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# Games Seki and Seki-I

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**Abstract.** Let $A : I \times J \to \mathbb{Z}_+$, be a non-negative integer $m \times n$ matrix each row and column of which contain a strictly positive entry. The game *Seki* is defined as follows. Two players $R$ and $C$ take turns and it is specified who begins; this player is called the *first*, while the opponent *second*.
By one move a player can either reduce a strictly positive entry of $A$ by 1 or pass. If both players pass then the game results in a draw. Player $R$ (respectively, $C$) wins if a row (respectively, column) appears every entry of which is 0.
After a move, such a row and column may appear simultaneously. In game Seki we assume that the player who made this last move is the winner. Yet, we also study another version of the game, Seki-I, in which the above case is defined as a draw.
If neither $R$ nor $C$ wins, even being first, $A$ is called a *seki matrix (SM or SM-I)*. Furthermore, $A$ is called a *complete seki matrix (CSM or CSM-I)* if $A$ is a seki matrix and, moreover, each player *must* pass, that is, if (s)he makes an active move, the opponent wins.
Both Seki and Seki-I are difficult games. We cannot solve them and present only some partial results and conjectures mostly related to CSMs and CSM-Is. The latter family looks simpler, yet, game Seki, unlike Seki-I, is closely connected with the so-called seki (shared life) positions in the classical game of Go. Both Seki and Seki-I are of independent interest as combinatorial games.

**Key words:** combinatorial games, games with positive incentive, Go, seki, shared life, complete seki, draw, integer matrix, integer doubly stochastic matrix.

# 1 Introduction; main results and conjectures

Game Seki was introduced in 1981 by Andrey I. Gol'berg (1954-1985) and the author [3]. Some computational and theoretical results were summarized in the preprint [1].

## 1.1 On $1 \times 1$ matrices

Let us begin with a $1 \times 1$ matrix $A$ (whose entry $A(1, 1)$ is integer and strictly positive). Every such matrix is a SM-I but not CSM-I, since Seki-I always results in a draw.

Let us consider Seki. If $A(1, 1) = 1$ then the first player wins by one move. If $A(1, 1) = 2$ then $A$ is a CSM. Finally, if $A(1, 1) \geq 3$ then $A$ is a SM but not a CSM.

## 1.2 CSM-I $\Rightarrow$ CSM $\Rightarrow$ SM $\Rightarrow$ SM-I

Let SM, SM-I, CSM, and CSM-I denote the corresponding classes of matrices.

**Claim 1** *The following strict containments hold: CSM-I $\subset$ CSM $\subset$ SM $\subset$ SM-I.*

**Proof.** A complete seki is, in particular, a seki, that is, CSM $\subseteq$ SM, by definition.

Let us also recall that if a 0-row and 0-column appear simultaneously, after a move, then the player who made this move wins in Seki, while the result of Seki-I is defined as a draw. Hence, it is more difficult to win in Seki-I than in Seki. This observation implies both right and left containments: SM $\subseteq$ SM-I and CSM-I $\subseteq$ CSM.

Already $1 \times 1$ matrices suffice to demonstrate that all three containments are strict. Indeed, if $A(1, 1) = 1$ then $A$ is a SM-I but not SM; if $A(1, 1) = 2$ then $A$ is a CSM but not CSM-I; finally, if $A(1, 1) = 3$ then $A$ is a SM but not CSM. $\square$

Let us also remark that a SM can be reduced to a CSM, in contrast to a SM-I.

**Claim 2** *For each $m \times n$ SM $A$ there is a $m \times n$ CSM $A'$ such that $A' \leq A$. Yet, for Seki-I the similar claim fails.*

**Proof.** Indeed, both games Seki and Seki-I result in a draw if both players pass their turns. The inverse also holds for Seki, but not for Seki-I. For example, any positive integer $1 \times 1$ matrix is a SM-I but none of them is a CSM-I. In contrast, if $A$ is a SM then a CSM $A'$ will result from $A$ in several moves. Obviously, $A$ and $A'$ are of the same size and $A' \leq A$. $\square$

**Remark 1** *All considered matrices are defined up to permutations of their rows and columns. It is also clear that $A$ is SM, SM-I, CSM, or CSM-I whenever the transposed matrix $A^T$ is. Indeed, the game is not changed if the matrix is transposed and the players $R, C$ are swaped.*

## 1.3 On $2 \times 2$, $1 \times n$, and $2 \times n$ matrices

First, let us consider the $1 \times n$ matrices. The case $n = 1$ was already studied: there is no CSM-I; only $A(1,1) = 2$ is a CSM; every $A(1,1)$ but 1 is a SM; each $A(1,1)$ is a SM-I.

**Proposition 1** *There is no SM-I (and hence, no SM either) of size $1 \times n$ with $n > 1$.*

**Proof.** Obviously, in this case, player $C$ wins in Seki-I (and hence, in Seki too). The winning strategy is simple, it is enough just to reduce a minimum entry all time. □

Second, it is not difficult to verify that the following four $2 \times 2$ matrices are CSMs:

$$\begin{array}{cccc} A_1^2 & A_2^2 & A_3^2 & A_4^2 \\ 1\,1 & 2\,0 & 2\,1 & 2\,2 \\ 1\,1 & 0\,2 & 1\,2 & 2\,2 \end{array}$$

Among them, only the first one is a CSM-I, while three others are not.

**Proposition 2** *Except for the listed above, there is no other $1 \times n$ or $2 \times n$ CSM or CSM-I. Furthermore, there is no $m \times n$ SM with $2 \geq m < n$.*

The case $m = 1$ follows from Proposition 1. The rest will be proven in Section 2.3.

For example, it is easy to verify that the following matrices are SMs but not CSMs:

$$\begin{array}{cccccc} 2\,1 & 2\,1 & 3\,2 & 3\,2 & 3\,1 & 3\,0 \\ 1\,1 & 1\,3 & 2\,2 & 2\,3 & 1\,3 & 1\,3 \end{array}$$

Let us finally notice that there are many $2 \times n$ SM-Is with $n > 2$. For example, neither $R$ nor $C$, even being first, wins in Seki-I in the following $2 \times 3$ matrix.

$$\begin{array}{c} 1\,1\,1 \\ 2\,3\,3 \end{array}$$

## 1.4 The importance of being first and strategy stealing

Let us also note that in both games Seki and Seki-I, for each player $R$ or $C$, to be the first is not worse than to be the second.

**Claim 3** *If a player can win (make a draw) being second, (s)he can do the same being first. In particular, the second player cannot win in a symmetric matrix.*

**Proof.** By the definition, in both games, Seki and Seki-I, the players are allowed to pass their turns. Hence, if a player wins (makes a draw) being second then, being first, (s)he can just pass and apply the same strategy.

Let us assume indirectly that the second player wins in a symmetric matrix. Then, the first player can just pass and then win. Indeed, becoming second, (s)he can apply the same (or, more accurately, the transposed) winning strategy. Thus, we obtain a contradiction. □

**Remark 2** *The games of this type were first considered in 1953 by John Milnor [5] who called them the* games with positive incentive.

## 1.5 Prime (complete) seki matrices

Given several non-negative integer matrices $A_\ell : I_\ell \times J_\ell \to \mathbb{Z}_+$, where $\ell \in [k] = \{1, \dots, k\}$, their direct sum $A = \oplus_{\ell\in[k]} A_\ell$ is standardly defined as follows: $A : I \times J \to \mathbb{Z}_+$, where $I = \cup_{\ell\in[k]} I_\ell$, $J = \cup_{\ell\in[k]} J_\ell$, and all $2k$ sets $I_\ell$, $J_\ell$, $\ell \in [k]$ are assumed pairwise disjoint. Furthermore, $A(i, j) = A(i_\ell, j_\ell)$ if $i \in I_\ell, j \in J_\ell$ for some $\ell \in [k]$ and $A(i, j) = 0$ otherwise.

