V_{n-1}$, since otherwise (x, y) is a nontrivial convex combination of other points of P_n, contradicting our assumption that it is a vertex. □

Proof *of Theorem 2.11.* It is easy to verify that the claim is true for $n = 3$. To complete the proof by induction on n, assume for the rest of the proof that condition (I) holds, and assume indirectly that (x, y) is a vertex of P_n different from $(\mathbf{1_n}, e^i)$ for all $i \in V_n$. Then, by Lemmas 2.14 and 2.15 we have that $y_i > 0$ for all $i \in V_n$. Let us fix this (x, y) for the rest of the proof, and we derive a contradiction in several steps, proving that such a vertex does not exist, which will then imply the statement of the theorem.

For a subset $S \subseteq V_n$ let us define the set function $f(S) = x(S) - (|S| - 1)y(S)$, and call S a *tight set* if $f(S) = 1$. Our proof is focusing on the structure of tight sets.

Let us note that if (x, y) is a vertex then it is the *unique* solution to the set of equations $f(S) = 1$ corresponding to tight sets.

Let us observe next that

$$f(A) + f(B) = f(A \cap B) + f(A \cup B) + y(A \setminus B)|B \setminus A| + y(B \setminus A)|A \setminus B| \qquad (2.14)$$

holds for arbitrary subsets $A, B \subseteq V_n$. Note also that since $(x, y) \in P_n$, we have that $f(X) \geq 1$ for all subsets $X \subseteq V_n$, $X \neq \emptyset$, and $f(\emptyset) = 0$ by definition.

(i) If $A, B \subseteq V_n$ are tight sets such that $A \cap B \neq \emptyset$, then we have either $A \subseteq B$ or $B \subseteq A$. To see this claim, note that by (2.14) and by the inequalities $f(A \cap B) \geq 1$ and $f(A \cup B) \geq 1$ we get

$$2 \geq 2 + y(A \setminus B)|B \setminus A| + y(B \setminus A)|A \setminus B|$$

from which, by the positivity of y, we get that $f(A \cap B) = f(A \cup B) = 1$ and either $A \setminus B = \emptyset$ or $B \setminus A = \emptyset$.

(ii) If $A \subseteq V_n$ is a tight set and $|A| > 1$, then $y(A) = \frac{x(A)-1}{|A|-1} \geq 1$. To see this, note that $(x, y) \in P_n$ implies in particular that $x_i \geq 1$ for all indices $i \in V_n$, and thus the equality $x(A) - (|A| - 1)y(A) = 1$ implies the claim.

(iii) If $A \subseteq V_n$ is a tight set and $|A| > 1$, then $A \cap B \neq \emptyset$ for all tight sets $B \subseteq V_n$. To see this claim, assume indirectly that $B \neq \emptyset$ is a tight set, disjoint from A. Then, by (ii) we have $y(A) \geq 1$, which together with $f(A \cup B) \geq 1$ and $|B| \geq 1$ implies by (2.14) that

$$2 = f(A) + f(B) = f(A \cup B) + y(A)|B| + y(B)|A| \geq 2 + 2y(B)$$

contradicting $y(B) > 0$.

Properties (i) and (iii) imply that the nontrivial tight sets must form a chain, and all such sets must include all tight sets of size 1. Thus we cannot have more than n tight sets, contradicting the fact that the corresponding equalities must define uniquely the $2n$ components of (x, y). This contradiction proves the theorem. □

From Theorem 2.11 we immediately derive that

$$\mathrm{UB}_2(\mathcal{F}_{dec}) = \sum_{i \in V} p_i - \max_{i \in V} \sum_{j \neq i} p_{ij}. \qquad (2.15)$$

Note that the right-hand side of (2.15) coincides with a well-known upper bound $\mathrm{UB}_2(Ko)$ introduced by Kounias [36]. Furthermore, the right-hand side of (2.15) is clearly not smaller than the Hunter-Worsley bound (2.7). Thus, when $m = 2$ we have

$$\mathrm{UB}_2(\mathcal{F}_{sub}) = \mathrm{UB}_2(\mathcal{F}_N) = \mathrm{UB}_2(HW) \leq \mathrm{UB}_2(\mathcal{F}_{dec}) = \mathrm{UB}_2(Ko). \qquad (2.16)$$

