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Abstract. Given two integers $n \geq 0$ and $k \geq 3$, two players alternate turns taking stones from a pile of n stones. By one move a player is allowed to take any number of stones $k' \in \{1, \ldots, k-1\}$. In particular, it is forbidden to pass.
Furthermore, it is not allowed to take the same number of stones as the opponent by the previous move. The player who takes the last stone wins in the normal version of the game and loses in the misère version; (s)he also wins, in both cases, when the opponent has no legal move, that is, after the move from 2 to 1.
An integer $k \in Z_{\geq 0}$ is called *vile* if the maximum power of 2 that is still a divisor of k is even, or in other words, if the binary representation of k ends with an even number of zeros; otherwise, if this number is odd, k is called *dopey*.
In this short note, we will solve (both the normal and misère versions of) the game when k is vile and obtain partial results when k is dopey.
In the normal version, the set of P-positions is an arithmetic progression ak, where $a = 0, 1, \ldots,$ if k is vile. When k is dopey, for the P-positions $n_0, n_1, \ldots,$ we have $n_0 = 0, n_1 = k + 1$, and $n_{i+1} - n_i$ is either k or $k + 1$ for all $i \in \mathbb{Z}_{\geq 0}$. Yet, in the latter case, it seems not easy to choose among k and $k + 1$. We conjecture that the differences $n_{i+1} - n_i$ between the successive P-positions form a periodical sequence.
In the misère version of the game, the P-positions are shifted by $+1$ with respect to the P-positions of the corresponding normal version.

Keywords: vile and dopey numbers, impartial games, subtraction games, games of memory one, normal and misère versions

The rules of the game under consideration are defined in the abstract. For $k = 4$ and $n = 2000$ this subtraction game was suggested as a problem at *http : //www.braingames.ru/* on July 12, 2006. Surely, its most important feature is the rule:

(*) **A player is not allowed to take the same number of stones as the opponent by the previous move.** In particular, to play, one should remember the last move.

1 The corresponding impartial games, of memory zero

In other words, rule (*) requires a one move memory, unlike the standard impartial games, which are of memory zero. The latter are modeled by a directed acyclic graph (digraph) $G = (V, E)$ and the set of possible moves in a positions $v \in V$ consists of all directed edges $(v, w) \in E$; in particular, it does not depend on previous moves or positions. Such games are well studied; see for example [1]. To solve an impartial game G means to partition $V = V_P \cup V_N$ into the so called P- and N-positions such that

- (i) each move (v, w) from a P-position $v \in V_P$ leads to a N-position $w \in V_N$;
- (ii) for any N-position $w \in V_N$ there is a move (w, v) leading to a a P-position $v \in V_P$.

In particular, $V_T \subseteq V_P$, where $V_T \subseteq V$ is the set of terminal (that is, of out-degree 0) vertices of G. Frequently, V_P is referred to as the kernel of G. The kernel V_P (and the partition $V = V_P \cup V_N$) is defined by (i) and (ii) for any digraph and it is unique for acyclic digraphs [5]. Furthermore, it is both obvious and well known that if the game begins in an N- or P-position then the Next player or, respectively, the Previous player wins. This follows immediately from properties (i) and (ii).

To solve the misère version of the game, we would rather modify the digraph G than the above theory. Let us add to G one new vertex v^* and the directed arc (v, v^*) from each terminal position $v \in V_T$. Obviously, the misère version for G is equivalent with the normal version for the obtained digraph G'.

First, let us ignore the rule (*) and solve the simplified impartial game.

We will assume that $k \geq 2$, since if $k = 1$ then there are no moves at all.

Proposition 1 *Given integers $n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 2}$, the P-positions of the considered game form the arithmetic progression $V_P = \{ak \mid a \in \mathbb{Z}_{\geq 0}\}$.*

Proof: Properties (i) and (ii) of the kernel are immediately implied by the rules of the game.

Now let us consider the game in question.

2 The normal and misère versions for $k \leq 2$ or $n \leq 2$

By definition, no moves at all are allowed when $k = 1$.

The case $k = 2$ is trivial too. Then, there are no moves for $n = 0$ and there is a unique move for each $n \geq 1$. In particular, there is no legal second move whenever $n \geq 2$.

Let us consider the normal version first. By definition, the first player loses when $n = 0$ and (s)he wins when $n \geq 1$, since the opponent has no second move.

The misère version requires a more detailed analysis. Definitely, the Next player wins for $n = 0$. Yet, if $n > 0$ and the Next player has no legal move then two options are possible: (j) (s)he wins and (jj) loses. Let us remark that both look reasonable for the misère version.