**Proposition 3** *If $A = \oplus_{\ell\in[k]} A_\ell$ then $A$ is SM, CSM, or CSM-I if and only if $A_\ell$ have the corresponding property for all $\ell \in [k]$. In particular, matrix $A$ is a draw in Seki if and only if $A_\ell$ is a draw for all $\ell \in [k]$. The "if" (but not "only if") part holds for Seki-I too.*

The proof is given Section 2.4. Let us note that the claim cannot be extended to SM-Is.

A SM, CSM, or CSM-I, is called *prime* if it is not a non-trivial ($k > 1$) direct sum. Obviously, by this definition, each SM, CSM, or CSM-I is the direct sum of prime ones.

## 1.6 Integer doubly stochastic matrices

Given a (non-negative integer) $m \times n$ matrix $A : I \times J \to \mathbb{Z}_+$, let $s^r_i$ and $s^c_j$ denote the sum of all entries of the row $i \in I$ and column $j \in J$, respectively. Then, $A$ is called an *integer doubly stochastic matrix* (IDSM) with the sum $s = s(A)$ if the sum of the entries in every row and every column of $A$ is a constant, that is, $s^r_i = s^c_j = s$ for all $i \in I$ and $j \in J$.

Obviously, an IDSM is a square matrix, $m = |I| = |J| = n$, whenever $s > 0$.

## 1.7 Conditions sufficient to win

Although it seems difficult to characterize the winning positions in Seki or Seki-I, yet, the following simple sufficient conditions will be instrumental.

**Lemma 1** *Being first, player $R$ wins in Seki-I whenever there is a row $i \in I$ such that $s^c_j - s^r_i \geq A(i, j)$ for every column $j \in J$.*

A similar, but a little bit more sophisticated, statement holds for game Seki.

**Lemma 2** *Being second, player $R$ wins in Seki whenever there is a row $i \in I$ such that $s^c_j - s^r_i \geq A(i, j)$ for every column $j \in J$ and the inequality is strict whenever $A(i, j) = 0$.*

Lemmas 1 and 2 will be proven in Sections 2.1 and 2.2, respectively. We will demonstrate that the inverse claims fail, that is, both conditions are only sufficient but not necessary for wining and that the last requirement of Lemma 2 is essential.

Of course, the similar two statements hold for the player $C$, as well.

## 1.8 Results and conjectures related to CSMs

The *hight* $h = h(A)$ of a matrix $A : I \times J \to \mathbb{Z}_+$ is defined as the maximum of its entries. The matrices of the hight 1 and 2 are called the $(0,1)$- and $(0,1,2)$-matrices.

Lemmas 2 and 1 imply the next two criteria, Theorems 1 and 2, respectively.

**Theorem 1** *An IDSM $A$ is a CSM whenever $h(A) \leq 2$ and $s(A) \geq 2$.*

**Proof.** Let player $C$ reduce an entry $(i,j)$. Then, player $R$ can reduce another entry, $(i,j')$ of the same row $i$ (distinct from $(i,j)$) unless $A(i,j) = s(A) = 2$. By these two moves, the sum $s^i$ was reduced by 2, the entries $A(i,j')$ and $A(i,j'')$ by 1 each, while all other entries of the row $i$ were not changed. It is easy to check that the conditions of Lemma 2 hold for the obtained matrix. Then, being now the second again, player $R$ wins.

Finally, it remains to notice that if $A(i,j) = s(A) = 2$ then $R$ still wins in one move just reducing further $A'(i,j) = 1$ to $A''(i,j) = 0$. $\square$

Let us notice that both conditions of the theorem are essential.

Indeed, if $s(A) = 1$ then the first player wins in one move.

If $h(A) \geq 3$ and player $C$ reduces an entry $(i,j)$ such that $A(i,j) \geq 3$ then the conditions of Lemma 2 may fail for the entry $(i,j)$ of the obtained matrix, since $A'(i,j) \geq 2$.

In fact, computer analysis shows that there are seven prime $3 \times 3$ CSMs of hight 3:

| $A^3_1$ | $A^3_2$ | $A^3_3$ | $A^3_4$ | $A^3_5$ | $A^3_6$ | $A^3_7$ |
|---|---|---|---|---|---|---|
| 033 | 133 | 301 | 320 | 320 | 320 | 033 |
| 303 | 313 | 022 | 212 | 203 | 213 | 331 |
| 330 | 331 | 121 | 023 | 032 | 033 | 312 |

Moreover, it also shows that there are no other $3 \times 3$ prime CSMs of hight at most 10.

Even without a computer, one can check that these seven matrices are CSMs. However, this verification would require some time consuming case analysis.

Let us notice that all seven matrices are symmetric, yet, the last two are not IDSMs. Furthermore, $A^3_6$ (respectively, $A^3_7$) results from (respectively, in) an IDSM, by one move.

Hence, there is a symmetric IDSM (for example, one obtained from $A^3_7$ by reducing its central entry by 1) in which the first player, $R$ or $C$, wins. It is obvious, since $A^3_7$ is a CSM. In contrast, by Claim 1, the second player cannot win in a symmetric matrix.

Let $H(n)$ denote the maximum hight of a $n \times n$ CSM. As we already know, $H(1) = H(2) = 2$, while $H(3) \geq 3$. In Section 3.1, we will see that $H(n) \geq 3$ for all $n \geq 3$.

**Conjecture 1** *Function $H(n)$ is well defined (that is, $H(n) < \infty$) for all integer $n \geq 1$; furthermore, it is monotone non-decreasing and unbounded. Perhaps, $H_n = n$ for all $n \geq 2$.*

Yet, all known CSMs are of hight at most 3. Also, they are all quadratic.

**Conjecture 2** *Each SM (and hence, each CSM too) is a square matrix, $m = |I| = |J| = n$; in other words, for any $m \times n$ matrix with $m \neq n$, somebody wins in game Seki.*

In the case $\min(m, n) \leq 2$, Conjecture 2 follows from Propositions 1 and 2.

There is no $1 \times n$ SM-I for $n > 1$ and no $2 \times n$ SM for $n > 2$. Yet, it is not known whether a $3 \times 4$ SM (or CSM, or CSM-I) exists. However, let us recall that

(i) there are non-square (for example, $2 \times 3$) SM-Is;

(ii) every SM can be reduced to a CSM of the same size;

(iii) the strict containments of Claim 1 hold.

**Conjecture 3** *Any prime* $(0, 1, 2)$ *CSM is an IDSM.*

This conjecture was proven in [1] for the $(0, 1)$ CSMs and it was verified by a computer code for all $n \times n$ matrices with $n \leq 5$.

Yet, even being proven in general, this conjecture would not clarify the structure of the CSMs, since already a prime CSM of hight greater than 2 may be not an IDSM, as we know.

## 1.9 Results and conjectures related to CSM-Is

In contrast, the family of all known CSM-Is looks much simpler than that of the CSMs.

**Theorem 2** *A* $(0, 1)$ *IDSM* $A$ *is a CSM-I whenever* $s(A) \geq 2$.

**Proof.** Let $C$ reduce an entry $A(i, j)$. Then $R$ easily wins, just reducing the entries of the same row $i$ in an arbitrary order. This statement can be also derived from Lemma 1. $\square$

The condition $s \geq 2$ is essential again. Indeed, if $s = 1$ then the first player makes a draw in one move. Hence, in this case $A$ is a SM-I but not CSM-I.