Our computational results described in Section 4 below seem to indicate that $\mathrm{UB}_m(\mathcal{F}_{dec})$, for $m > 2$, and $\mathrm{LB}_m(\mathcal{F}_{dec})$ are very tight bounds in practice. In the next section we show that $\mathrm{LB}_m(\mathcal{F}_{dec})$ actually dominates all polynomially computable lower bounds of order two and three that we are aware of. Furthermore $\mathrm{UB}_3(\mathcal{F}_{dec})$ dominates almost all known polynomially computable upper bounds of degree two and three, with the notable exception of $\mathrm{UB}_3(\mathcal{F}_{sub})$.

3 Comparison with Known Bounds

In this section we first recall several of the known bounds from the literature. As customary, for $k = 1, 2, ...$ we denote by

$$S_k = \sum_{\substack{I \subseteq V \\ |I|=k}} p_I$$

the so called binomial moments of the events A_i, $i \in V$; and we set $S_0 = 1$ by definition.

The first lower bound we recall for the case of $m = 2$ was introduced by Chung and Erdős [10]:

$$\text{LB}_2(CE) \;=\; \frac{S_1^2}{S_1 + 2S_2}. \tag{3.1}$$

A sharp lower bound based on S_1 and S_2 was later established by Dawson and Sankoff [12] (see also [19, 34]) and it is given by

$$\text{LB}_2(DS) \;=\; \frac{2}{h+1} S_1 - \frac{2}{h(h+1)} S_2 \tag{3.2}$$

where

$$h \;=\; 1 + \left\lfloor \frac{2S_2}{S_1} \right\rfloor.$$

The following theorem, provided in [39], establishes dominance between these bounds.

Theorem 3.1 ([39]) $\text{LB}_2(DS) \;\geq\; \text{LB}_2(CE)$. □

De Caen [13] provided another lower bound, still for $m = 2$, that is incomparable with the DS bound, and at least as good as $\text{LB}_2(CE)$

$$\text{LB}_2(DeC) \;=\; \sum_{i=1}^{n} \frac{p_i^2}{p_i + \sum_{j \neq i} p_{ij}} \geq \text{LB}_2(CE). \tag{3.3}$$

De Caen's bound is very simple, but it does not give a very good approximation for the probability of the union in many cases. An improvement of the De Caen's lower bound is the one obtained by Kuai, Alajaji and Takahara [41]:

$$\text{LB}_2(KAT) \;=\; \sum_{i=1}^{n} \left(\frac{\theta_i}{\sum_{j=1}^{n} p_{ij} + (1-\theta_i) p_i} p_i^2 + \frac{(1-\theta_i)}{\sum_{j=1}^{n} p_{ij} - \theta_i p_i} p_i^2 \right) \tag{3.4}$$

where

$$\theta_i = \frac{\sum_{j \neq i} p_{ij}}{p_i} - \left\lfloor \frac{\sum_{j \neq i} p_{ij}}{p_i} \right\rfloor, \; i = 1, ..., n.$$

Indeed, they show that their bound is always at least as good as the ones of De Caen and of Dawson and Sankoff:

Theorem 3.2 ([41]) $\mathrm{LB}_2(KAT) \;\geq\; \max\left(\mathrm{LB}_2(DeC), \mathrm{LB}_2(DS)\right).$ □

For the case of $m = 2$, Kwerel [34] introduced an upper bound of the form

$$\mathrm{UB}_2(Kw) \;=\; S_1 - \frac{2}{n} S_2. \tag{3.5}$$

An incomparable upper bound, which we mentioned earlier, was introduced by Kounias [36]:

$$\mathrm{UB}_2(Ko) \;=\; S_1 - \max_{k \in V} \sum_{i \neq k} p_{ik}. \tag{3.6}$$

Another already mentioned upper bound was introduced by Hunter [30] and Worsley [62]:

$$\mathrm{UB}_2(HW) \;=\; S_1 - \max_{T \in \mathcal{T}} \sum_{(i,j) \in T} p_{ij}, \tag{3.7}$$

where the maximization is taken over the set $\mathcal{T}$ of all spanning trees of the complete graph on the vertex set V.