If $n = 1$ (respectively, $n = 0$) then the Next player loses (respectively, wins) for both cases, (j) and (jj). Yet, if $n > 1$ then (s)he wins in case (j) but loses in case (jj).

Let us also notice that, after the move $(2, 1)$, there is no legal move at all, for any k. (Moreover, this is the only such case, otherwise at least one move is legal.) Hence, for $n = 2$ and any $k \geq 2$, the Next player wins in case (j) and loses in case (jj). Finally, it is easily seen that options (j) and (jj) become equivalent when $k > 2$ and $n > 2$ (or $n < 2$).

From now on we will assume that $k \geq 3$.

Furthermore, we will return to the misère version of the game only in the last subsection, while in all others only the normal version of the game is considered, by default.

3 Evil, odious, vile, and dopey integers

Given an an integer $k \in Z_{\geq 0}$, its 2-rank $r_2(k)$ is defined as the maximum power of 2 that is still a divisor of k, or in other words, $r_2(k)$ is the number of zeros at the end of the binary representation of k. Then, k is called *vile* or *dopey* if $r_2(k)$ is even or odd, respectively.

For example, $4 = 2^2$ and $12 = 3 \times 2^2$ are vile, while $32 = 2^5$ and $160 = 5 \times 2^5$ are dopey.

Remark 1 *These names were coined recently by Fraenkel [3] with the following motivation.*

"The vile numbers are those whose binary representations end in an even number of 0s, and the dopey numbers are those that end in an odd number of 0s. No doubt their names are inspired by the evil and odious numbers, those that have an even and an odd number of 1s in their binary representation respectively. To indicate that we count 0s rather than 1s, and only at the tail end, the "ev" and "od" are reversed to "ve" and "do" in "vile" and "dopey". "Evil" and "odious" where coined" in [1].

4 Case $n = k \geq 3$

Proposition 2 *If $n = k \geq 3$ then, beginning with n, the Next player wins (in the normal version of the game) if and only if k is dopey and the previous move was not $(\frac{3}{2}n, n)$.*

Although, the last move, $(\frac{3}{2}n, n)$, is winning for the Previous player when k is dopey, yet, in this case $(\frac{3}{2}n, n + 1)$ is another his winning move.

Proof: First, let us consider a couple of examples.

Case 1: $n = k = 4 = 2^2$ is vile. The Next player cannon win. If (s)he plays $(4, 1)$ or $(4, 3)$ then the opponent gets 0 by the next move. If (s)he plays $(4, 2)$ then the opponent responds with $(2, 1)$ and there is no more legal move. The same arguments work for any vile n.

Case 2: $n = k = 8 = 2^3$ is dopey. The Next player has a winning option $(8, 4)$ (unless the previous move was not $(12, 8)$), as the above analysis shows. However, this winning option is unique. In particular, if the previous move was indeed $(12, 8)$ then $(8, 4)$ becomes illegal and, against any other move, the Next player wins getting 0 immediately.

Let us note, yet, that $(12, 9)$ is another winning move. Indeed, response $(9, 8)$ loses, as the above analysis shows, and against any other response, the opponent wins in one move.

It is easy to see that exactly the same arguments work for any dopey (or vile) n. Indeed, the only thing that matters in the above arguments is the parity of the number of times that n can be divided by 2, in other words, whether n vile or dopey. □

5 P- and N-positions in the non-zero memory games

The concepts of P- and N-positions are easy to extend from the impartial games (of memory zero) to the games with the non-zero memory, in which the set of possible moves in a position v depends not only on v but on the history too.

Standardly, in the digraph $G = (V, E)$ of such a game, $v \in V$ is an N-position if the Next player wins when the game begins in v; otherwise v is a P-position.

Yet, several new features appear in the games with memory.

Now, P-positions do not form the kernel of G. More precisely, only the property (i) (there are no moves between P-positions) still holds, while (ii) may fail. Although, we can still claim that the Next player can avoid entering P-positions and force the opponent to enter one, yet, not necessarily in two moves; such a forcing sequence may be longer.

Let us also notice that the Next player wins (loses) in an N-position (respectively, in a P-position) v if the game begins in it, while v may become a losing (winning) position for the Next player when it appears in the middle of the play; see Proposition 2, for example.

6 The main result for vile positions

For vile k we obtain the same solution as for the case of memory zero; see Proposition 1.

Theorem 1 *If k is vile, the P-positions form an arithmetic progression* $A = \{ak \mid a \in \mathbb{Z}_{\geq 0}$.