However, there are more CSM-Is. The following $(0, 2)$-matrices are called 2-*cycles*:

$$\begin{matrix} 2&2&0\\ 2&0&2\\ 0&2&2 \end{matrix} \qquad \begin{matrix} 2&2&0&0\\ 2&0&2&0\\ 0&2&0&2\\ 0&0&2&2 \end{matrix} \qquad \begin{matrix} 2&2&0&0&0\\ 2&0&2&0&0\\ 0&2&0&2&0\\ 0&0&2&0&2\\ 0&0&0&2&2 \end{matrix}$$

More precisely, a 2-cycle $C^n$ is defined for each $n \geq 3$ as the (unique) $n \times n$ matrix $A$ such that the first and the last entries of its main diagonal, as well as all entries of the two neighbor diagonals of $A$, equal 2 (that is, $A(1, 1) = A(n, n) = A(i, i + 1) = A(i + 1, i) = 2$ for $i = 1, \ldots, n - 1$), while all remaining entries of $A$ equal 0.

**Theorem 3** *Every* 2*-cycle is a prime CSM-I.*

Obviously, a 2-cycle is prime. The rest will be proven in Sections 2.5.

**Conjecture 4** *Each CSM-I is the direct sum of* 2*-cycles and* $(0, 1)$ *IDSMs of hight at least* 2*; in other words, all prime CSM-I are listed by Theorems 2 and 3. In particular, each CSM-I is a square matrix.*

# 2 Proofs and comments

## 2.1 Lemma 1

Let $S1$ denote the condition of Lemma 1 and let row $i$ satisfy $S$. We will show that $R$ can win just reducing the entries of row $i$ in any order. Indeed, let $R$ play at $(i, j)$ and $C$ answer at $(i', j')$. We will show that $S1$ still holds for the same row $i$ in the reduced matrix $A'$.

**Case 1**: $i' = i$. It is obvious. Indeed, $s_i^r$ is reduced by 2, while $s_j^c$ by at most 1 for each $j$ unless $j' = j$. In the last case $s_j^c$ is reduced by 2 but $A(i, j)$ too.

**Case 2**: $i' \neq i$ but $j' = j$. Then $s_j^c$ is reduced by 2 but $s_i^r$ and $a_{i,j}$ are reduced by 1 each. Hence, $S1$ still holds for $j' = j$. Obviously, it holds for all other $j'' \in J$ as well.

**Case 3**: $i' \neq i$ and $j' \neq j$. It is again obvious. Indeed, $s_j^c$, $s_{j'}^c$, and $s_i^r$ are reduced by 1 each, while for any other column $j'' \in J$ the sum $s_{j''}^c$ did not change. Still, we have to consider separately the case $s_i^r = A(i, j) = 1$. Then, $s_j^c \geq 1$ for all $j \in J$ and, moreover, $s_i^c \geq 2$. Thus, no zero column appears after eliminating $A(i, j)$ and $R$ wins in one move. $\square$

Let player $C$ make a move in a 2-cycle $C^n$. It is easily seen that $S1$ fails in the obtained matrix. Yet, player $R$ begins and wins, since $C^n$ is a CSM-I (see Section 2.5 for the proof). Thus condition $S1$ is only sufficient but not necessary for winning.

## 2.2 Lemma 2

Let $S2$ denote the condition of Lemma 2 and let row $i$ satisfy $S2$. If $C$ plays at $(i', j')$ then $R$ responds at $(i, j)$ for some $j \neq j'$. This is possible unless $A(i, j') = 0$ for all $j' \in J$ but $j$. In the last case, $R$ plays at $(i, j)$. It is easy to check that $S2$ still holds for $i$ after these two moves (even when $A(i, j) = 1$). It remains to note that $C$ can pass. In this case, $R$ can reduce any entry of $i$ to keep $S2$ for it. Thus, $S2$ holds in all cases; hence, $R$ eliminates row $i$ in $s_i^r$ moves and win before a column is eliminated, or simultaneously, in the worst case. $\square$

Condition $S2$ means that $R$ can take a burden of deleting $a_{i,j}$ and still win.

In Section 2.3.2 we will show that for $2 \times n$ matrices $S2$ is not only necessary but also sufficient for the player $R$ to win, being second. As a corollary we derive that there is no $2 \times n$ seki for $n > 2$. Furthermore, a $2 \times 2$ matrix is a SM if and only if $S2$ fails for both players $R$ and $C$. From this observation we will derive that there are only four $2 \times 2$ CSMs $A_\ell^2$, $\ell = 1, 2, 3, 4$, given above; see Lemma 5 of Section 2.3.4.

Let us also note that the extra strictness requirement in $S2$ is essential. Indeed, if $a_{i,j} = 0$ and $s_i^r = s_j^c$ then not $R$ but $C$ wins. Indeed, $C$ can begin and delete column $j$ before $R$ could eliminate row $i$ or any other row.

What would be good sufficient conditions for the first player to win? Let us try to replace $S2$ by the weaker inequality $s_j^c - s_i^r \geq a_{i,j} - 1$. Yet, the obtained condition fails to be sufficient, as, for example, the following three matrices show:

$$\begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix}\qquad\begin{matrix}1&1&1\\1&2&0\\1&0&2\end{matrix}\qquad\begin{matrix}1&1\\1&1\\ &\end{matrix}$$

For each of them the inequality $s^c_j - s^r_i \geq a_{i,j} - 1$ holds for the first row, $i = 1$, and every column $j$. Moreover, all entries of the first row are strictly positive. Yet, these three matrices are CSMs, by Theorem 1. Hence, $R$ cannot win even if (s)he begins.

Of course, condition $S2$, which is sufficient even for the second player to win, is also sufficient for the first player to win. However, such a condition is too strong and is not necessary even for the $2 \times 2$ matrices. In fact, the best what we can suggest is a triviality:

**Lemma 3** *Playing first, $R$ wins whenever (s)he can reduce the original matrix $A$ to a matrix $A'$ satisfying condition $S2$ (which is sufficient for $R$ to win in $A'$ being second).* □

Somewhat surprisingly, this lemma is efficient enough. For example, let us consider

$$\begin{matrix}1&k\\k&k\end{matrix}$$

This matrix is a CSM if $k = 1$. Yet, if $k > 1$ then $R$ begins and wins by reducing $A(1, 2) = k$. Indeed, row 1 of the obtained matrix $A'$ satisfies $S2$, since $(k + 1) - (1 + (k - 1)) = 1 > 0$ and $(k + (k - 1)) - (1 + (k - 1)) = k - 1 > 0$. Hence, $R$ wins, by eliminating the first row, even if $C$ begins. More generally, if $0 < \ell < k$ then $S2$ holds for the matrix

$$\begin{matrix}\ell&k-\ell\\k&k\end{matrix}$$

## 2.3 Proposition 2

### 2.3.1 Simple corollaries of conditions $S1$ and $S2$ for $m = 2$

Obviously, for each matrix $A : I \times J \to \mathbb{Z}_+$ we have

$$\sum_{i\in I} s^r_i = \sum_{j\in J} s^c_j = \sum_{i\in I, j\in J} A(i, j). \tag{1}$$

Condition $S1$ claims that there is a $i \in I$ such that $s^c_j \geq s^r_i + A(i, j)$ for each $j \in J$.

Condition $S2$ is stronger than $S1$, it requires additionally that $s^c_j > s^r_i$ whenever $A(i, j) = 0$.

In particular, if $m < n$ then $S1$ implies that there are $i \in I$ and $j \in J$ such that $s^r_i > s^c_j$.

In case $m = 2$ we derive a much stronger corollary. If row 1 satisfies $S1$ then $A(2, j) \geq s^r_1$ for each $j \in J$ and, hence,

$$s^r_2 = \sum_{j=1}^{n} A(2, j) \geq n s^r_1 \tag{2}$$

In other words, for $m = 2$, condition $S1$ implies that the sum of one row is at least $n$ times the sum of the other. Of course, this follow from $S2$ too, since $S2 \Rightarrow S1$.

#### 2.3.2 Case $2 = m \leq n$

Let us show that for $m = 2$ condition $S2$ is not only sufficient but also necessary for $R$, being second, to win. More precisely, the following claim holds.

