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Abstract. An impartial game is called (strongly) miserable if its normal and misère versions differ just slightly; more precisely, if some (all) zeros and ones of the Sprague-Grundy (SG) functions swap, while all larger values remain the same.
We obtain necessary and sufficient conditions for (strong) miserability and show that NIM, Wythoff's NIM, and game Euclid are miserable but not strongly miserable, while all subtraction games and the Fraenkel extension NIM(a) of Wythoff's NIM(1) for $a > 1$ are strongly miserable.
We also show that the sum of (strongly) miserable games is miserable but not necessarily strongly miserable.
SG theory enables us to solve efficiently the sum of games with known SG functions. If all summands are miserable, one can replace some of them by the corresponding misère versions, compute the modified SG functions making use of the above criteria, and solve the sum of the obtained games.

Keywords: combinatorial games, impartial games, standard and misère versions, Sprague-Grundy function, NIM, Wythoff's NIM, Fraenkel's NIM, game Euclid, game Mark, subtraction games, swap positions.

1 Summary

Let us briefly summarize our main results. We assume that readers are familiar with basic definitions of combinatorial game theory. Some of them will be recalled in Section 2. We also recall in Section 4 the definitions and main properties of several popular games, which are used here as examples, namely, NIM, Euclid, Wythoff, and Fraenkel's games.

We study the Sprague-Grundy (SG) functions g^N and g^M of the normal and misère versions G^N and G^M of impartial games. Such a game is modeled by an acyclic directed graph (digraph) $G = (V, E)$. It may be infinite but the set of successors of any position $v \in V$ should be finite; in particular, the set of terminal positions V_T is not empty.

Furthermore, let $V_i^L = \{v|\ g^L(v) = i\}$ denote the set of positions with the SG value i, where symbol L is either N or M and $i \in \mathbb{Z}_{\geq 0} = \{0, 1, \dots\}$. Given two subsets of positions $V', V'' \subseteq V$, we say that V'' is *reachable* from V' if for every $v' \in V'$ there is a move $e = (v', v'') \in E$ such that $v'' \in V''$. For example, V_i^L is reachable from V_j^L whenever $i < j$, by the definition of the SG functions g^L; in particular, V_0^L is reachable from V_1^L.

Game G is called *miserable* if g^N and g^M "coincide" up to the swap of some positions of V_0^N and V_1^N; more precisely, if there are subsets $V_0' \subseteq V_0^N$ and $V_1' \subseteq V_1^N$ such that $g^N(v) + g^M(v) = 1$ for $v \in V' = V_0' \cup V_1'$ and $g^N(v) = g^M(v)$ otherwise; or in other words, $g(v) = (0, 1)$ for $v \in V_0'$, $g(v) = (1, 0)$ for $v \in V_1'$, and $g(v) = (g^N(v), g^M(v)) = (k, k),\ k \geq 0$, for $v \notin V'$. The positions from V_0', V_1', and V' will be called *0-swap, 1-swap*, and swap positions, respectively.

A concept very similar to miserability (but not exactly the same) was introduced in [24], where the following sufficient conditions were obtained.

Theorem 1 *([24]). A game is miserable if there exist $V_0' \subseteq V_0^N$ and $V_1' \subseteq V_1^N$ such that*

- *(i0) V_0' is reachable from V_1';*
- *(i1) V_1' is reachable from $V_0' \setminus V_T$;*
- *(i2) from any $v \notin V' = V_0' \cup V_1'$ either each set V_0' and V_1' or none of them is reachable.*

Moreover, then $g(v) = (0, 1)$ if and only if $v \in V_0'$ and $g(v) = (1, 0)$ if and only if $v \in V_1'$.

Conversely, the last two statements, (i0) and (i1) (but not necessarily (i2)) hold for each miserable game.

Thus, the above three conditions are sufficient but not necessary for miserability.

A miserable game is uniquely defined by the subsets $V_0' \subseteq V_0^N$ and $V_0' \subseteq V_0^N$. In their turn, these subsets are uniquely defined as the sets of 0- and 1-swap positions of the game.

Let us also note that the induced digraph $G[V_0' \cup V_1']$ is bipartite (and acyclic).

Furthermore, $V_T \subseteq V_0'$ holds for all (not only for miserable) games, just by definitions.

See Remark 1 of Section 3 for more details.

Finally, if we set $V_0' = V_0^N$ and $V_1' = V_1^N$ then conditions (i0) and (i2) will automatically hold , yet, (i1) may fail. Requiring (i1), in this case, we obtain a stronger concept.

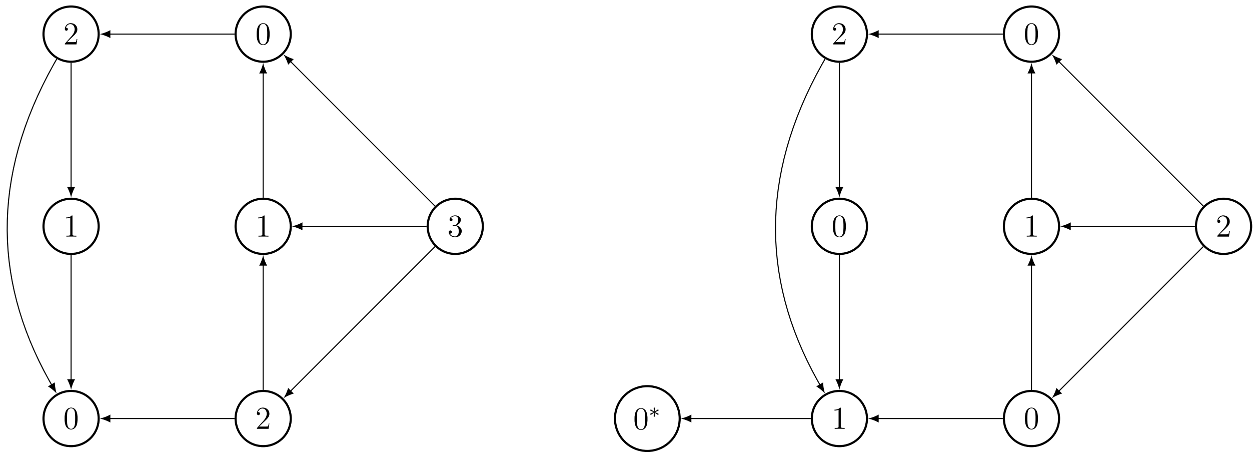

Figure 1: This game is not miserable: $g(v) = (g^N(v), g^M(v))$ takes values $(2, 0)$ and $(3, 2)$. Let us note, that conditions of Theorem 2 as well as (i2) of Theorem 1 fail. Indeed, from the $(2, 0)$-position there is a move to the $(0, 1)$- but not to the $(1, 0)$-position; also from the $(0, 0)$-position there is a unique move, which leads to the $(2, 2)$-position.

A game will be called *strongly miserable* if $V'_0 = V^N_0$ and $V'_1 = V^N_1$, that is, when all 0- and 1-values of the SG function are swaped. In this case, the above criterion can be simplified. Indeed, as we know, V^N_i is reachable from V^N_j whenever $i < j$. In particular, both V^N_0 and V^N_1 are reachable from any V^N_j with $j > 1$ and V^N_0 is reachable from V^N_1. Thus, it remains to verify only that V^N_1 is reachable from $V^N_0 \setminus V_T$. We will extend this observation as follows.