Proof: We have to show that, beginning from ak, the Next player (N) loses. Let us proceed by induction on a. The claim holds for $a = 0$, since $n = ak = 0$ is a P-position. Let us assume that it holds for $n = bk$, $\forall\, b \in \{0, 1, \dots, a-1\}$ and show that it holds for $b = a$.

Let us begin with the case of odd k, or in other words, $r_2(k) = 0$. Then, obviously, for any move of player N from $n = ak$ there is a (unique) response of P to $(a-1)k$.

In general, the arguments are a little more sophisticated. Given a position $n \notin A$, let us set $b = \lfloor n/k \rfloor$. Then, bk is reachable from n by one move unless the previous move was $(2n - bk, n)$. Let us assume that this is case and (n, n') is the next move. Obviously, $n' \notin A$ still holds. Furthermore, bk is reachable from n' by one move unless $n' = \frac{1}{2}(n+bk)$ or $n' < bk$. In the latter case, $(b-1)k$ is reachable from n' by one move unless $n' = \frac{1}{2}(n + (b-1)k)$. It is easily seen that in both cases the 2-rank is reduced by 1, that is, $r_2(n) - r_2(n') = 1$.

The case of the first move, $n = ak$, should be considered separately. As we already know, if k is odd then, after any move of N, player P will reach $(a-1)k$ by the next move (and win, by the inductive assumption). If k is even then $(n, n - k/2)$ is the only move that does not lose immediately. In this case, $r_2(k) - r_2(n - k/2) = 1$, too. Let us note, however, that $r_2(ak) > r_2(k)$ when a is even. We assume that k is vile, that is, its 2-rank $r_2(k)$ is even. By the first move, the 2-rank is reduced to $r_2(k) - 1$ (from some value $r_2(ak) \geq r_2(k)$) and then, by each move, the the 2-rank is reduced by 1. (If not then the player who made the last move loses, since the opponent enters A by the next move.)

Hence, in an even number of moves, the 2-rank will be reduced to 0, that is, the play will enter an odd position, in which player N will have to move. If the last move was $(2, 1)$ then N has no legal move at all. Otherwise, in response to any move of N, the opponent (P) can enter a position bk. In both cases, player N, who started from $n = ak$, loses. □

7 Results and conjectures for dopey positions

Proposition 3 *When k is dopey, for the P-positions $n_0, n_1, \ldots$, we have $n_0 = 0, n_1 = k+1$, and $n_{i+1} - n_i$ is either k or $k+1$ for all $i \in \mathbb{Z}_{\geq 0}$.*

Proof: By definition, $n_0 = 0$ is a P-position (in the normal version of the game). The next one is $n_1 = k + 1$. Indeed, the move $(k+1, k)$ loses, by Proposition 2, while against any other move the opponent terminates at 0.

Similar arguments prove the last claim, as well. If $n_i + k$ is an N-position then $n_i + k + 1$ is an P-position. Indeed, the move to $n_i + k$ loses, by assumption, while against any other move the opponent can enter the previous P-position n_{i-1}. □

Yet, it seems not easy to choose among k and $k+1$. Computations show that $V_P =$ $\{0, 7, 13, 20, 26, 33, 39, \ldots\}$ for $k = 6 = 3 \times 2$; $\{0, 11, 22, 32, 43, 54, 64, \ldots\}$ for $k = 10 = 5 \times 2$; $\{0, 9, 17, 25, 34, 42, 50, \ldots\}$ for $k = 8 = 2^3$; $\{0, 25, 50, 74, 99, 124, 149, \ldots\}$ for $k = 24 = 3 \times 8$.

For example, let us consider the case $k = 10$, when a player can subtract at most 9. By definition, 0 is a P-position (in the normal version of the game). Furthermore, 10 is an N-position, since $(10, 5)$ is a winning move. The next P-position is 11. Indeed, after $(11, 10)$ the opponent wins with $(10, 5)$ and after any other move (s)he just terminates at 0. In contrast, 21 is an N-position with the winning move $(21, 16)$. Indeed, response $(16, 11)$ becomes illegal and after any other the Next player wins (playing at 11 after $15, 14, 13$, or 12, at 5 after 10, and at 0 after $9, 8$, or 7).

Conjecture 1 *The successive differences in V_P form a periodical sequence.*

This period is just 1 for a vile k. Yet, it may be much longer when k is dopey; for example, if $k = 24 = 3 \times 2^3$ then

$V_P = \{0, 25, 50, 74, 99, 124, 149, 174, 198, 222, 247, 272, 296, 321, 346, 371, 396, 420, 444, 469, \ldots\}$

Perhaps, $(25, 25, 24, 25, 25, 25, 25, 24, 24)$ is the period in this case.