**Lemma 4** *Let $A$ be $m \times n$ matrix with $2 = m \leq n$ and let $C$ begin. Player $R$ wins if and only if condition $S2$ holds. Moreover, $A$ can be a SM only if $m = n = 2$. If $2 = m < n$ then $R$ wins when $S2$ holds and $C$ wins if it does not.*

**Proof.** By Lemma 2, condition $S2$ is always sufficient for $R$, being second, to win. To prove that it is necessary too, we assume that $S2$ fails and will show that $C$ can always maintain this situation, that is, $C$ has a move (not pass) such that for any response of $R$ $S2$ still does not hold for the obtained matrix. Then, either $A$ is a SM or $C$ wins. Indeed, obviously, $S2$ holds for a matrix that contains a 0-row but no 0-column. Hence, by maintaining $S2$, player $C$ guarantees that no 0-row appears before a 0-column.

Without loss of generality, let us assume that $s^r_1 \leq s^r_2$. Then $S2$ fails (for row 1, i. e., there is a column $j \in J$ such that $s^c_j - s^r_1 < A(1, j)$) unless $s^r_2 \geq n s^r_1$.

The problem that $C$ has to resolve is not too difficult; for example, the following simple strategy works: whenever possible, $C$ reduces $A(2, j)$. It is not difficult to check that, after any move of $R$, condition $S2$ still fails for the obtained matrix.

If $A(2, j) = 0$ then $C$ can reduce $A(1, j)$. If $A(1, j) = 1$ then (s)he wins in one move. If $A(1, j) \geq 2$ then either $C$ can eliminate column $j$ and win, or the matrix will be decomposed into the direct sum of a $1 \times 1$ matrix $(1, j)$, with $A(1, j) \geq 2$, and the corresponding $1 \times (n-1)$ matrix. There are two subcases. If $n > 2$ then $C$ wins in every $1 \times (n - 1)$ matrix and, hence, in the original matrix, too. Yet, if $n = 2$ then both matrices may be SMs. □

#### 2.3.3 The standard form of a $2 \times 2$ matrix

Given a $2 \times 2$ matrix $A : I \times J \to \mathbb{Z}_+$, where $I = J = \{1, 2\}$, without any loss of generality we will assume that

$$s^r_1 \leq s^r_2, \;\; s^c_1 \leq s^c_2, \;\; s^r_1 \leq s^c_1 \tag{3}$$

In terms of the entries of $A$, we can rewrite this system of inequalities as follows:

$$A(2, 2) - A(1, 1) \geq A(2, 1) - A(1, 2) \geq 0. \tag{4}$$

#### 2.3.4 Listing all $2 \times 2$ CSMs

By definition, the first player cannot win in a CSM; moreover, if (s)he does not pass then the opponent wins. By Lemma 4, for $2 \times 2$ matrices condition $S2$ is necessary and sufficient for the first player to win. This provides a characterization of the $2 \times 2$ CSMs from which we will derive a much simpler one.

**Lemma 5** *There exist only four* $2 \times 2$ *CSMs* $A^2_\ell$, $\ell = 1, 2, 3, 4$, *introduced in Section 1.3.*

**Proof.** Let $A$ be a $2 \times 2$ CSM. Without any loss of generality, we can assume that (4) holds for $A$ and that $C$ begins. Then, $R$ can enforce $S2$ after every move of $C$; in particular, after one in $(2, 2)$. There are two cases: $S2$ will hold for row 1 or 2.

**Case 1**: $S2$ holds for row 1. Then, if $R$ beings, (s)he can still enforce $S2$ for row 1 by the same move and win. Indeed, $S2$ for row 1 is obviously respected by any increase of the entries of row 2. Hence, $A$ is not a CSM and we get a contradiction.

**Case 2**: $S2$ holds for row 2. By two moves row 2 could be reduced by at most 2.

Hence, by (2), $s^r_1 \geq 2(s^r_2 - 2)$ and by (3), $s^r_2 \geq s^r_1$. Thus, $s^r_2 \geq 2s^r_2 - 4$, that is, $s^r_2 \leq 4$. Summarizing, we obtain that $s^r_1 \leq s^r_2 \leq 4$. There are just a few matrices satisfying these inequalities and it is easy to verify that, except for four matrices $A^2_\ell$, $\ell = 1, 2, 3, 4$ of Section 1.3, there is no other CSM among them. □

**Corollary 1** *Matrix* $A^2_1$ *is a unique CSM-I among all* $1 \times n$ *and* $2 \times n$ *matrices.*

**Proof.** Let us recall that CSM-I $\subset$ CSM, by Claim 1, and $A^2_\ell$ is a CSM-I only for $\ell = 1$. □

## 2.4 Proposition 3

Suppose, in Seki or Seki-I a player, $R$ or $C$, does not win in $A_\ell$ for all $\ell \in [k]$, even being first. Then, (s)he cannot win in $A$ either. Indeed, the opponent guarantees at least a draw always replying optimally in the same $A_\ell$, where the previous move was made.

For Seki, the inverse also holds. If a player, say $R$, wins in $A_{\ell_0}$ then $A$ cannot be a draw. Indeed, if $R$ is the second then (s)he always responds optimally in the same subgame $A_\ell$, where the previous move was made by $C$. If $R$ is the first then (s)he starts optimally in $A_{\ell_0}$ and after this applies the same strategy. Obviously, $R$, wins whenever $A_{\ell_0}$ is finished. Otherwise, if some other subgame $A_\ell$ is finished before $A_{\ell_0}$ then either $R$ or $C$ wins, but $A_\ell$ cannot result in a draw. Indeed, a draw in Seki means that both players pass their turns. Yet, $R$ will always proceed in $A_{\ell_0}$ rather than pass.

However, the last argument of the proof does not work for the game Seki-I. For example, if $A_\ell$ is a $1 \times 1$ matrix with an entry 1 then the first player can force a draw in one move. For this reason, Proposition 3 cannot be extended to SM-Is.

Yet, the claim still holds for the CSM-Is, since they are defined it terms of winning rather than a draw. A matrix is a CSM-I if after any active move of a player the opponent wins. By the above arguments, $A$ has this property if and only if $A_\ell$ have it for all $\ell \in [k]$. □

## 2.5 Theorem 3

Now, let us recall Remark 1 and show that the 2-cycles are transitive.

**Lemma 6** *Let $(i,j),(i',j')$ be two positive entries of a 2-cycle $C^n$, that is, $C^n(i,j) = C^n(i',j') = 2$, where $i,i',j,j' \in [n] = \{1,\ldots,n\}$. Then, there are permutations of the rows and columns of $C^n$ that transform $(i,j)$ to $(i',j')$ and $C^n$ to itself.*

**Proof.** For the beginning, let us consider the case $j = i' = i+1, j' = i$, or in other words, $(i,j) = (i,i+1),(i',j') = (i+1,i)$. Then, the inverse orderings $(n,\ldots,1)$ of the rows and columns of $C^n$ have the desired properties. Indeed, these permutations transform $(i,i+1)$ to $(n-i,n-i-1)$ for $1 \leq i < n$ and swap $(1,1)$ and $(n,n)$.

Let us remark that the transposition of $C^n$ also satisfies the required conditions; it swaps $(i,i+1)$ with $(i+1,i)$ for $1 \leq i < n$ and keeps $(1,1)$ and $(n,n)$.

Since permutations form a group, it remains to transform $(i,i+1)$ to $(1,1)$ keeping $C^n$.