Theorem 2 *The next five properties are equivalent:*

- *(i) game G is strongly miserable;*
- *(i') the pairs $g(v) = (g^N(v), g^M(v))$ take only values $(0, 1), (1, 0)$, and (k, k) for $k \geq 2$;*
- *(ii) $V^N_0 \cap V^M_0 = \emptyset$, that is, $g(v) \neq (0, 0)$;*
- *(iii) V^N_1 is reachable from $V^N_0 \setminus V_T$;*
- *(iii') V^M_1 is reachable from V^M_0.*

Property (iii) was introduced by Ferguson in [14], where he proved that it holds for all subtraction games; see Section 4.7 for more details.

It is also clear that the 0- and 1-swap sets V'_0 and V'_1 are unique when exist.

Hence, to demonstrate that a miserable game is not strongly miserable it is sufficient to show a position $v \in (V^N_0 \setminus V'_0) \cup (V^N_1 \setminus V'_1) = (V^N_0 \cup V^N_1) \setminus V'$.

Another way is to show a 0-swap position from which there is no move to a 1-swap one.

An example of a non-misearable game is shown in Figure 1. It is easy to verify that all properties of Theorem 2 and (i2) of Theorem 1 fail. Indeed, from the $(2,0)$-position there is a move to the $(0,1)$- but not to the $(1,0)$-position; also from the $(0,0)$-position there is a unique move, which leads to the $(2,2)$-position.

We will consider several important examples. The classical game of NIM is miserable. This was demonstrated already in 1901 by Bouton at the very end of his seminal paper [7].

The positions of NIM are non-negative integer n-vectors $V = \{(k_1, \dots, k_n) \mid k_i \in \mathbb{Z}_{\geq 0}\}$. The set of swap positions $V' = \{(k_1, \dots, k_n) \mid k_i \leq 1\}$ consists of 2^n (0,1)-vectors. It is partitioned into 0- and 1-swap positions, $V' = V'_0 \cup V'_1$, that have even and odd sum of coordinates, respectively. It is easy to verify that all conditions of Theorem 1 hold. Yet, NIM is not strongly miserable already for $n = 2$. Indeed, both containments $V'_0 \subset V_0^N$ and $V'_0 \subset V_0^N$ are strict; furthermore $g^M(2,2) = g^N(2,2) = 0$, while $g^N(2,1) = 3$ and $g^N(2,0) = 2$; hence, conditions (ii) and (iii) of Theorem 2 fail.

The n-pile NIM is the sum of one-pile NIMs and above arguments can be naturally extended to the sum of arbitrary impartial games as follows.

Theorem 3 *The sum of miserable games is miserable.*

Let us notice that the sum of strongly miserable games is miserable, by Theorem 3, but not necessarily strongly miserable, as the example of NIM shows. Indeed, the one-pile NIM is strongly miserable: $V_0^N = V_1^M = \{0\}$, $V_1^N = V_0^M = \{1\}$, while $V_i^N = V_i^M = \{i\}$ for all $i > 1$. Yet, the n-pile is miserable but it is not strongly miserable when $n > 1$.

There are several interesting extensions of the two-pile NIM. In Section 4.3 we will show that game NIM(a) introduced by Fraenkel [16, 17] is strongly miserable when $a > 1$. (We will verify condition (ii).) Yet, in case $a = 1$ Fraenkel's game turns into the famous Wythoff [39] game, which is miserable but not strongly miserable. The corresponding sets 0- and 1-swap positions are finite: $V'_0 = \{(0,0),(1,2),(1,2)\}$ and $V'_1 = \{(0,1),(1,0),(2,2)\}$.

Yet, Wythooff's game is not strongly miserable. Both containments $V'_0 \subset V_0^N$ and $V'_0 \subset V_0^N$ are strict. Furthermore $g^M(3,5) = g^N(3,5) = 0$, while all moves from $(3,5)$ lead to positions whose SG values are at least 2. Hence, condition (ii) and (iii) fail.

In Section 4.4, we recall a two-parametrical extension NIM(a,b) recently introduced in [25] and get similar results: for all $b \in \mathbb{Z}_{>0}$, game NIM(a,b) is strongly miserable when $a > 1$ and it is miserable but not strongly when $a = 1$. In the latter case

$V'_0 = \{(0,0),(b,b+1),(b+1,b)\}$ and $V'_1 = \{(0,1),(1,0),(b+1,b+1)\}$.

It was shown in [24] that game Euclid [8] is miserable. Positions of this game are positive integer pairs (x,y). Two players move alternately. By one move a player can subtract any positive multiple of the smaller number from the larger one, provided the difference is still positive. Obviously, the game terminates in the position (z,z), where $z = gcd(x,y)$.

For game Euclid, the set of swap positions is infinite and defined by the Fibonacci pairs: $V' = \{(F_i, F_{i+1}) \mid i \in \mathbb{Z}_{\geq 0}\}$, it is partitioned by the 0- and 1-swap positions, $V' = V'_0 \cup V'_1$, that correspond to even and odd i, respectively.

We should remark that all positions $\{(\ell x, \ell y), (\ell y, \ell x) \mid \ell = 1, 2, \dots\}$ are equivalent.

Furthermore, game Euclid is not strongly miserable. Both containments $V_0' \subset V_0^N$ and $V_0' \subset V_0^N$ are strict. Furthermore, from $(3, 4)$ there is only one move, which leads to $(3, 1)$. Yet, $g^M(3, 4) = g^N(3, 4) = 0$, while $g^N(3, 1) = 2$. Hence, condition (ii) and (iii) fail.

SG theory enables us to solve efficiently the sum of games with known SG functions. If all summands are miserable, one can replace some of them by their misère versions, compute the modified SG functions making use of the above criteria, and solve the obtained game.

Misère impartial games were considered many papers [1, 2, 4, 11, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 33, 34, 36, 40] beginning with the pioneer Bouton's article [7].

2 Sprague-Grundy theory of impartial games

The first examples of the impartial games (and their solutions) appeared in the papers by Bouton [7] in 1901 and Wythoff [39] in 1907. The general theory was discovered independently by Sprague [37, 38] in 1935 and Grundy [21] in 1939.

2.1 Minimum excludant functions mex and mex_b

The *minimum excludant* function $mex(S)$ is defined for any subset $S \subset \mathbb{Z}_{\geq 0}$ of the non-negative integers as the minimum $z \in \mathbb{Z}_{\geq 0}$ such that $z \notin S$; in particular, $mex(\emptyset) = 0$.

We will also need the following generalization suggested in [25]. Given an integer $b \geq 1$ and a finite set $S \subset \mathbb{Z}_{\geq 0}$ of m non-negative integers $0 = s_0 < s_1 < \dots < s_{m-1}$, we define $mex_b(S) = s_i + b$, where s_i is the smallest number in S such that $s_{i+1} - s_i > b$. By convention, we assume that $s_m = \infty$ and $mex_b\{\emptyset\} = 0$. Obviously, $mex_1 = mex$.