It is easy to see that $2S_2 = \sum_{k \in V} \sum_{i \neq k} p_{ik}$, i.e., $\frac{2}{n} S_2$ is the average weight of stars, which are special spanning trees. Thus we can conclude that

$$\mathrm{UB}_2(HW) \leq \mathrm{UB}_2(Ko) \leq \mathrm{UB}_2(Kw). \tag{3.8}$$

For the case of $m = 3$, efficient lower and upper bounds were provided by Kwerel [35]:

$$\mathrm{LB}_3(Kw) \;=\; \frac{h + 2n - 1}{(h+1)n} S_1 - \frac{2(2h + n - 2)}{h(h+1)n} S_2 + \frac{6}{h(h+1)n} S_3 \tag{3.9}$$

where

$$h = 1 + \left\lfloor \frac{2((n-2)S_2 - 3S_3)}{(n-1)S_1 - 2S_2} \right\rfloor,$$

and

$$\mathrm{UB}_3(Kw) \;=\; S_1 - \frac{2(2h-1)}{h(h+1)} S_2 + \frac{6}{h(h+1)} S_3 \tag{3.10}$$

where

$$h = 2 + \left\lfloor \frac{3S_3}{S_2} \right\rfloor.$$

Boros and Prékopa [3] provided a closed form upper bound for the case of $m = 4$, based on the linear programming model of Prékopa [46]:

$$UB_4(BP) = S_1 - \frac{2((h-1)(h-2)+(2h-1)n)}{h(h+1)n} S_2 + \frac{6(2h+n-4)}{h(h+1)n} S_3 - \frac{24}{h(h+1)n} S_4, \tag{3.11}$$

where

$$h = 1 + \left\lfloor \frac{(n-2)S_2 + 3(n-4)S_3 - 12S_4}{(n-2)S_2 - 3S_3} \right\rfloor.$$

In the above cited work of Prékopa [46] the following concise linear programming model is shown to provide polynomial time computable upper and lower bounds on $\Pr(\bigcup_{i \in V} A_i)$ for arbitrary values of m:

$$\begin{array}{rrcll} \max\ (\min) & \sum_{i \in V} x_i & & & \\ s.t. & \sum_{i \in V} \binom{i}{k} x_i & = & S_k & \text{for all } k = 1, ..., m, \\ & x_i & \geq & 0 & \text{for all } i = 1, ..., n. \end{array} \tag{3.12}$$

We denote the maximum and minimum values of the above LP by $\text{UB}_m(P)$ and $\text{LB}_m(P)$, respectively. It has been proved in [3, 46] that all of the bounds based on the binomial moments as input, are dominated respectively by $\text{UB}_m(P)$ and $\text{LB}_m(P)$. In fact the LP (3.12) is obtained from (2.1) by aggregating constraints and variables.

In [55] the authors provide a substantially improved family of bounds, based on a linear programming model which is obtained from (2.1) again by aggregation of constraints and variables, but still utilizes more information from the individual p_I values than the binomial moments based (3.12). To state this model, let us introduce

$$S_{ik} = \frac{1}{k} \sum_{\substack{I \subseteq V \\ i \in I \\ |I| = k}} p_I$$

for all $i \in V$ and $k = 1, ..., m$, and note that the equalities

$$S_k = \sum_{i \in V} S_{ik}$$

hold for all $k = 1, ..., m$. Let us then consider the linear programming model

$$\begin{array}{llll} \max\ (\min) & \sum_{i\in V}\sum_{j\in V} y_{ij} & & \\ s.t. \quad \sum_{j\in V}\binom{j}{k} y_{ij} & = & S_{ik} & \text{for all } i\in V \text{ and } k=1,...,m, \\ y_{ij} & \geq & 0 & \text{for all } i,j\in V. \end{array} \tag{3.13}$$

Let us denote by $\mathrm{UB}_m(PG)$ and $\mathrm{LB}_m(PG)$ the maximum and minimum values, respectively in the above LP. The following relations were proved in [55].