The following (unique) permutations of the rows and columns solve the problem:

$$(f(i), f(i+2), f(i-2), f(i+4), f(i-4), \ldots);\ (f(i+1), f(i-1), f(i+3), f(i-3), f(i+5), \ldots),$$

where the first and second permutations correspond to the rows and columns, respectively, and the function $f : \mathbb{Z} \to [n]$ is defined by the formula:

$$f(j) = \begin{cases} j, & \text{if } \ j \in [n] = \{1,\ldots,n\}; \\ 2n+1-j, & \text{if } \ j > n; \\ 1-j, & \text{if } \ j < 1. \end{cases} \tag{5}$$

For example, the permutations $(3,5,1,7,2,6,4)$ and $(4,2,6,1,7,3,5)$ of, respectively, the rows and columns of $C^7$ transform $(3,4)$ to $(1,1)$ and keep $C^7$ itself. □

Now, we are ready to show that $C^n$ is a CSM-I, that is, for each active move of a player there is a winning reply of the opponent in Seki-I. By Lemma 6, without loss of generality, we can assume that $C$ plays at $(1,1)$. Let us show that $R$ wins naturally replying at $(1,2)$. Obviously, by this (s)he threatens to eliminate the first row in two moves: $(1,2)$ and $(1,1)$. Still, $C$ has a (unique) defense, playing at $(2,1)$. Now, against $R$'s $(1,2)$, $C$ will just repeat $(2,1)$ forcing a draw. Yet, $R$ has a stronger move, at $(2,3)$. Now $C$ cannot play at $(1,1)$,$(1,2)$,$(2,1)$, or $(2,3)$, since in response $R$ would win in one move. Also $C$ cannot pass, since $R$ will response at $(1,2)$ and, after $C$'s forced $(2,1)$, wins playing at $(1,3)$.
To prevent this plan $C$ has only two (equivalent) options: to play either at $(3,2)$ or at $(4,3)$ if $n > 3$, respectively, at $(3,3)$ if $n = 3$. Yet, if $C$ plays at $(3,2)$ then $R$ wins in two moves: first $(2,3)$ and then $(2,1)$; if $C$ plays at $(3,3)$ or $(4,3)$ then $R$ still wins by the same two moves but in the inverse order: first $(2,1)$ and then $(2,3)$. □

# 3 Miscellaneous observations

## 3.1 Semi-complete seki matrices

A *semi-complete seki matrix (semi-CSM)* is defined as a SM in which one player must pass (otherwise the opponent wins), while the other can make an active move (such that the reduced matrix is still a SM). Only two $3 \times 3$ semi-CSMs are known:

| $A^3_8$ | $A^3_9$ |
|---|---|
| 0 3 3 | 0 3 5 |
| 4 0 3 | 3 0 5 |
| 4 3 0 | 4 4 0 |

For example, $A^3_8$ player $C$ can reduce $A(1,2)$ from 3 to 2. If $R$ would pass then $C$ wins playing at $(3,2)$. Instead, $R$ should reduce $A(3,1)$ from 4 to 3 getting a (not complete) SM.

Unlike $C$, player $R$ must pass. Indeed, due to obvious symmetry, $R$ has only three distinct moves: $(3,2)$, $(2,1)$, and $(1,2)$. It is not difficult to verify that, in all three cases, $C$ wins answering, respectively, at $(1,2)$, $(1,2)$, and $(3,2)$ (or, less obviously, at $(2,1)$).

We leave the complete case analysis of $A^3_8$ and $A^3_9$ to the careful reader.

Of course, the transposed two matrices are semi-CSMs too. To show this it would suffice to swap the players $R$ and $C$. Computations show that there are no other semi-CSMs among the $3 \times 3$ matrices of height $h \leq 10$. Let us also note that neither $A^3_8$ nor $A^3_9$ is an IDSM.

In contrast, there are many $4 \times 4$ semi-CSMs; moreover, some of them are simultaneously IDSMs. For example, there are ten $4 \times 4$ IDSMs of height 3 that are semi-CSMs.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0023 | 0023 | 0033 | 0123 | 0123 | 0123 | 0133 | 0312 | 0233 | 1133 |
| 2300 | 2201 | 3300 | 2301 | 2310 | 2121 | 3301 | 2112 | 3302 | 3311 |
| 1121 | 2210 | 1122 | 2121 | 2112 | 2211 | 2122 | 2121 | 3131 | 2222 |
| 2111 | 1121 | 2211 | 2121 | 2121 | 2211 | 2221 | 2121 | 2222 | 2222 |

## 3.2 More examples of $4 \times 4$ CSMs of hight $3$

There are thirteen $4 \times 4$ IDSMs of height 3 in which $R$ always wins, being first or second:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0023 | 0113 | 0113 | 0113 | 0113 | 0123 | 0123 | 0123 | 0123 | 0123 | 0123 | 0222 | 0223 |
| 1310 | 1130 | 3110 | 1220 | 2120 | 0321 | 0213 | 0213 | 0321 | 2103 | 1230 | 0222 | 3220 |
| 2012 | 2102 | 0221 | 2012 | 2210 | 3012 | 2220 | 3120 | 3012 | 2220 | 2202 | 3111 | 2113 |
| 2210 | 2210 | 2111 | 2210 | 1112 | 3210 | 2220 | 3210 | 3210 | 2220 | 3111 | 3111 | 2221 |

However, somewhat surprisingly, there exist only three $4 \times 4$ IDSMs of height 3 in which

$R$ wins being first and the game is a draw when $R$ is second.

| | | |
|---|---|---|
| 0 1 3 3 | 0 2 2 3 | 0 2 3 3 |
| 3 1 0 3 | 2 2 3 0 | 3 3 1 1 |
| 2 3 2 0 | 2 1 1 3 | 3 1 2 2 |
| 2 2 2 1 | 3 2 1 1 | 2 2 2 2 |

In contrast, there are sixty $4 \times 4$ IDSMs of height 3 in which the first player wins.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0001 | 0023 | 0023 | 0023 | $\cdots$ | 0233 | 1133 | 1133 |
| 0010 | 0230 | 0320 | 2300 | $\cdots$ | 3212 | 1313 | 1322 |
| 0100 | 3101 | 3011 | 1130 | $\cdots$ | 3221 | 3221 | 3212 |
| 1000 | 2201 | 2201 | 2102 | $\cdots$ | 2222 | 3221 | 3221 |

On the other hand, there are also many $4 \times 4$ CSMs that are not IDSMs. In some of them sums ($s^r_i$ and $s^c_j$) may differ by 2. Five examples follow:

| | | | | |
|---|---|---|---|---|
| 3 0 0 2 | 0 0 3 3 | 0 0 3 3 | 0 0 3 3 | 0 0 3 3 |
| 0 0 3 3 | 0 3 1 1 | 3 3 0 0 | 3 3 0 0 | 0 3 0 3 |
| 0 3 2 1 | 3 1 1 2 | 1 3 1 3 | 0 3 1 3 | 2 2 2 0 |
| 2 3 1 1 | 3 1 2 1 | 3 1 3 1 | 3 1 3 0 | 3 1 1 1 |

## 3.3 Generating complete seki matrices recursively

### 3.3.1 $(3, 1, 1, -1)$- and $(3, 2, 2, -2)$-extensions

Let us recall four $3 \times 3$ CSMs $A^3_i; i = 3, 4, 5, 6$ and consider three $2 \times 2$ SMs $A^2_j, j = 1, 2, 3$:

| $A^2_1$ | $A^3_3$ | $A^2_2$ | $A^3_5$ | $A^3_4$ | $A^2_3$ | $A^3_6$ |
|---|---|---|---|---|---|---|
| | 3 0 1 | | 3 2 0 | 3 2 0 | | 3 2 0 |
| 2 2 | 0 2 2 | 2 3 | 2 0 3 | 2 1 2 | 3 3 | 2 1 3 |
| 2 2 | 1 2 1 | 3 2 | 0 3 2 | 0 2 3 | 3 3 | 0 3 3 |

It is easy to notice that the $3 \times 3$ matrices CSMs $A^3_3$, $A^3_4$, $A^3_5$ and $A^3_6$ can be viewed as extensions of the $2 \times 2$ SMs $A^2_1$, $A^2_2$, $(A^2_2)^T$, and $A^2_3$, respectively.