2.2 Impartial games, normal and misère versions

An *impartial game* is a two-person game in which both players have perfect information, there are no moves of chance, and the allowed moves are the same for both players. Such a game is modeled by a directed acyclic graph $G = (V, E)$, each vertex $v \in V$ of which is a position and each arc $e = (v, v') \in E$ is a move of the corresponding game. As we already mentioned, G may be infinite but the set of successors of any position $v \in V$ should be finite; in particular, the set of the terminal positions V_T is not empty.

We assume that a token is initially placed in a vertex $v_0 \in V$, and two players take turns moving the token from a current vertex v to one of its immediate successors v' (that is, $(v, v') \in E$). The game ends when the token reaches V_T.

The player who has then to move (but cannot) is claimed the winner in the misère version of the game G^M and looser in the normal version G^N. The goal of this paper is a joint analysis of these two versions.

2.3 Sprague-Grundy function

Given a digraph $G = (V, E)$, the Sprague-Grundy (SG) function $g^N : V \to \mathbb{Z}_{\geq 0}$ is defined recursively by the formula $g^N(v) = mex\{g(w) | (v, w) \in E\}$, that is, $g^N(v)$ takes the minimum value not taken by any immediate successor of v. This recursion starts in the terminal positions, all of which receive the value 0, that is, $g^N(v) = 0$ for all $v \in V_T$.

Lemma 1 *([37, 21]). Set V_i^N is reachable from V_j^N whenever $i < j$, yet, $g(v) \neq g(v')$ for any move $(v, v') \in E$.*

Proof: It follows immediately from the definition of the minimum excludant. □

The misère SG function g^M is defined by the same formula as g^N but the initialization is different: by convention $g^M(v) = 1$ (while $g^N(v) = 0$) for all terminal positions $v \in V_T$. Let us modify the original digraph $G = G^N$ adding one new vertex v^* and the edge (v, v^*) for every $v \in V_T$ and denote the obtained digraph G^M; see Figure 1 as an example.

Then, obviously, the misère SG function of G^N is the normal SG function of G^M, that is, $g^M(G^N) = g^N(G^M)$. In other words, the normal-misère swap is an involution: $g^{MM} = g^N$.

For example, digraph G in Figure 1 is not miserable. It is easy to see that the SG functions $g^M(G)$ and $g^N(G)$ differ a lot. For the (strongly) miserable games relations between them are summarized by Theorems 2 and 1.

2.4 P-positions and kernels

The zeros of the SG function g^N are called the *P-positions*.

Lemma 2 *The subset $P \subseteq V$ of all P-positions of a digraph $G = (V, E)$ is*

- *(p)* independent*: $(v, v') \in E$ for no $v, v' \in P$, and*
- *(pp)* absorbing*: for any $v \notin P$ there is a move $(v, v') \in E$ such that $v' \in P$.*

Proof: It follows immediately from Lemma 1. □

Properties (p) and (pp) show that a player cannot stay in P and, being out of P, can always enter it. Hence, a player who enters P (the *Previous* one) wins the normal version of the game. Indeed, (s)he can always re-enter P, while the opponent must always leave it. The play will terminate in V_T in a finite number of moves, since we assume that any vertex has a finite number of successors. Furthermore, $V_T \subseteq P$, by (pp).

In graph theory, an independent and absorbing vertex-subset $P \subseteq V$ of a digraph $G = (V, E)$ is called a *kernel*. This concept was introduced in 1944 by von Neumann and Morgenstern in their seminal book [30], where it was shown that an acyclic digraph has a unique kernel, which can be found by the following recursive algorithm.

Set $V_0^1 = V_T$ (obviously $V_0^1 \subseteq P$). Let V_1^1 denote the set of all vertices from which V_0^1 is reachable (obviously, $V_1^1 \subseteq V \setminus P$). Delete $V_0^1 \cup V_1^1$ from V and repeat, etc., thus, getting sets V_0^i and V_1^i in every step $i = 1, 2, \ldots$ Then, $P = \cup_{i=1}^{\infty} V_0^i$, while $N = V \setminus P = \cup_{i=1}^{\infty} V_1^i$. The latter set is referred to as the set of N-positions (in which the Next player wins).

2.5 Sum of games

Given n (vertex-disjoint) digraphs $G_i = (V_i, E_i),\ i \in [n] = \{1, \ldots, n\}$, their sum $G = G_1 \oplus \ldots \oplus G_n = (V, E)$ is defined by the vertex-set $V = V_1 \times \ldots \times V_n = \{(v_1, \ldots, v_n) \mid v_i \in V_i, i \in [n]\}$ and the set E of directed edges that contains all pairs (v, v') such that $v = (v_1, \ldots, v_n)$ and $v' = (v'_1, \ldots, v'_n)$ differ by exactly one coordinate $i \in [n]$ and $(v_i, v'_i) \in E_i$. In other words, there are n impartial games and a token in each. In a current position $v = (v_1, \ldots, v_n)$, a player chooses an arbitrary game $i \in [n]$ and move $(v_i, v'_i) \in E_i$ in it. Two player take turns. The play begins in a given initial position $v^0 = (v^0_1, \ldots, v^0_n) \in V$ and (in a finite number of moves) ends in a terminal position $v^T = (v^T_1, \ldots, v^T_n) \in V$, where v^T_i is a terminal position of V_i for all $i \in [n]$. The normal and misère versions are defined standardly.

The fundamental property of the SG function is given by the equality

$$g^N(v) = g^N(v_1) \oplus \ldots \oplus g^N(v_n), \text{ for all } v = (v_1, \ldots, v_n), \ \ v_i \in V_i, ; i \in [n], \tag{1}$$

where $\oplus$ is the so-called *bitwise XOR (or NIM) sum*: we take the binary representations of $g^N(v_i)$ for all $i \in [n]$ and take their bitwise binary sum. For example,

$3 \oplus 5 = 011_2 + 101_2 = 110_2 = 6$, Similarly, $5 \oplus 6 = 3$, $6 \oplus 3 = 5$, and $3 \oplus 5 \oplus 6 = 0$.

Thus, one can get $g^N(v_1, \ldots, v_n)$ just computing $g^N(v_i)$ for all $i \in [n]$. Recall that the zeros of g^N are exactly the P-positions of the sum. Let us also remark that getting the sets P_i of all P-positions for all $i \in [n]$ is not sufficient for computing the P-positions of the sum, since $v \in P$ and $v \notin P$ are both possible for $v = (v_1, v_2)$ when $v_1 \notin P_1$ and $v_2 \notin P_2$.

Finally, let us note that some summands may be in the normal, while some other in the misère version. One can bring them all to the normal version just making use of the transformation in Figure 1 and formula $g^M(G^N) = g^N(G^M)$.

3 Proofs of the main theorems

Proof of Theorem 1. Although, it has already appeared in [24], we will repeat the proof here, because it is very short and requires minor corrections.

Let us recall that, by the definition, a game is miserable if the pairs $g(v) = (g^N(v), g^M(v))$ take values $(0, 1)$ (when $v \in V'_0$, in particular, for $v \in V_T$)), $(1, 0)$ (when $v \in V'_1$), and equal values for all other positions $v \notin V' = V'_0 \cup V'_1$; in other words, if $g^N(v) = g^M(v)$ whenever one of these two values is > 1. (Note that $(0, 0)$- and $(1, 1)$-positions may exist too).