Theorem 3.3 $\mathrm{UB}_m(PG) \leq \mathrm{UB}_m(P)$ *and* $\mathrm{LB}_m(PG) \geq \mathrm{LB}_m(P)$ *for all* $m \geq 1$. □

In [55] the authors also show that $\mathrm{LB}_2(PG)$ is the same as $\mathrm{LB}_2(KAT)$ and that $\mathrm{UB}_2(PG)$ coincides with $\mathrm{UB}_2(Kw)$. In view of these results and of Theorem 3.2, we can conclude that bounds $\mathrm{UB}_m(PG)$ and $\mathrm{LB}_m(PG)$ dominate all other bounds cited above (assuming the same input is used), with the notable exceptions of $\mathrm{UB}_2(Ko)$ and $\mathrm{UB}_2(HW)$ in case of $m = 2$.

We shall show next that bounds based on the family $\mathcal{F}_{dec}$ dominate the PG bounds for all $m \geq 1$.

Theorem 3.4 *For all* $m \geq 1$ *we have*

$$\mathrm{UB}_m(\mathcal{F}_{dec}) \leq \mathrm{UB}_m(PG) \quad \textit{and} \quad \mathrm{LB}_m(\mathcal{F}_{dec}) \geq \mathrm{LB}_m(PG).$$

Proof Let us first introduce $x_{ij} = jy_{ij}$ for all $i, j \in V$, and rewrite (3.13) equivalently as

$$\begin{array}{llll} \max\ (\min) & \sum_{i\in V}\sum_{j\in V} \frac{1}{j} x_{ij} & & \\ s.t. \quad \sum_{j\in V}\binom{j-1}{k-1} x_{ij} & = & \sum_{\substack{i\in I\subseteq V\\ |I|=k}} p_I & \text{for all } i\in V \text{ and } k=1,...,m, \\ x_{ij} & \geq & 0 & \text{for all } i,j\in V. \end{array} \tag{3.14}$$

The dual of this linear programming problem can be written as

$$\begin{array}{ll} \min\ (\max) & \sum_{i\in V}\sum_{k=1}^{m} u_{ik}\left(\sum_{\substack{i\in I\subseteq V\\ |I|=k}} p_I\right) \\ s.t. & \sum_{k=1}^{m}\binom{j-1}{k-1} u_{ik} \left\{\begin{array}{c}\geq\\ \leq\end{array}\right\} \frac{1}{j} \quad \text{for all } i,j\in V. \end{array} \tag{3.15}$$

Thus, the minimum and maximum values in the above linear programs are equal, respectively, to $\mathrm{UB}_m(PG)$ and $\mathrm{LB}_m(PG)$.

Let us denote by $u^* = (u^*_{ik} \mid i \in V,\ k = 1, ..., m)$ the optimal solution to the above LP(s), and define

$$w^*_I = \sum_{i \in I} u^*_{i,|I|}$$

for all $I \subseteq V$, $1 \le |I| \le m$.

Then we have

$$\sum_{\substack{I \subseteq V \\ 1 \le |I| \le m}} w^*_I p_I \ = \ \sum_{i \in V} \sum_{k=1}^{m} u^*_{ik} \left(\sum_{\substack{i \in I \subseteq V \\ |I| = k}} p_I \right) \tag{3.16}$$

and for a subset $S \subseteq V$ we have

$$w^*(S) \ = \ \sum_{i \in S} \sum_{k=1}^{m} \binom{|S|-1}{k-1} u^*_{ik}. \tag{3.17}$$

Thus, in the minimization problem the feasibility of u^* in (3.15) for $j = |S|$ implies that

$$w^*(S) \ = \ \sum_{i \in S} \sum_{k=1}^{m} \binom{|S|-1}{k-1} u^*_{ik} \ \ge \ \sum_{i \in S} \frac{1}{|S|} \ = \ 1, \tag{3.18}$$

while in the maximization problem the corresponding inequality implies that

$$w^*(S) \ = \ \sum_{i \in S} \sum_{k=1}^{m} \binom{|S|-1}{k-1} u^*_{ik} \ \le \ \sum_{i \in S} \frac{1}{|S|} \ = \ 1. \tag{3.19}$$