More precisely, given an $m \times n$ matrix $A : I \times J \to \mathbb{Z}_+$, we define two new $(m+1) \times (n+1)$ matrices $A'$ and $A''$ as follows. Let us add one new row $i_0$ to $I$ and one new column $j_0$ to $J$ and set $A'(i_0, j_0) = A''(i_0, j_0) = 3$. Then, let us choose an entry $(i^*, j^*)$ in $A$ such that $A(i^*, j^*) \geq 1$ (respectively, $A(i^*, j^*) \geq 2$) and reduce it by 1, that is, set $A'(i^*, j^*) = A(i^*, j^*) - 1$ (respectively, by 2, that is, set $A''(i^*, j^*) = A(i^*, j^*) - 2$). All other entries of $A$ remain the same, that is, $A(i, j) = A'(i, j) = A''(i, j)$ whenever $i \in I \setminus \{i^*\}$ or $j \in J \setminus \{j^*\}$. Finally, let us define $A'(i^*, j_0) = A'(i_0, j^*) = 1$ (respectively, $A''(i^*, j_0) = A''(i_0, j^*) = 2$) and $A'_{i,j_0} = A'_{i_0,j} = A''(i, j_0) = A''(i_0, j) = 0$ for all $i \in I \setminus \{i^*\}$ and $j \in J \setminus \{j^*\}$.

The obtained two matrices $A'$ and $A''$ will be called the $(3, 1, 1, -1)$- and $(3, 2, 2, -2)$-*extension* of $A$ at $(i^*, j^*)$, respectively.

For example, $A'' = A_5^3$ is the $(3,2,2,-2)$-extension of $A_2^2$ at $(1,1)$ and $A' = A_3^3$ is the $(3,1,1,-1)$-extension of $A_1^2$ at $(2,2)$. Let us notice that both last matrices are CSMs.

The $(3,1,1,-1)$-extension was suggested by Andrey Gol'berg in 1981. Being applied to a CSM, it frequently (but not always) results in another CSM. For example, computations show that *all* $(3,1,1,-1)$-extension of the ten $3\times 3$ CSMs, $A_i^3$, $i = 1,2,3,4,5$ and the following CSMs $B_j^3$, $j = 6,7,8,9,11$, result in CSMs.

| $A_1^3$ | $A_2^3$ | $A_3^3$ | $A_4^3$ | $A_5^3$ | $B_6^3$ | $B_7^3$ | $B_8^3$ | $B_9^3$ | $B_{11}^3$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 3 3 | 1 3 3 | 3 0 1 | 3 2 0 | 3 2 0 | 0 2 2 | 0 2 2 | 1 1 2 | 1 2 2 | 2 2 2 |
| 3 0 3 | 3 1 3 | 0 2 2 | 2 1 2 | 2 0 3 | 2 0 2 | 2 1 1 | 1 2 1 | 2 1 2 | 2 2 2 |
| 3 3 0 | 3 3 1 | 1 2 1 | 0 2 3 | 0 3 2 | 2 2 0 | 2 1 1 | 2 1 1 | 2 2 1 | 2 2 2 |

Furthermore, the $(3,1,1,-1)$-extensions of $A_6^3$ at $(i^*,j^*)$ are CSMs whenever $i^*+j^* \geq 4$, that is, at $(1,3)$, $(1,3)$, $(2,3)$, $(3,1)$, $(3,2)$, and $(2,2)$. However, at $(1,1)$ it is not a CSM (in the obtained matrix, both players, $R$ and $C$, win being first). Finally, the $(3,1,1,-1)$-extension of $A_6^3$ is not defined at $(1,2)$ and at $(2,1)$, since $A_6^3(1,2) = A_6^3(2,1) = 0$.

Similarly, the $(3,1,1,-1)$-extensions of $A_7^3$ at $(i^*,j^*)$ are CSMs whenever $\min(i^*,j^*) \geq 2$, that is, at $(2,2)$, $(2,3)$, $(3,2)$, and $(3,3)$. Yet, applied at $(1,2)$ and $(2,1)$ it results in semi-CSMs rather than in CSMs; moreover, at $(1,3),(3,1)$ we do not obtain CSMs either.

In [3] it was erroneously announced that $(3,1,1,-1)$-extensions of the CSMs are always CSMs; more precisely, if $A'$ is a $(3,1,1,-1)$-extension of $A$ at $(i^*,j^*)$ then $A'$ is a CSM whenever $s^r_{i^*} \geq 4$, $s^c_{j^*} \geq 4$ and $A$ is a CSM.

The above analysis of the CSMs $A_6^3$ and $A_7^3$ shows that this claim was an overstatement. Yet, among all $2\times 2$ and $3\times 3$ CSMs there are only the above five counterexamples: $(3,1,1,-1)$-extension of $A_7^3$ at $(1,2)$, $(1,3)$, $(2,1)$, $(3,1)$ and of $A_6^3$ at $(1,1)$ are not CSMs.

Let us notice that the required conditions $s^r_{i^*} \geq 4$, $s^c_{j^*} \geq 4$ are absolutely necessary. Indeed, if $A$ is a CSM and $s^r_{i^*} \leq 3$ (respectively, $s^c_{j^*} \leq 3$) then, being first, $R$ (respectively $C$) wins in $A'$ in at most 3 moves.

Now, let us proceed with the $(3,2,2,-2)$-extensions. As we already mentioned, all $(3,2,2,-2)$-extensions of the (not complete $2\times 2$ SMs) $A_2^2$ and $A_3^2$ are CSMs. The same holds for *all* $(3,2,2,-2)$-extensions of the following seven $3\times 3$ matrices:

| $A_1^3$ | $A_2^3$ | $A_4^3$ | $A_5^3$ | $B_9^3$ | $B_{11}^3$ | $B_{12}^3$ |
|---|---|---|---|---|---|---|
| 0 3 3 | 1 3 3 | 3 2 0 | 3 2 0 | 1 2 2 | 2 2 2 | 3 1 1 |
| 3 0 3 | 3 1 3 | 2 1 2 | 2 0 3 | 2 1 2 | 2 2 2 | 1 2 2 |
| 3 3 0 | 3 3 1 | 0 2 3 | 0 3 2 | 2 2 1 | 2 2 2 | 1 2 2 |

Let us notice that the first six of them are CSMs and consider the next nine $3\times 3$ matrices:

| $D_1^3$ | $D_2^3$ | $D_3^3$ | $D_4^3$ | $D_5^3$ | $D_6^3$ | $D_7^3$ | $D_8^3$ | $D_9^3$ |
|---|---|---|---|---|---|---|---|---|
| 2 0 0 | 2 0 0 | 3 0 1 | 3 0 1 | 3 0 1 | 3 0 2 | 3 0 1 | 3 0 2 | 3 1 2 |
| 0 2 3 | 0 3 3 | 0 3 2 | 0 3 3 | 0 2 3 | 0 3 3 | 0 3 1 | 0 3 2 | 1 2 3 |
| 0 3 2 | 0 3 3 | 1 2 2 | 1 3 2 | 1 3 1 | 2 3 1 | 1 1 3 | 2 2 2 | 2 3 1 |

For the first four of them, $D^3_i$, $i = 1, 2, 3, 4$, the $(3, 2, 2, -2)$-extensions at $(i^*, j^*)$ are CSMs whenever $\min(i^*, j^*) \geq 2$, yet, the first player wins for $(i^*, j^*) = (1, 1)$ and the extension is not defined at $(1, 2)$, $(1, 3)$, $(2, 1)$, $(3, 1)$. The same holds for $D^3_5$, except, the extension is not defined at $(3, 3)$ either. The same holds for $D^3_6$, except, the first player wins when $(i^*, j^*)$ is $(1, 3)$, $(3, 1)$ or $(1, 1)$. For $C^3_7$ the extension is a CSM at $(3, 3)$, the first player wins when $(i^*, j^*)$ is $(1, 1)$ or $(2, 2)$, and the extension is not defined at the remaining entries. For $C^3_8$, the extension at $(3, 3)$ is a CSM, the first player wins when $(i^*, j^*)$ is $(1, 1)$ or $(2, 2)$, furthermore, the extension at $(1, 3)$, $(2, 3)$, $(3, 1)$, $(3, 2)$ is a not complete SM, and it is not defined at $(1, 2)$ and $(2, 1)$. Finally, for $C^3_9$, the extension at $(2, 3)$, $(3, 2)$, and $(1, 1)$ a CSM, it is a not complete SM at $(1, 3)$, $(3, 1)$, and $(2, 2)$, and not defined at $(1, 2)$, $(2, 1)$, and $(3, 3)$.