The "if" statements follow immediately from the definitions. Indeed, if $g(v) = (0, 1)$ (respectively, $(1, 0)$) then there is a move (v, w) such that $g(w) = (k, 0)$ (respectively, $(0, k)$). In both cases, $k \neq 0$ and $k < 2$; hence $k = 1$. □

The "only if" statements will be proven by induction. From (i0), (i1) and (i2) we will derive simultaneously that $g(v) = (0, 1)$ for each $v \in V'_0$, $g(v) = (1, 0)$ for each $v \in V'_1$, and $g(v) = (k, k)$ for each $v \notin V' = V'_0 \cup V'_1$, for some $k \geq 0$.

Indeed, by the definition, $g(v) = (0, 1)$ for any terminal position $v \in V_T$.

Let us assume that for $g(w) = (\ell', \ell'')$ either $\ell' = \ell''$, or $(\ell', \ell'') = (0, 1)$ (if and only if $v \in V_0'$), or $(\ell', \ell'') = (1, 0)$ (if and only if $v \in V_1'$).

Then, let us consider a position $v \in V \setminus W$ each immediate successor of which is in W, that is, $w \in W$ for every move (v, w). It is easy to verify that conditions of Theorem 1 imply the same property for the pair $g(v) = (k', k'')$, that is, either $k' = k''$, or $(k', k'') = (0, 1)$ (if and only if $v \in V_0'$), or $(k', k'') = (1, 0)$ (if and only if $v \in V_1'$). $\square$

Remark 1
- *(j) Let us repeat that $g(v) = (0, 1)$ for any terminal position $v \in V_T$, or in other words, $V_T \subseteq V_0'$, in every impartial game.*
- *(jj) The conditions of Theorem 1 are sufficient but not necessary for miserability.*

  Indeed, given a miserable game, let us add to it one new position v^* and three new moves from v^* to v_1, v_2, and v_0 such that $g(v_0) = (0, 0), g(v_1) = (1, 0)$, and $g(v_2) = (1, 1)$. Then, obviously, $g(v^*) = (2, 2)$ and the obtained game is miserable. Yet, our condition fails, since V_1' is, while V_0' is not reachable, from v^*.
- *(jjj) A miserable game is uniquely defined by the subsets $V_0' \subseteq V_0^N$ and $V_0' \subseteq V_0^N$.*

  In their turn, these subsets are also uniquely defined as the sets of $(0, 1)$- and $(1, 0)$-positions, or in other words, as 0- and 1-swap positions of the game.
- *(jv) Conditions (i0) and (i2) of the theorem automatically hold if we set $V_0' = V_0^N$ and $V_0' = V_0^N$; yet, in this case (i1) may fail. If it holds too then the game is not only miserable but strongly miserable.*
- *(v) To demonstrate that a miserable game is not strongly miserable it is enough to show a position $v \in (V_0^N \setminus V_0') \cup (V_1^N \setminus V_1') = (V_0^N \cup V_1^N) \setminus V'$.*
- *(vj) By the definition of the SG function, no move can stay within V_0' or V_1'. Hence, the induced directed subgraph $G[V'] = G[V_0' \cup V_1']$ is bipartite (and of course, it is acyclic).*

Proof of Theorem 2. Obviously, (i') is just a reformulation of (i) and it implies (ii).

It is also easy to show that (i) implies (iii). Indeed, let us assume indirectly that there is a position $v \in V_0^N \setminus V_T$ from which V_1^N is not reachable, that is, $g^N(v) = 0$ and $g^N(v') = 1$ for no move $(v, v') \in E$. Then, by (i), $g^M(v') = 0$ for no move $(v, v') \in E$. Hence, by Lemma 1, $g^M(v) = 0$ and $g(v) = (g^N(v), g^M(v)) = (0, 0)$ in contradiction with (i).

By similar arguments, we prove that (i) implies (iii').

It remains to show that, conversely, (i') results from (ii) or (iii) or (iii').

We will prove this indirectly, by induction. First, let us recall that $g(v) = (0, 1)$ for any $v \in V_T$, by definition of the SG function, and hence, (i') holds in this case. Second, let us assume indirectly that (i') fails for a position $v \in V$ but holds for all its (immediate) successors and show that (iii), (iii') and (ii) also fail.

Case 1: $g(v) = (0, 0)$. Then (ii) fails for v, by definition. If (iii) holds for v then there is a move (v, v') such that $g^N(v') = 1$ and, by (i'), $g^M(v') = 0$. Yet, $g^M(v) = 0$ too and, by Lemma 1, $g^M(v') \neq 0$, which is a contradiction.

Similarly, if (iii') holds for v then there is a move (v, v') such that $g^M(v') = 1$ and, by (i'), $g^N(v') = 0$. Yet, $g^N(v) = 0$ too and, by Lemma 1, $g^N(v') \neq 0$, which is a contradiction.

Case 2: $g(v) = (k, 0)$ (respectively, $g(v) = (k, 1)$) , where $k > 1$. By Lemma 1, there is a move (v, v') such that $g^N(v') = 1$ (respectively, $g^N(v') = 0$). Then, by (i'), $g(v') = (1, 0)$ (respectively, $g(v') = (0, 1)$) and $g^M(v) = g^M(v')$, in contradiction with Lemma 1.

Case 3: $g(v) = (k, \ell)$, where $k > \ell > 1$. By Lemma 1, there is a move (v, v') such that $g^N(v') = \ell$. Then, by (i'), $g^M(v') = g^N(v') = \ell$. Hence, $g^M(v) = g^M(v') = \ell$, in contradiction with Lemma 1.

Similar arguments work for the symmetric cases: $k = g^M(v) > g^N(v) \in \{0, 1, \ell\}$.

Let us also remark that cases 2 and 3 are impossible just by the induction hypothesis, while the assumptions (iii), (iii'), and (ii) are irrelevant. □

Proof of Theorem 3. NIM is the simplest example of a sum. Indeed, the n-pile NIM is the sum of n one-pile NIMs. Bouton's arguments for NIM (see the proof Lemma 4 below) can be easily applied to any sum.

Let $G = G_1 \oplus \ldots \oplus G_n$ be the sum of n games. By definition, $V(G) = V(G_1) \times \ldots \times V(G_n)$. If all n summands $G_i,\ i \in [n] = \{1, \ldots, n\}$, are miserable then the sum is miserable too, with $V'(G) = V'(G_1) \times \ldots \times V'(G_n)$; in other words, the set of the (swap) positions of the sum is the direct product of the sets of the (swap) positions of the summands. Furthermore, the partition into 0- and 1-swap positions, $V'(G) = V'_0(G) \cup V'_1(G)$, is defined as follows. A swap position $v = (v_1, \ldots, v_n\} \in V'(G)$ is a 0-swap position, $v \in V'_0(G)$ (respectively, a 1-swap position, $v \in V'_1(G)$) if v_i is a 1-swap position, $v_i \in V_1(G_i)$, for an even (respectively, odd) number of summands $i \in [n]$. All conditions of Theorem 1 are easy to verify.

Indeed, $V'_0(G)$ is reachable from $V'_1(G)$ and $V'_1(G)$ is reachable from $V'_0(G) \setminus V_T(G)$.