8 The misere version

In the misère version, the P-positions are shifted by $+1$ with respect to the corresponding normal version. Propositions 1, 2, 3, and Theorem 1 should be reformulated accordingly.

This transformation is obvious, since the first P-position 0 in the normal version of the considered subtraction game is replaces with 1 in its misère version.

Moreover, the concept of *strong miserability* recently introduced in [4] for the impartial games can be extended to the games with memory. All subtraction games are strongly miserable [2, 4], that is, the sets of P-positions of the normal and misère versions are disjoint.

Remark 2 *Let us recall that* $(2, 1)$ *is a special move, after which there is no legal move for any* $k \geq 2$. *(This move can be treated as a winning move in both the normal and misère versions of the game; see subsection 2.) In contrast, its* $+1$*-shift* $(3, 2)$ *is a regular move, winning in the misère version but losing in the standard one, for any* $k \geq 3$

9 Further generalizations and computer experiments

One can introduce a positive integer parameter $m \in \mathbb{Z}_{\geq 0}$ and require $|n' - n''| > m$, where n' and n'' are the numbers of stones taken by two successive moves. In particular, these two numbers should be just distinct when $m = 0$; this is the case considered in the paper.

For $0 \leq m \leq 4$ and $k \leq 128$, the results of some computing experiments are given below.

Let us note, however, that there are no accurate proofs that confirm these observations.

For $m = 1$, the set of P-positions $P_k = \{0\}$ for $2 \leq k \leq 5$, while

$P_k = \{ka \mid a \in \mathbb{Z}_{\geq 0}\}$ for $k \in \{6, 8, 10, 14, 24, 25, 30 - 33, 40 - 42, 54 - 56, 94 - 101, 118 - 128\}$

form the arithmetic progressions with difference k.

Furthermore, $P_k = \{0, 14 + ka\}$ for $k \in \{11, 13, 14\}$ and

$P_7 = \{0, 8 + 7a\}$; $P_9 = \{0, 9, 28 + 9a\}$; $P_{12} = \{0, 14 + 38a, 26 + 38a\}$;
$P_{15} = \{0, 36 + 96a, 84 + 96a, 99 + 96a, 114 + 96a\}$,
$P_{16} = \{104a, 22 + 104a, 38 + 104a, 54 + 104a, 72 + 104a, 88 + 104a\}$,
$P_{17} = \{57a, 22 + 57a, 40 + 57a\}$,
$P_{18} = \{58a, 22 + 58a, 40 + 58a\}$,
$P_{19} = \{102a, 22 + 102a, 64 + 102a, 83 + 102a, \}$,
$P_{20} = \{106a, 22 + 106a, 66 + 106a, 86 + 106a, \}$,

$P_{21} = \{116a, 22 + 116a, 52 + 116a, 73 + 116a, 94 + 116a\}$,
$P_{22} = \{96a, 22 + 96a, 52 + 96a, 74 + 96a\}$,
$P_{23} = \{118a, 23 + 118a, 49 + 118a, 72 + 118a, 95 + 118a\}$;
$P_{26} = \{0, 26, 52, 82 + 26a\}$,
$P_{27} = \{0, 30 + 147a, 57 + 147a, 87 + 147a, 114 + 147a, 150 + 147a\}$,
$P_{28} = \{0, 30, 58, 88, 118 + 30a\}$,
$P_{29} = \{89a, 30 + 89a, 59 + 89a\}$;
$P_{34} = \{0, 34, 68, 143 + 34a\}$,
$P_{35} = \{0, 35, 70, 120 + 35a\}$,
$P_{36} = \{154a, 36 + 154a, 82 + 154a, 118 + 154a\}$,
$P_{37} = \{0, 37, 81 + 196a, 118 + 196a, 155 + 196a, 192 + 196a\}$,
$P_{38} = \{156a, 38 + 156a, 80 + 156a, 118 + 156a\}$,
$P_{39} = \{79a, 39 + 79a\}$;
$P_{57} = \{485a, 57 + 485a, 183 + 485a, 240 + 485a, 297 + 485a, 371 + 485a, 428 + 485a\}$,
$P_{58} = \{361a, 58 + 361a, 187 + 361a, 245 + 361a, 303 + 361a\}$,
$P_{59} = \{204a, 86 + 204a, 145 + 204a\}$,
$P_{60} = \{206a, 86 + 206a, 146 + 206a\}, \ldots$, where $a \in \mathbb{Z}_{\geq 0}$