Therefore, by (3.16), (3.18) and (3.19) we can conclude that in both problems (2.4) the vector $w^* = (w^*_I \mid 1 \le |I| \le m)$ is a feasible solution in the case of $\mathcal{F} = \mathcal{F}_{dec}$. Since the bounds $\mathrm{UB}_m(PG)$ and $\mathrm{LB}_m(PG)$ correspond by (3.16) to objective functions values of feasible solutions, the optimum values in those problems must provide dominating bounds, as claimed in the theorem. □

We summarize the strict dominance relations among the bounds considered with the graphs in Figure 1. Here an arc between node i and node j means that bound i *strictly dominates* bound j (i.e., i gives a value that is always at least as good as j and strictly better than j in some instances). Nonadjacent nodes represent incomparable bounds, and isolated nodes represent bounds incomparable with all other bounds.

The weak dominance relations have been proved in this section or in previous works referenced here. Incomparability between bounds and strict dominance follow from numerical examples that will be described in the next section.

4 Numerical Results

We present the results of some computational experiences that we performed to evaluate the quality of our bounds and of the bounds that we have presented in the previous section. In particular, for $m = 2, 3$, we compare the $\mathrm{UB}_m(\mathcal{F}_{sub})$ and $\mathrm{UB}_m(\mathcal{F}_{dec})$ upper bounds to the ones proposed by Kwerel ($\mathrm{UB}_m(Kw)$), by Prekopa and Gao ($\mathrm{UB}_m(PG)$), and to another interesting, but not polynomial, upper bound, for $m = 3$, by Bukszár and Prékopa based on so-called t-cherry trees [48] that we denote $\mathrm{UB}_3(BP)$.

For $m = 2$, we compare our lower bounds with the $\mathrm{LB}_2(CE)$, $\mathrm{LB}_2(DS)$, $\mathrm{LB}_2(DeC)$ and $\mathrm{LB}_2(KAT)$ bounds. In the case of lower bounds of degree 3, we compare our bounds to the one provided by Prekopa and Gao ($\mathrm{LB}_3(PG)$).

We construct instances of the probability bounding problem as follows. Given integers n, m, and $k \geq n$, we randomly generate probabilities z_t, $t = 1, \ldots, k$, such that $\sum_{t=1}^{k} z_t \leq 1$, and a $k \times n$ matrix $R = \{r_{ti}\}$ with $\Pr(r_{ti} = 0) = \Pr(r_{ti} = 1) = 1/2$.

We then consider events A_i, $i = 1, \ldots, n$, corresponding to the columns of R, with probability $\Pr(A_i) = \sum_{\{t: r_{ti}=1\}} z_t$. Thus, $\Pr\left(\bigcap_{1 \leq i_1 < \ldots < i_s \leq n} A_{i_h}\right) = \sum_{\{t: r_{ti_1} = \ldots = r_{ti_s} = 1\}} z_t$, for $s = 2, \ldots, m$.

For each pair (n, m) we generate 100 instances. Let *opt* be the tightest possible bound provided by the Linear Program (2.1). Then, for each bound considered and for each instance, we compute the (percentage) Relative Error, which is $100 \times \frac{|bound - opt|}{opt}$. In the following tables, for each bound we report the mean, the standard deviation, and the maximum value of the Relative Error. We finally remark that the strict domination among the bounds described in Figure 1 follows from the weak domination results described in the previous section and from the numerical results presented in the Tables 1 – 5 below.

Table 1: Mean, Standard Deviation and Maximum value of the (percentage) Relative Error of the Upper Bounds for $m = 2$.