Thus, $(3, 2, 2, -2)$-extensions respect CSMs very frequently but not always.

### 3.3.2 Two infinite recursive sequences of CSMs

Let us apply the $(3, 1, 1, -1)$- and $(3, 2, 2, -2)$)-extensions at $(1, 1)$ recursively beginning, respectively, with the $2 \times 2$ matrices $A^2 = A^2_1$ and $B^2 = A^2_2$. We obtain the following two infinite matrix sequences $A^n, B^n$ for $n = 2, 3, \ldots$

$$
\begin{matrix}22\\22\end{matrix}\quad
\begin{matrix}310\\112\\022\end{matrix}\quad
\begin{matrix}3100\\1210\\0112\\0022\end{matrix}\quad
\begin{matrix}31000\\12100\\01210\\00112\\00022\end{matrix}\quad
\begin{matrix}310000\\121000\\012100\\001210\\000112\\000022\end{matrix}\ \ldots\qquad
\begin{matrix}32\\23\end{matrix}\quad
\begin{matrix}320\\212\\023\end{matrix}\quad
\begin{matrix}3200\\2120\\0212\\0023\end{matrix}\quad
\begin{matrix}32000\\21200\\02120\\00212\\00023\end{matrix}\ \ldots
$$

**Proposition 4** *Except for $B^2 = A^2_2$, both sequences contain only CSMs.*

**Proof.** By construction, these two sequences contain only IDSMs with the sums $s(A^n) = 4$ and $s(B^n) = 5$, for all $n \geq 2$. Moreover, except for $A(1, 1) = 3$ and $B(1, 1) = B(n, n) = 3$, all other entries take only values $0, 1$, and $2$. Thus, conditions of Theorem 1 and Lemma 2 "almost" hold. Moreover, by the same arguments we can show that after "almost" every active move of a player the opponent wins in 3 (respectively, in 4 ) moves. However, there are two important exceptions when the opponent cannot win immediately; instead (s)he can reduce $A^n$ (respectively, $B^n$) to the $(n - 1) \times (n - 1)$ matrix from the same sequence, in which one entry is also reduced by 1. Thus, we can finish the proof by induction on $n$, since in both sequences the $3 \times 3$ matrices are known CSMs: $A^3 = A^3_3$ and $B^3 = A^3_4$, respectively.

**Case 1**. If a player, say $R$, begins at $(1, 1)$ then $C$ answers at $(2, 1)$. It is easily seen that $R$ must play at $(1, 2)$. (In case of the second sequence, $C$ will repeat by playing at $(2, 1)$ once more and again $R$ must answer at $(1, 2)$. As a result of this exchange of moves, the the original $n \times n$ matrix, $A^n$ or $B^n$, is reduced to the direct sum of two matrices: a $1 \times 1$ CSM with the entry 2 and the $(n - 1) \times (n - 1)$ matrix from the same sequence in which the first entry is 2 rather than 3. Since $C$ has to move in the obtained game, (s)he can enforce the same reduction again, etc.

**Case 2**. If a player, say $R$, begins at $(2, 1)$ then $C$ answers at $(1, 1)$, threatening to eliminate the first column by the next 2 (respectively, 3) moves. The only defense of $R$ is to play at $(1, 2)$. (In case of the second sequence, $C$ will proceed by eliminating of $(2, 1)$ and $R$ must eliminate $(1, 2)$ in return). The above exchange of moves results in exactly the same reduction as in Case 1. Thus, $C$ wins again.

Let us notice that the matrices of the second sequence are double-symmetric, that is, $B(i, j) = B(j, i)$ and $B(i, j) = B(n-i+1, n-j+1)$ for all $n \geq 3$ and $i, j \in [n] = \{1, \dots, n\}$. Hence, we can substitute $(1, 2)$,$(2, 1)$, and $(1, 1)$ by $(n-1, n)$,$(n, n-1)$, and $(n, n)$, respectively.

It remains to verify that for any other move of a player, the opponent can win in 3 (respectively, in 4) moves by applying the strategy suggested in the proof of Lemma 2. $\square$

**Corollary 2** $H(n) \geq 3$ *whenever* $n \geq 3$

**Proof.** For every $n \geq 2$, each of the above two sequences contains a (unique) $n \times n$ matrix, which is a CSM of hight 3 when $n \geq 3$. $\square$

# 4 Go and Bridge versus Seki and Whistette

This short last section is addressed to the readers familiar with (at least the rules of) the games Go and/or Bridge. I am glad to thank all others for their attension.

## 4.1 Game Seki and shared life positions in Go

Given an $m \times n$ matrix $A : I \times J \to \mathbb{Z}_+$, let us consider a position in Go with $m$ White and $n$ Black groups that are indexed by $I$ and $J$, respectively, and let $A(i, j)$ be the number of common free points (so-called liberties or dame) between the White group $i \in I$ and Black group $j \in J$. Thus, players $R$ and $C$ in Seki correspond to Black and White in Go. Indeed, $R$ (respectively, $C$) wants to delete all positive entries of a row (respectively, of a column); accordingly, Black (respectively, White) wants to surround completely a White (respectively, Black) group. Several examples are given in Figures 1-4.

Let us remark that the rules of Go correspond only to Seki but not to Seki-I.

We refer the reader to the Appendix of [1] for more details and examples illustrating the relations between Seki and shared life positions of Go.

## 4.2 Games Single-Suit and Bridge

The relation between Go and Seki is somewhat similar to the relation between Bridge and the Single-suit two-person game [4]. This game was introduced in 1929 by Emanuel Lasker who called it Whistette. It is defined as follows: $2n$ cards with the numbers $1, \dots, 2n$ are shuffled and divided into two hands $A$ and $B$ of $n$ cards each, held by players $R$ and $C$. Let us say, $C$ begins. He chooses one of his cards and put in on the table. Then $R$, after seeing