Given a position $v = (v_1, \ldots, v_n) \notin V'(G) = V'(G_1) \times \ldots \times V'(G_n)$; if v_i are swap positions, $v_i \in V'(G_i)$, for all $i \in [n]$ then v is a swap position too, $v \in V'(G)$; otherwise, if $v_i \notin V'(G_i)$ for a unique $i \in [n]$ then both sets $V'_0(G)$ and $V'_1(G)$ are reachable from v, since both $V'_0(G_i)$ and $V'_1(G_i)$ are reachable from v_i in G_i; in contrast, if $v_i \notin V'(G_i)$ for at least two $i \in [n]$ then none of these two sets is reachable from v, neither $V'_0(G)$ nor $V'_1(G)$. □

4 Examples

4.1 NIM

The ancient game of NIM is played as follows. There are n piles of $x_1, \ldots, x_n$ stones. Two players move alternately. By one move a player chooses a pile $i \in [n] = \{1, \ldots, n\}$ and takes y_i stones such that $0 < y_i \leq x_i$. The game is over as soon as there are no stones left.

Standardly, the player who took the last stone wins in the normal version and loses in the misère one. Both versions were solved by Bouton [7] in 1901.

Obviously, the n-pile NIM is the sum of n one-pile NIMs. The SG function of the one-pile NIM is trivial: $g^N(x_i) = x_i$. Hence, $g^N(x_1, \ldots, x_n) = x_1 \oplus \ldots \oplus x_n$. For example,

$$g^N(0,2) = g^N(2,0) = 0 \oplus 2 = 2, g^N(1,2) = g^N(2,1) = 1 \oplus 2 = 3, g^N(2,2) = 2 \oplus 2 = 0.$$

These simple computations imply the following claim.

Lemma 3 *Already the two-pile NIM is not strongly miserable.*

Proof: Only positions $(0,2),(1,2)$ and $(2,0),(2,1)$ are reachable from $(2,2)$. It is easy to verify that $g^M(2,2) = g^N(2,2) = 0$, while $g^N(2,1) = 3$ and $g^N(2,0) = 2$; hence, conditions (ii) and (iii) of Theorem 2 fail. □

However, as it was shown by Bouton in 1901, miserability holds even for n piles.

Lemma 4 *([7]). Game NIM is miserable.*

Proof: Clearly, the n-pile NIM is (strongly) miserable and it is the sum of n one-pile NIMs. Thus, the claim follows from Theorem 3. Still, we repeat here the original Bouton proof (in terms of Sections 1), since it is of high historical value, elegant, and takes just a few lines.

The sets of 0- and 1-swap positions $V'_0, V'_1 \subseteq V = \{x = (x_1, \ldots, x_n) \mid x_i \in \mathbb{Z}_{\geq 0}, i \in [n]\}$ are defined as follows.

The set of swap positions $V' = \{x = (x_1, \ldots, x_n) \mid x_i \leq 1, i \in [n]\}$ consists of 2^n positions, for which $x_i = 0$ or $x_i = 1$ for all $i \in [n]$. Obviously, in such a position a player can only eliminate a non-empty pile. Hence, all moves are equivalent and the result of a play that begins in swap positions depends only on parity but not on the players' skills. Moreover, the SG function $g^N(x)$ of a swap position x is either 0, when the number of non-empty piles is even, or 1, when it is odd. By definition, these two types of positions form the 0- and 1-swap positions, V'_0 and V'_1, respectively. Obviously, inclusions $V'_0 \subseteq V^N_0$ and $V'_1 \subseteq V^N_0$ hold.

It is also easy to verify all conditions of Theorem 1. Indeed, a 0-swap position can be reached from any 1-swap position and a 1-swap position can be reached from any non-terminal 0-swap position by elimination of one pile, that is, by one move.

Then, let us consider a non-swap position, $x = (x_1, \ldots, x_n) \notin V'$. If x has at least two coordinates of cardinality at least 2 then no swap position can be reached from x. Otherwise, if there is exactly one such coordinate, then it can be reduced either to 0 or to 1; the corresponding two moves result either in V'_0 and V'_1, or vice versa. □

The above two lemmas are summarized by the next statement.

Proposition 1 *Game NIM is miserable but not strongly misearable.* □

4.2 Wythoff's NIM

In 1907, Wythoff [39] introduced the following interesting modification of the two-pile NIM.

Two piles contain x and y stones. Two players move alternately. By one move, a player can take either any number of stones from one pile (and nothing from the other), or equal numbers of stones from both. It is not allowed to pass one's turn.

In other words, $V = \{(x, y) \in \mathbb{Z}^2_{\geq 0}\}$ and from a position (x, y) there are moves to

(j) $\{(x', y) \mid 0 \leq x' < x\}$, (jj) $\{(x, y') \mid 0 \leq y' < y\}$, and
(jjj) $\{(x', y') \mid 0 < x - x' = y - y' \leq \min(x, y)\}$.

Let us notice that positions (x, y) and (y, x) are equivalent, due to obvious symmetry. For this reason, we assume that $x \leq y$, unless it is explicitly said otherwise, and we will keep this assumption through subsections 4.2 – 4.5.

Wythoff proved that the P-position (x_n, y_n) are given by the following recursion

$$x_n = mex\{x_i, y_i \mid 0 \leq i < n\}, \quad y_n = x_n + n; \quad n \in \mathbb{Z}_{\geq 0}, \tag{2}$$

which can be solved explicitly as follows;

$$x_n = \lfloor \phi n \rfloor, \quad y_n = x_n + n = \lfloor (\phi + 1)n \rfloor; \quad n \in \mathbb{Z}_{\geq 0}, \tag{3}$$

where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is the golden section; see also [12].

Somewhat surprisingly, no explicit formula is known for the SG function of the Wythoff game; see [4, 5, 32] for partial results in this direction.

However, the zeros of the SG function are well defined by (2) or (3).

Lemma 5 *(see, for example,[4]). The normal and misère SG functions of Wythoff's NIM differ only in six positions given below and coincide in all other positions:*

$g^N(0,0) = g^N(1,2) = g^N(2,1) = g^M(0,1) = g^M(1,0) = g^M(2,2) = 0,$
$g^M(0,0) = g^M(1,2) = g^M(2,1) = g^N(0,1) = g^N(1,0) = g^N(2,2) = 1,$
$g^N(x,y) = g^M(x,y)$ *for all* $(x,y) \notin \{(0,0),(1,2),(2,1);(0,1),(1,0),(2,2)\} = V'.$
In other words, the Wythoff game is miserable with the sets of 0- *and* 1-*swap positions* $V'_0 = \{(0,0),(1,2),(2,1)\}$ *and* $V'_1 = \{(0,1),(1,0),(2,2)\}$*, respectively.*

Proof: It is not difficult to verify all conditions of Theorem 1.

Indeed, V'_0 is reachable from V'_1 and V'_1 is reachable from $V'_0 \setminus \{(0,0\} = \{(1,2),(2,1)\}$, according to the rules of the game.