Events	$\mathrm{UB}_2(HW)$			$\mathrm{UB}_2(Ko)$			$\mathrm{UB}_2(Kw) \equiv \mathrm{UB}_2(PG)$		
n	Mean	Std	Max	Mean	Std	Max	Mean	Std	Max
6	9.15	7.17	28.88	17.51	11.17	47.32	40.70	18.88	76.63
7	15.81	8.40	46.79	26.69	13.64	63.70	44.68	22.20	94.27
8	20.94	11.15	57.01	32.37	17.39	75.36	46.60	23.17	104.20
9	24.79	11.78	60.57	35.60	17.86	75.85	47.11	27.58	130.38
10	31.25	14.28	84.49	41.88	20.01	86.79	48.19	23.10	106.20
11	44.37	15.55	78.81	71.86	25.95	133.01	170.25	20.14	205.64
12	58.15	16.60	100.33	92.38	28.22	149.05	196.63	20.42	230.89
13	64.88	19.01	103.08	103.00	36.57	207.98	216.09	26.16	262.06
14	73.62	18.01	123.03	119.69	29.95	190.52	244.44	24.65	287.28
15	84.18	20.30	135.67	125.89	33.29	200.78	261.90	25.66	311.31

Table 2: Mean, Standard Deviation and Maximum value of the (percentage) Relative Error (with respect to the Hunter Bound) of the Upper Bounds for $m = 2$ and $n \geq 20$.

Events	$UB_2(Ko)$			$UB_2(Kw)$		
n	Mean	Std	Max	Mean	Std	Max
20	27.01	10.11	56.89	99.73	18.96	173.19
30	25.69	8.77	45.97	98.49	19.13	189.23
40	24.27	7.24	43.06	90.90	13.92	131.73
50	21.51	7.26	35.24	82.88	12.86	150.17
60	21.73	4.97	31.69	75.59	7.10	96.17
70	18.78	6.21	32.02	74.66	10.03	108.35
80	18.01	4.48	29.53	69.01	6.42	81.13
90	17.99	4.81	27.44	66.34	7.58	98.62
100	16.69	4.22	29.50	61.48	4.25	76.79

In Table 2 we report some results for instances with $n \geq 20$ and $m = 2$. Since the dimension of the primal problem grows exponentially with n, we could not solve it for $n \geq 20$. Thus, in the Relative Error formula we use the value of the Hunter bound as a base instead of *opt* for computing the relative errors. This is justified by the results that show that this bound always dominates both the Kounias and the Kwerel bounds.

Table 3: Mean, Standard Deviation and Maximum value of the (percentage) Relative Error of the Upper Bounds for $m = 3$.

Events	$UB_3(\mathcal{F}_{sub})$			$UB_3(\mathcal{F}_{dec})$			$UB_3(PG)$			$UB_3(Kw)$			
n	Mean	Std	Max	Mean	Std	Max	Mean	Std	Max	Mean	Std	Max	Mean
6	15.45	6.31	26.83	7.39	3.30	15.80	9.63	3.69	18.51	26.09	12.08	72.33	5.90
7	19.41	8.20	39.68	9.55	4.88	24.21	12.26	5.18	28.23	22.34	11.22	58.91	8.23
8	19.59	9.01	43.40	11.10	4.19	21.51	13.93	5.10	28.25	22.74	10.43	55.45	9.93
9	21.24	11.14	47.06	12.24	6.09	27.54	15.16	6.55	30.60	21.01	10.64	54.40	12.29
10	48.47	15.41	77.84	18.52	6.42	37.37	21.25	7.12	43.79	37.61	14.10	67.72	17.74
11	54.71	21.02	84.60	18.32	7.34	38.34	21.20	8.45	43.72	34.87	15.63	63.96	19.79
12	60.64	22.44	98.16	19.98	7.20	41.95	23.20	8.22	47.62	36.01	15.02	67.04	23.30
13	68.11	24.94	99.35	20.71	7.87	43.34	24.03	9.12	49.63	35.82	13.76	69.81	24.14
14	79.08	21.23	98.40	21.52	7.79	36.21	24.90	8.64	40.82	38.25	13.26	65.42	26.65
15	71.47	32.23	115.11	22.80	6.58	30.37	27.49	6.90	36.24	37.98	11.35	56.87	25.38