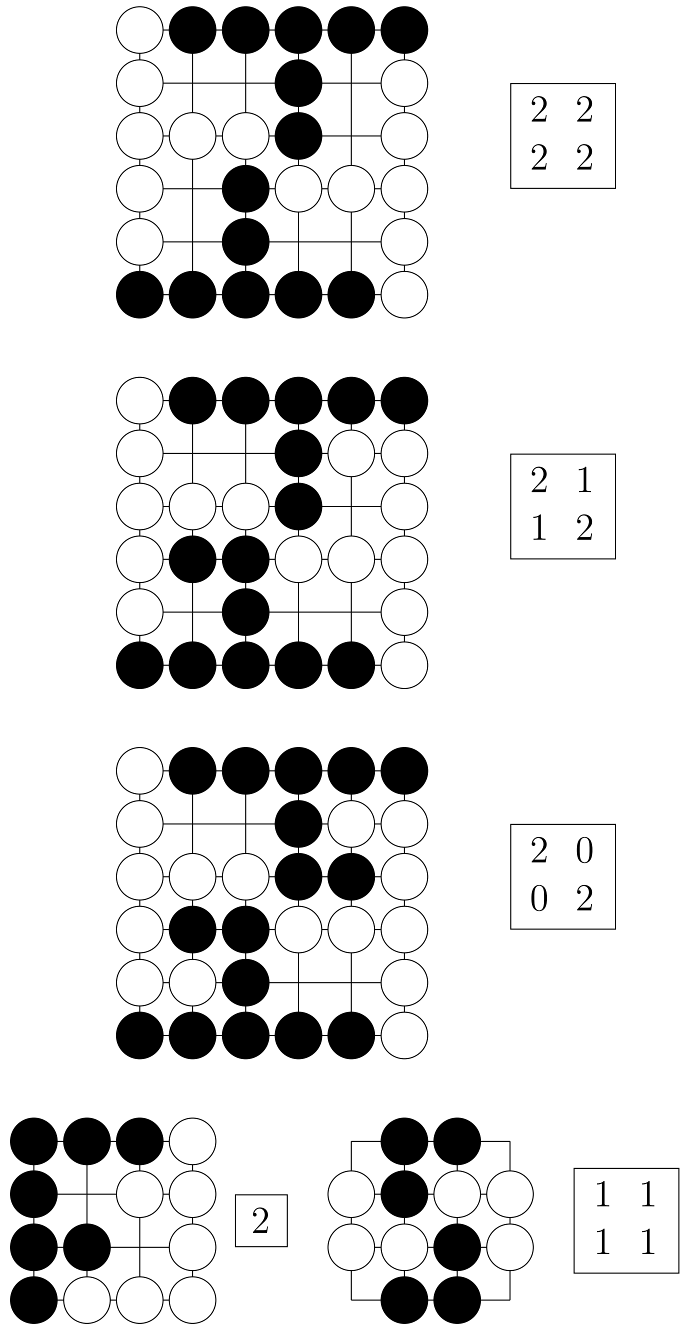
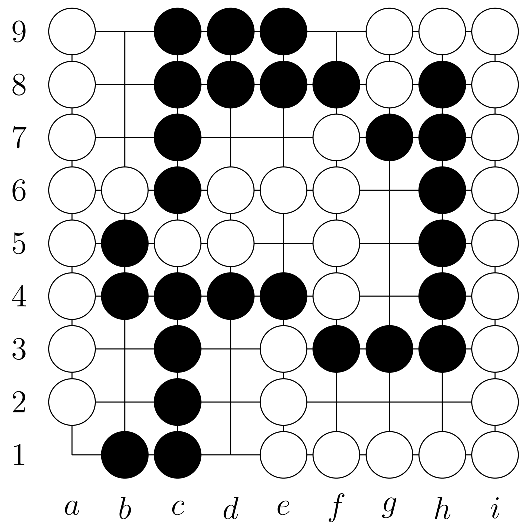



Figure 1: Five standard complete seki positions in GO and the corresponding $1 \times 1$ and four $2 \times 2$ complete seki matrices.



$$\boxed{\begin{matrix} 0 & 3 & 3 \\ 3 & 3 & 1 \\ 3 & 1 & 2 \end{matrix}}$$

Figure 2: A complete seki position corrersponding to the matrix $A_7^3$.

this card, selects one of her own. The player with the higher card wins the trick and gets the lead. Two cards of the trick are removed and play continues until there are no more cards. The goal of each player is to win as many tricks as possible.

The game is not trivial already for $n = 3$. For example, let $C$ and $R$ get $6, 4, 2$ and $5, 3, 1$ (or in Bridge terms, A,Q,10 and K,J,9), respectively. If $C$ leads with 6 then he will win only the first trick, yet, if $C$ leads with 2 or 4 then he will get two tricks. For another example, let $C$ and $R$ get $5, 4, 2$ and $6, 3, 1$ (or in Bridge terms, K,Q,10 and A,J,9), respectively. If $C$ leads with 5 or 4 and $R$ wins with 6 then she will get no more tricks, yet, if $R$ discards 1 in the first trick then she will win the remaining two tricks. For larger $n$, say $n = 20$, Whistette becomes complicated. Many interesting results are obtained in [4]. Recently, Johan Wästlund [6] got a polynomial algorithm solving Whistette.

In some situations, Bridge is reduced to Whistette. Let us consider, for example, the six-card end-play in which North and West have $3, 6, 10, J, Q, A$ and $4, 5, 7, 8, 9, K$, respectively, one of them is on lead, and there are no trumps left. However, such situations are very rare. For example, a positions with $n > 6$ cannot appear at all, since there are only 13 cards in each suit. Thus, it is not necessary for a Bridge expert to be good at Whistette.

Similarly, the games Go and Seki demand very different skills. Unlike Bridge and Go, Whistette and Seki are pretty boring games, yet, they reveal deeper and nicer mathematical properties and they are sufficiently complicated too. At least, there are many positions that would be difficult to analyze even for advanced Bridge and Go players.

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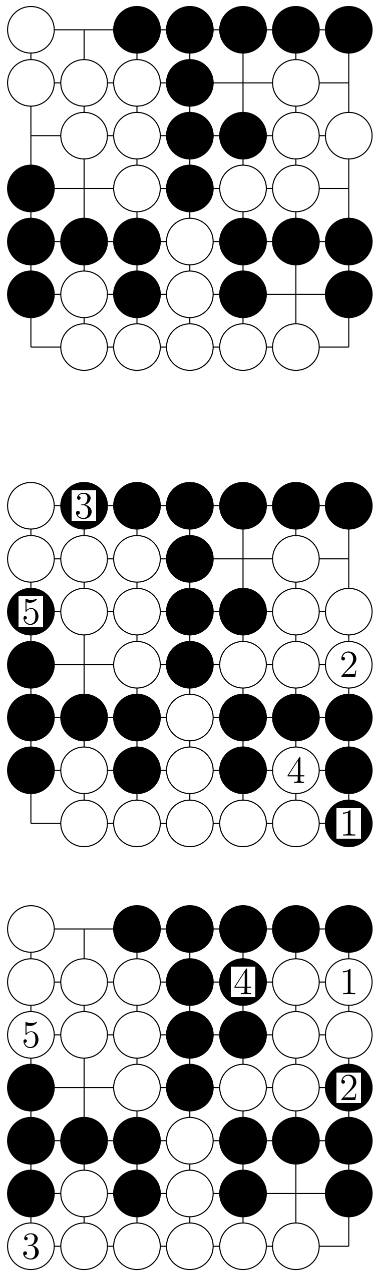
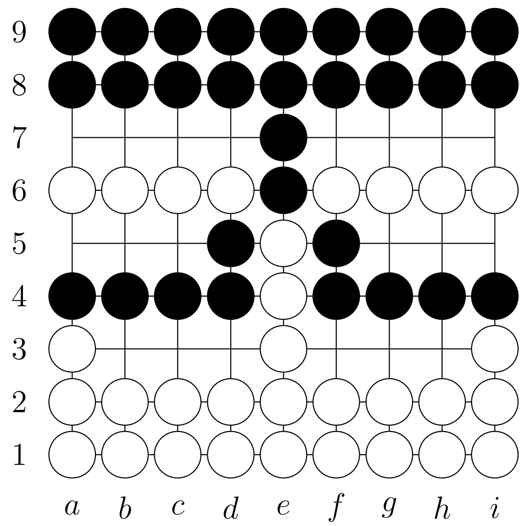



| | | |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 2 | 0 |
| 2 | 0 | 1 |

Figure 3: A non-seki position corresponding to a CSM.




| | | |
|---|---|---|
| 0 | 3 | 3 |
| 4 | 0 | 3 |
| 4 | 3 | 0 |

Figure 4: A semi-complete seki position corresponding to $A_8^3$.

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