Furthermore, it is easy to verify that each "column" $x = 0, x = 1, x = 2$, each "row" $y = 0, y = 1, y = 2$, and each "diagonal" $y = x \pm 1$ and $y = x$ contain exactly two positions of $V' = V'_0 \cup V'_1$, one from V'_0 and one from V'_1; while all other rows, columns, and diagonals contain none. Let us also notice that both V'_0 and V'_1 are reachable from $(1,1)$. Thus, from every non-swap position $(x,y) \notin V'$, either each set V'_0 and V'_1, or none of them is reachable. $\square$

Lemma 6 *Wythoff's game is not strongly miserable.*

Proof: The zeros of the SG function g^N, given by (2) or (3), form an infinite set. By Lemma 5 this set and the set of zeros of the misère SG function "almost" coincide; more precisely, their symmetric difference is V'. Thus $g(x,y) = (g^N(x,y), g^M(x,y)) = (0,0)$ for infinitely many positions (x,y) defined by (2) or (3), e.g., for $(3,5), (4,7), (6,10), \ldots$, which contradicts condition (ii) of Theorem 2.

4.3 Frankel's NIM(a)

In 1982, Fraenkel generalized the Wythoff NIM keeping (j, jj) and replacing (jjj) by

(jjj') $\{(x', y') \mid 0 \leq x' \leq x,\ 0 \leq y' \leq y,\ x' + y' < x + y, \text{ and } |(x - x') - (y - y')| < a\}$,

where a is a strictly positive integer parameter; see [16] and also [17].

In other words, in Fraenkel's game a player can take either any strictly positive number of stones from one pile, and nothing from the other, or "approximately" equal numbers of stones from both piles; more precisely, the difference must be at most $a - 1$.

Obviously, Fraenkel's NIM(a) turns into Wythoff's NIM when $a = 1$. Fraenkel showed that the recursive solution (2) of NIM(1) should by just slightly modified to solve NIM(a):

$$x_n = mex\{x_i, y_i \mid 0 \leq i < n\}, \quad y_n = x_n + an; \quad n \in \mathbb{Z}_{\geq 0}. \tag{4}$$

Moreover, he solved this recursion and got the following explicit formula for (x_n, y_n).

Let $\alpha = \alpha(a) = \frac{1}{2}(2 - a + \sqrt{a^2 + 4})$ be the (unique) positive root of the quadratic equation $z^2 + (a-2)z - a = 0$, or equivalently $\frac{1}{z} + \frac{1}{z+a} = 1$. In particular, $\alpha(1) = \frac{1}{2}(1 + \sqrt{5})$ is the golden section and $\alpha(2) = \sqrt{2}$. The explicit solution is given by the following formula

$$x_n = \lfloor \alpha n \rfloor, \quad y_n = x_n + an \equiv \lfloor n(\alpha + a) \rfloor; \quad n \in \mathbb{Z}_{\geq 0}. \tag{5}$$

Fraenkel's proof is based on the following "folk-theorem" going back at least to Betty [3].

Lemma 7 *([3]) Let α and β be positive irrationals satisfying $\alpha^{-1} + \beta^{-1} = 1$ then two sets $A = \{n\alpha \mid \mathbb{Z}_{>0}\}$ and $B = \{n\beta \mid \mathbb{Z}_{>0}\}$ partition the set $\mathbb{Z}_{>0}$ of strictly positive integers.* $\square$

The very elegant and short proof given in [16] Fraenkel attributes to Ostrovsky.

As mentioned in [16], the explicit formula (5) solves the game $NIM(a)$ in linear time, in contrast to recursion (4) providing only an exponential algorithm.

A recursion solving the misère version of NIM(a) can be found in page 69 of [17].

$$x_n = mex\{x_i, y_i \mid 0 \leq i < n\}, \quad y_n = x_n + an + 1; \quad n \in \mathbb{Z}_{\geq 0}; \quad a \in \mathbb{Z}_{\geq 2} = \mathbb{Z}_{\geq 0} \setminus \{0, 1\}. \tag{6}$$

Let us recall that NIM(1) is the Wythoff game, which is (not strongly) miserable.

Lemma 8 *Game $NIM(a)$ is strongly miserable when $a > 1$.*

Proof: One can easily verify condition (ii) of Theorem 2 just comparing (4) and (6). Indeed, $y_n = x_n + an$ for the normal version and $y_m = x_m + am + 1$ for the misère one. If $(x_n, y_n) = (x_m, y_m)$ then $y_n - y_m = x_n - x_m + a(n-m) - 1 = a(n-m) - 1 = 0$. Hence, $a(n-m) = 1$, which is impossible for $a > 1$. □

4.4 Game NIM(a, b)

Further generalization was suggested in [25]. Given two positive integers a and b, we keep (jjj') and replace two sets defined by (j) and (jj) by one set defined as follows:

(j–jj) $\{(x', y') \mid 0 \le x' \le x,\ 0 \le y' \le y,\ \ x'+y' < x+y,\ \text{ and } [x-x' < b \text{ or } y-y' < b]\}$.

In other words, in NIM(a, b) a player can take either (j–jj) any positive number of stones from one pile and at most $b-1$ from the other, or (jjj') any "approximately equal" numbers of stones from both piles; more precisely, these two numbers may differ by at most $a-1$.

We still assume that two player move alternately, it is not allowed to pass one's turn, the player who has no move loses in the normal version and wins in the misère one.

Obviously, NIM$(a, 1)$ is exactly the Fraenkel NIM(a), since $0 \le z - z' < 1$ iff $z = z'$.

The recursive solutions of NIM(a, b), for the normal and misère versions, were recently obtained in [25]. They are similar to the corresponding Fraenkel recursions (4) and (6) for NIM(a), but *mex* is replaced by mex_b. For the normal version the P-positions are given by

$$x_n = mex_b\{x_i, y_i \mid 0 \le i < n\},\ y_n = x_n + an;\ \ n \in \mathbb{Z}_{\ge 0},\ a, b \in \mathbb{Z}_{>0}. \tag{7}$$

while for the misère version and $a \ge 2$ we have

$$x_n = mex_b\{x_i, y_i \mid 0 \le i < n\},\ y_n = x_n + an + 1;\ n \in \mathbb{Z}_{\ge 0};\ b \in \mathbb{Z}_{>0},\ a \in \mathbb{Z}_{\ge 2}. \tag{8}$$

Case $a = 1$ for the misère version appears to be special for all b, not only for $b = 1$, and it is similar to NIM$(1, 1)$ = NIM(1), that is, to the classical Wythoff NIM.

Lemma 9 *([25]). The normal and misère SG functions of NIM$(1, b)$ differ only in six positions given below and coincide in all other positions:*

$g^N(0,0) = g^N(b, b+1) = g^N(b+1, b) = g^M(0,1) = g^M(1,0) = g^M(b+1, b+1) = 0,$
$g^M(0,0) = g^M(b, b+1) = g^M(b+1, b) = g^N(0,1) = g^N(1,0) = g^N(b+1, b+1) = 1,$
$g^N(x,y) = g^M(x,y)$ *for all* $(x,y) \notin \{(0,0),(b,b+1),(b+1,b);(0,1),(1,0),(b+1,b+1)\} = V'$.

In other words, game NIM$(1, b)$ is miserable with the 0- and 1-swap positions $V'_0 = \{(0,0),(b,b+1),(b+1,b)\}$ and $V'_1 = \{(0,1),(1,0),(b+1,b+1)\}$.

Proof: It is easy to see that Lemma 9 turns into Lemma 5 when $b = 1$; so we will mimic the proof of the latter and verify all conditions of Theorem 1.