Note that in Table 3 the mean relative error of $\text{UB}_3(\mathcal{F}_{sub})$ is always greater than the ones provided by $\text{UB}_3(\mathcal{F}_{dec})$ and $\text{UB}_3(BP)$. Nevertheless, the following example with $n = 4$ shows a case where the $\text{UB}_3(\mathcal{F}_{sub})$ and $\text{UB}_3(BP)$ bounds are tighter than $\text{UB}_3(\mathcal{F}_{dec})$. Let:
$p_1 = 0.4539, p_2 = 0.2585, p_3 = 0.3094, p_4 = 0.2106, p_{12} = 0.1971, p_{13} = 0.2164, p_{14} = 0.1461, p_{23} = 0.2047, p_{24} = 0.0605, p_{34} = 0.1087, p_{123} = 0.1433, p_{124} = 0.0605, p_{134} = 0.1087, p_{234} = 0.0605$.
In this case we have $\text{UB}_3(\mathcal{F}_{sub})$=$\text{UB}_3(BP)$=0.6114 which is also optimal, $\text{UB}_3(\mathcal{F}_{dec})$=0.6239, $\text{UB}_3(PG)$=0.6410 and $\text{UB}_3(Kw)$=0.6719. On the other hand, for the probability distribution
$p_1 = 0.4673, p_2 = 0.2909, p_3 = 0.2011, p_4 = 0.3329, p_{12} = 0.1511, p_{13} = 0.1421, p_{14} = 0.1954, p_{23} = 0.1246, p_{24} = 0.1662, p_{34} = 0, p_{123} = 0.0742, p_{124} = 0.0768, p_{134} = 0, p_{234} = 0$,
we obtain $\text{UB}_3(\mathcal{F}_{sub}) = \text{UB}_3(BP) = 0.7143$, while $\text{UB}_3(Kw)$=0.6639, which is also optimal. This means that the bounds $\text{UB}_3(\mathcal{F}_{sub})$ and $\text{UB}_3(BP)$ are incomparable with the $\text{UB}_3(\mathcal{F}_{dec})$, $\text{UB}_3(PG)$ and $\text{UB}_3(Kw)$ bounds.

The following two tables report the comparisons between the lower bounds for $m = 2, 3$.

Table 4: Mean, Standard Deviation and Maximum value of the (percentage) Relative Error of the Lower Bounds for $m = 2$.

Events	$LB_2(\mathcal{F}_{sup})$			$LB_2(\mathcal{F}_{dec})$			$LB_2(CE)$			$LB_2(DS)$			$LB_2(Dec)$			$LB_2(KAT)$		
n	Mean	Std	Max	Mean	Std	Max	Mean	Std	Max	Mean	Std	Max	Mean	Std	Max	Mean	Std	M
6	28.39	9.08	49.78	4.45	2.95	14.12	9.37	3.89	22.68	7.89	3.95	21.67	8.71	3.57	21.18	7.23	3.70	2
7	29.12	9.02	52.97	4.87	2.68	13.67	9.20	3.32	17.02	8.18	3.41	15.73	8.44	2.98	16.31	7.32	3.02	1
8	28.74	9.01	46.08	5.58	2.77	14.05	9.86	3.75	19.99	9.14	3.77	19.15	9.10	3.41	18.54	8.25	3.37	1
9	26.81	8.76	44.28	4.72	2.36	11.61	8.41	3.27	17.77	7.74	3.30	17.23	7.84	3.03	17.10	7.11	3.03	1
10	26.91	8.27	43.80	4.90	2.53	14.27	8.25	3.48	19.87	7.72	3.57	19.71	7.67	3.24	19.35	7.11	3.25	1
11	25.84	7.48	41.96	4.35	2.16	10.80	7.08	2.80	15.20	6.61	2.84	15.10	6.63	2.64	14.46	6.15	2.65	1
12	27.71	8.01	41.79	4.45	1.95	9.74	6.92	2.51	12.23	6.53	2.51	11.96	6.48	2.33	11.69	6.09	2.33	1
13	27.17	8.46	42.53	4.45	1.83	9.17	6.61	2.32	12.96	6.25	2.32	12.57	6.23	2.18	12.37	5.89	2.19	1
14	27.36	6.45	45.54	3.99	1.75	9.74	6.23	2.31	15.22	5.92	2.33	14.84	5.87	2.17	14.09	5.58	2.20	1
15	27.12	6.80	42.53	4.14	1.77	9.66	6.14	2.24	11.64	5.85	2.25	11.33	5.79	2.14	11.27	5.53	2.15	1

Table 5: Mean, Standard Deviation and Maximum value of the (percentage) Relative Error of the Lower Bounds for $m = 3$.