Set V'_0 is reachable from V'_1 and V'_1 is reachable from $V'_0 \setminus \{(0,0)\} = \{(b, b+1), (b+1, b)\}$, in accordance with the rules of the game. Then, let us introduce the set of positions

$$V'' = \{(x,y) \mid 0 \le x \le 2b, \text{ or } 0 \le y \le 2b, \text{ or } |x-y| \le 1\} \subseteq V$$

consisting of $2b+1$ rows $0 \le y \le 2b$, columns $0 \le x \le 2b$, and $2a+1 = 3$ diagonals $|x-y| \le 1$. It is easy to see that both $V'_0 = \{(0,0),(b,b+1),(b+1,b)\}$ and $V'_1 = \{(0,1),(1,0),(b+1,b+1)\}$ are reachable from $V'' \setminus V'$ but none of these two sets is reachable from $V \setminus V''$. □

Similarly, we extend Lemma 6 from the Wythoff NIM$(1,1)$ to NIM$(1,b)$ for all $b \in \mathbb{Z}_{>0}$.

Lemma 10 *Game NIM$(1,b)$ is not strongly miserable for all $b \in \mathbb{Z}_{>0}$.*

Proof: The zeros of the SG function g^N, given by (7) or (8), form an infinite set. By Lemma 9 this set and the set of zeros of the misère SG function "almost" coincide; more precisely, their symmetric difference is V'. Thus $g(x,y) = (g^N(x,y), g^M(x,y)) = (0,0)$ for infinitely many positions (x,y) defined by (7) or (8), which contradicts condition (ii) of Theorem 2. □

Thus, game NIM$(1,b)$ is miserable but not strongly miserable.

Lemma 11 *For any $b \in \mathbb{Z}_{\ge 0}$, game NIM(a,b) is strongly miserable when $a > 1$.*

Proof: We can just copy the proof of Lemma 8 for the Fraenkel game NIM(a). □

The case $a = 0$ is also of interest. In this case the moves of type (jjj') become impossible, since $a - 1 = -1 < 0$. Hence, the obtained game NIM$(0,b)$ is a simple generalization of the standard two-pile NIM. It is easy to verify [25] that

$x_n = y_n = bn$ for all $n \ge 0$ in the normal version, while

$x_0 = y_0 = (0,1)$, $x'_0 = y'_0 = (1,0)$, and $x_n = y_n = bn + 1$ for all $n \ge 1$ in the misère version.

These recursions imply the next claim.

Lemma 12 *The game NIM$(0,b)$ is strongly miserable when $b > 1$; in this case the sets P_N and P_M are the zeros and ones of the SG function; in particular, $P_N \cap P_M = \emptyset$.*

In contrast, the game NIM$(0,1)$ is miserable but not strongly miserable; in this case

$P_N \setminus P_M = \{(0,0),(1,1)\}$ *and* $P_M \setminus P_N = \{(0,1),(1,0)\}$.

Finally, the last four lemmas are summarized by the following statement.

Proposition 2 *([25]). For any $a, b \in \mathbb{Z}_{\ge 0}$, the game NIM$(a,b)$ is strongly miserable unless $a = 1, b \ge 1$ or $b = 1, a \le 1$, in which cases it is miserable but not strongly.* □

Let us note that no explicit formula is known not only for the SG function of NIM(a,b) but even for its P-positions. It is shown in [25] that $b \le x_{n+1} - x_n \le 2b$. Thus, for $b = 1$ only values 1 and 2 are taken, which enables us to apply Betty's lemma [16, 17]. Yet, in general, differences $x_{n+1} - x_n$ demonstrate some "pseudo-random" behavior and it looks unlikely that an explicit formula for x_n can be obtained.

In [25] it was conjectured that x_n can be computed in polynomial time and that limits $\ell(a,b) = \lim_{n\to\infty} x_n(a,b)/n$ exist and are irrational algebraic numbers for all $a, b \in \mathbb{Z}_{>0}$.

Both these conjectures were recently proven in [6]. A linear time algorithm was obtained, as well as the formula $\ell(a,b) = \frac{a}{r-1}$, where $r > 1$ is the unique positive real root (so called Perron root) of the polynomial

$$P(z) = z^{b+1} - z - 1 - \sum_{i=1}^{a-1} z^{\lceil ib/a \rceil},$$

which is the characteristic polynomial of a non-negative $(b+1) \times (b+1)$ integer matrix depending only on parameters a and b, and where $|r'| < r$ for any other root r' of $P(z)$.

Let us remark that the above formulas hold only for relatively prime a and b, while $x_n(ka, kb) = kx_n(a,b)$ and hence, $y_n(ka, kb) = ky_n(a,b)$ and $\ell(ka, kb) = k\ell(a,b)$ [25].

Finally, let us also remark that the proofs are based on the Perron-Frobenius theory of non-negative matrices and, in particular, on the Collatz-Wielandt formula; see Chapter 8 of the textbook [29]. It can be also shown that $P(z)$ satisfies all conditions of the Cauchy-Ostrovsky theorem; see theorems 1.1.3, 1.1.4 in the textbook [35].

4.5 Game Euclid

In 1969 Cole and Davie [8] introduced a game inspired by the Euclidean algorithm. The positions of this game are all pairs (x, y) of positive integers. Two players move alternately. By one move a player is allowed to subtract any positive multiple of the smaller number from the larger one, provided the difference is still positive. The game ends when no more moves are possible. It is easily seen that the game has a unique terminal position (z, z), where $z = gcd(x, y)$, the greatest common divisor of x and y. Again, positions (x, y) and (y, x) are equivalent. Also, positions $(\ell x, \ell y)$ are equivalent for all positive integer ℓ.

In [8] it was shown that (x, y) is a P-position if and only if $x < \phi y$ and $y < \phi x$, where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is the golden section.

Moreover, in 2004, Nivasch [31] got a very nice formula for the SG function:

$$g^N(x,y) = \lfloor |x/y - y/x| \rfloor \quad \forall\, x, y \in \mathbb{Z}_{>0}. \tag{9}$$

In [24], it was shown that the game is miserable with the 0- and 1-swap positions defined as follows. Let F_j be the jth Fibonacci number, $F_j = 1, 1, 2, 3, 5, 8, \ldots$ for $j = 0, 1, 2, 3, 4, 5, \ldots$. Let us call (x, y) a *Fibonacci position* of rank i if (x, y) or (y, x) equals $(\ell F_i, \ell F_{i+1})$, where $i \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{>0}$. It is easy to see that in a Fibonacci position there is a unique move when $i > 0$ and no move when $i = 0$. Moreover, this unique move leads to the Fibonacci position of rank $i - 1$, since $(\ell F_i, \ell F_{i+1} - \ell F_i) = (\ell F_i, \ell F_{i-1})$. Then the next move is again unique (if $i > 1$) and it leads to the Fibonacci position of rank $i - 2$, etc. until the play terminates in the Fibonacci position (ℓ, ℓ) of rank 0.

Lemma 13 *([28]). The following two claims are equivalent:*

- *(t)* (x, y) *is a Fibonacci position,*
- *(tt) beginning from* (x, y)*, each further move is forced.*

Proof: Obviously, each of these two claims is equivalent to
(ttt) y/x expands into a continued fraction whose every incomplete quotient is 1. □

In [28], Lengyel showed that the winning strategy in position (x, y) can be defined in terms of the incomplete quotient of y/x for game Euclid, as well as for several similar and more general games. The following obvious reformulations of Lemma 13 will be useful.