Events	$\mathrm{LB}_3(\mathcal{F}_{dec})$			$\mathrm{LB}_3(PG)$			$\mathrm{LB}_3(Kw)$		
n	Mean	Std	Max	Mean	Std	Max	Mean	Std	Max
6	0.90	1.33	6.22	4.87	2.81	14.17	5.63	3.17	14.69
7	1.17	1.37	6.62	4.43	2.20	10.67	5.07	2.56	11.65
8	2.02	1.52	6.97	4.92	2.08	10.86	5.40	2.28	12.02
9	1.56	1.22	4.79	4.35	1.96	9.91	4.75	2.16	10.69
10	1.72	1.04	5.23	4.04	1.32	7.67	4.38	1.44	8.21
11	0.81	0.78	3.67	3.37	1.72	8.73	3.71	1.90	9.44
12	0.98	0.83	3.72	3.27	1.48	8.02	3.61	1.70	8.67
13	0.91	0.79	3.37	3.17	1.51	8.31	3.45	1.62	8.66
14	0.76	0.73	3.20	2.88	1.63	8.27	3.12	1.79	9.30
15	0.77	0.53	2.58	2.89	1.44	6.86	3.11	1.56	7.09

Finally, the complete (strict) domination and incomparability diagram among both known and new bounds discussed in the paper is reported in Figure 1.

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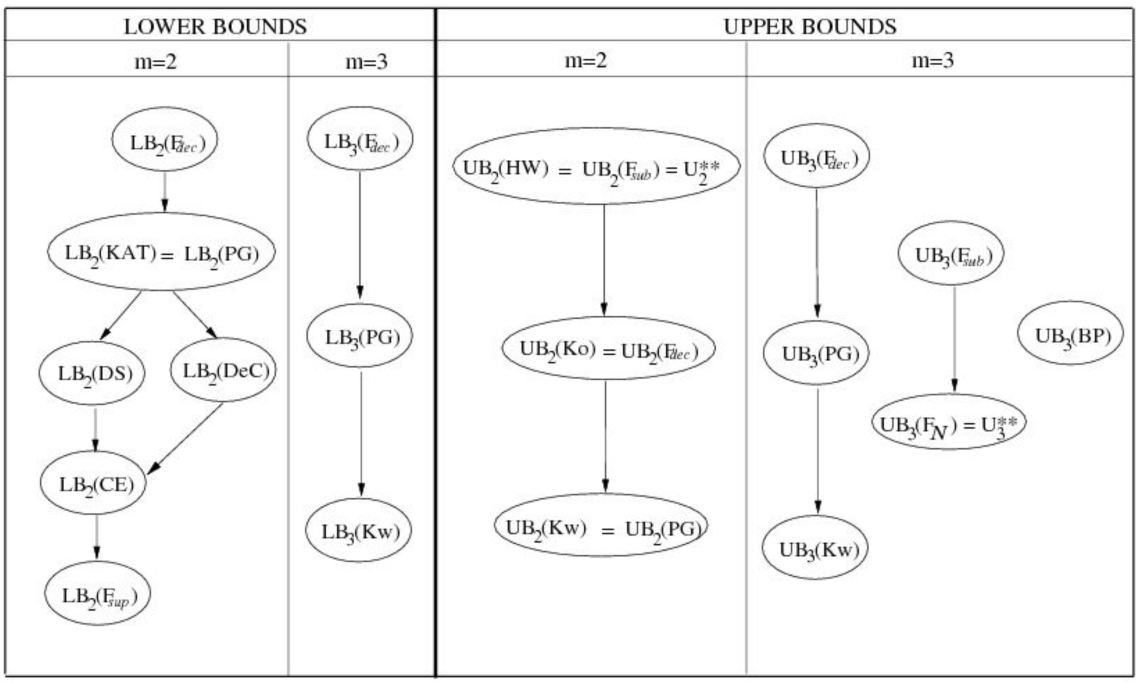

Figure 1: Dominance relationships among known bounds.