Lemma 14 *A Fibonacci position* (x, y) *of rank* i *is a P-position if and only if* i *is even.*
This position is terminal if $i = 0$ *and if* $i > 0$ *then there is unique move from it, which leads to a Fibonacci position of rank* $i - 1$*.* □

Lemma 15 *([24]). From a non-Fibonacci position, either no move enters a Fibonacci one, or the "longest" two moves enter two Fibonacci positions whose ranks differ by* 1*.*

For example, from (22,4) there are moves to (18,4),(14,4)(10,4),(6,4), and (2,4), the last two of which are Fibonacci positions of rank 2 and 1 respectively, while from (16,3) there are moves to (13,3),(10,3),(7,3),(4,3),and (1,3), none of which is a Fibonacci position.

Proof of Lemma 15. We will show that if a move from a position v leads to a Fibonacci position $v_i = (\ell F_i, \ell F_{i+1})$, of rank i, then there is another move from v that leads either to the Fibonacci position $v_{i-1} = (\ell F_i, \ell F_{i-1})$, of rank $i - 1$, or to the Fibonacci position $v_{i+1} = (\ell F_{i+2}, \ell F_{i+1})$, of rank $i + 1$.

Case 1: $v = (\ell F_i, \ell F_{i+1} + \ell' F_i)$. Then subtraction of $\ell' F_i$ from the second coordinate results in v_i. Yet, subtraction of $(\ell' + 1)F_i$ results in v_{i-1}.

Case 2: $v = (\ell F_i + \ell' F_{i+1}, \ell F_{i+1})$. Then subtraction of $\ell' F_{i+1}$ from the first coordinate results in v_i. Yet, subtraction of $(\ell' - 1)F_{i+1}$ results in v_{i+1}.

It remains to notice that $\ell, \ell' \in \mathbb{Z}_{>0}$ and that v_i is not reachable from other positions. □

Lemma 16 *Game Euclid is not strongly miserable.*

Proof: From position $(3, 4)$ there is only one move, which leads to $(3, 1)$. However, $g^N(3, 4)g^M(3, 4) = 0$, while $g^N(3, 1) = 2$. Hence, conditions (ii) and (iii) of Theorem 2 fail. Equivalently, both containments, $V'_0 \subset V^N_0$ and $V'_1 \subset V^N_1$, are strict. □

Four previous lemmas, result in the following claim.

Proposition 3 *Game Euclid is not strongly miserable but it is miserable with the* 0*- and* 1*-swap positions being the Fibonacci positions of even and odd rank, respectively.* □

4.6 Game 3-Euclid and its versions

Collins and Lengyel [9, 10] presented an extension of the above game to three dimensions that they called 3-Euclid. In 3-Euclid, a position is a triplet of positive integers. Each move is to subtract from one of the integers a positive integer multiple of one of the others as long as the result remains positive. Generally, from a position (x_1, x_2, x_3) where $x_1 \leq x_2 \leq x_3$, there are three types of moves in 3-Euclid: $i - j$ moves: subtracting a multiple of x_i from x_j, provided $x_i < x_j$, where $i, j \in \{1, 2, 3\}$ and $i < j$. Recently, Ho [26] considered two modifications of 3-Euclid, in which only $1 - 3$ and, respectively, $1 - 2$ and $1 - 3$ moves are allowed. He proved that the kernels of these two games coincide. Moreover, his results imply (see Corollary 3 of [26]) that both games are strongly miserable.

It is an open question whether the game 3-Euclid itself is (strongly) miserable.

4.7 Subtraction games

Given a (finite or infinite) non-empty subset of positive integers $S \subseteq \mathbb{Z}_{>0}$, a *subtraction game* G_S is defined by the set of positions $V = \mathbb{Z}_{\geq 0}$ and possible moves $(x, x - s)$, where $s \in V$ and $s \in S$. All subtraction games are strongly miserable, as the next statement shows.

Proposition 4 *([14]). Every subtraction game G_S satisfies the Ferguson property (iii).*

Since the proof in [14]) is very short and elegant, we copy it here for readers' convenience. This proof is based on the following lemma that is of independent interest.

Lemma 17 *([14] Let k be the smallest element of S. Then $g_S^N(x) = 0$ implies $g_S^N(x+k) = 1$. Conversely, $g_S^N(x) = 1$ implies $g_S^N(x-k) = 0$.*

Proof: Since $k \in S$, $g_S^N(x) = 0$ implies $g_S^N(x+k) \neq 0$. Assume the conclusion is false and find the smallest x such that $g_S^N(x) = 0$ and $g_S^N(x+k) > 1$. By the latter, there is an $s \in S$ such that $g_S^N(x+k-s) = 1$ and $x - s \geq 0$, since k is the smallest element of S. Furthermore, $g_S^N(x) = 0$ implies $g_S^N(x-s) > 0$. Thus, there exists an $s' \in S$ such that $g_S^N(x-s-s') = 0$. This, together with $g_S^N(x+k-s) = 1$ entails $g_S^N(x-s-s'+k) > 1$. Thus, $y = x-s-s' < x$ also satisfies $g_S^N(y) = 0$ and $g_S^N(y+k) > 1$ contradicting the choice of x as the smallest one.

Conversely, if $g_S^N(x) = 1$ and $g_S^N(x-k) \neq 0$, there is an $s \in S$ such that $g_S^N(x-k-s) = 0$. From the first part of the theorem, this implies $g_S^N(x-s) = 1$. It contradicts $g_S^N(x) = 1$. $\square$

Proof of Proposition 4. Given any nonterminal x such that $g_S^N(x) = 0$, one has $g_S^N(x-k) \neq 0$, where k is the smallest element of S. This implies that there is an $s \in S$ such that $g_S^N(x-k-s) = 0$. From the lemma, $g_S^N(x-s) = 1$. $\square$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$g^N(n)$	0	1	0	2	1	2	0	1	0	2	0	1	2	1	0	2	1	2	0
$g^M(n)$	1	0	1	2	0	2	0	1	2	1	0	1	2	1	0	2	0	1	0

Table 1: Game Mark is not miserable: $g(v) = (g^N(v), g^M(v))$ takes values $(0, 2)$ and $(2, 1)$.

4.8 Not miserable impartial games

Of course, "most" of the impartial games are not miserable. For example, let us consider game Mark recently introduced by Fraenkel [19]. The set of positions of Mark is $\mathbb{Z}_{\geq 0}$ and for any $n \in \mathbb{Z}_{\geq 0}$ the moves to $n - 1$ and $\lfloor n/2 \rfloor$ are allowed. The first SG values of the normal and misère versions of this game are given in Table 1. This table shows that Mark is not miserable. Indeed, the pairs $g(n) = (g^N(n), g^M(n))$ takes values $(0, 0), (0, 2)$ and $(2, 1)$.

Let us remark that Mark is not a subtraction game. Moreover, it has very different properties. For example, all subtraction games are strongly miserable, while Mark is not even miserable. It is also shown in [19] that the SG function of Mark is not periodic, unlike the SG functions of all subtraction games.

Another example of a non-misrable game is given by the digraphs G^N and G^M in Figure 1. For this game $g(v) = (g^N(v), g^M(v))$ takes values $(0, 0), (2, 0)$ and $(3, 2)$.

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