For $m = 2$ we have $P_k = \{0\}$ for $k \in \{2 - 9, 26\}$ and $P_{10} = \{0, 11\}$, while $P_k = \{41a\}$ for $28 \leq k \leq 41$ and $P_k = \{ka\}$ for $k \in \{11, 13 - 19, 41 - 79\}$; furthermore

$P_{12} = \{0, 13 + 12a\}$;
$P_{20} = \{0, 41 + 20a\}$, $P_{21} = \{0, 44 + 21a\}$, $P_{22} = \{69a, 47 + 69a, 91 + 69a\}$,
$P_{23} = \{0, 50 + 23a\}$, $P_{24} = \{0, 80 + 77a, 104 + 77a, 128 + 77a\}$,
$P_{25} = \{0, 80 + 186a, 105 + 186a, 130 + 186a, 155 + 186a, 180 + 186a, 211 + 186a\}$;
$P_{27} = \{0, 72 + 89a, 99 + 89a, 126 + 89a\}$; $P_{28} = \{127a, 70 + 127a, 99 + 127a\}$,
$P_{29} = \{0, 41 + 97a, 70 + 97a, 99 + 97a\}$, $P_{30} = \{101a, 41 + 101a, 71 + 101a\}$,
$P_{31} = \{0, 41 + 101a, 72 + 101a, 103 + 101a\}$, $P_{32} = \{106a, 41 + 106a, 74 + 106a\}$,
$P_{33} = \{107a, 41 + 107a, 74 + 107a\}$, $P_{34} = \{185a, 41 + 185a, 117 + 185a, 151 + 185a\}$,
$P_{35} = \{188a, 41 + 188a, 118 + 188a, 153 + 188a\}$,
$P_{36} = \{155a, 41 + 155a, 82 + 155a, 118 + 155a\}$,
$P_{37} = \{156a, 41 + 156a, 82 + 156a, 119 + 156a\}, \ldots$, where $a \in \mathbb{Z}_{\geq 0}$.

For $m = 3$ we have $P_k = \{0\}$ when $2 \leq k \leq 13$ and $P_k = \{0, 16\}$ for $k \in \{14, 15\}$, while $P_k = \{ka\}$ for $k \in \{16, 18 - 28, 60 - 116\}$ and $P_k = \{60a\}$ for $55 \leq k \leq 60$; furthermore,

$P_{17} = \{70a, 18 + 70a, 36 + 70a, 53 + 70a\}$;
$P_{29} = \{0, 60 + 29a\}$, $P_{30} = \{0, 63 + 30a\}$, $P_{31} = \{0, 66 + 31a\}$,
$P_{32} = \{0, 69 + 100a, 101 + 100a, 133 + 100a\}$, $P_{33} = \{0, 72 + 33a\}$,
$P_{34} = \{0, 75 + 108a, 109 + 108a, 143 + 108a\}$, $P_{35} = \{0, 118 + 112a, 153 + 112a, 188 + 112a\}$,
$P_{36} = \{0, 118 + 195a, 154 + 195a, 190 + 195a, 234 + 195a\}$,
$P_{37} = \{0, 166 + 202a, 203 + 202a, 240 + 202a, 286 + 202a\}$,
$P_{38} = \{0, 198 + 208a, 236 + 208a, 274 + 208a, 322 + 208a\}$,
$P_{39} = \{0, 106 + 215a, 145 + 215a, 184 + 215a, 234 + 215a\}$,
$P_{40} = \{0, 104 + 132a, 144 + 132a, 184 + 132a\}$,

$P_{41} = \{0, 103 + 277a, 144 + 277a, 185 + 277a, 239 + 277a, 322 + 277a\}$,
$P_{42} = \{0, 60 + 140a, 102 + 140a, 144 + 140a\}$, $P_{43} = \{0, 60 + 144a, 103 + 144a, 146 + 144a\}$,
$P_{44} = \{148a, 60 + 148a, 104 + 148a\}$,
$P_{45} = \{299a, 60 + 299a, 105 + 299a, 150 + 299a, 208 + 299a, 254 + 299a\}$,
$P_{46} = \{152a, 60 + 152a, 106 + 152a\}$, $P_{47} = \{155a, 60 + 155a, 108 + 155a\}$,
$P_{48} = \{156a, 60 + 156a, 108 + 156a\}$,
$P_{49} = \{483a, 60 + 483a, 169 + 483a, 218 + 483a, 276 + 483a, 385 + 483a, 434 + 483a\}$,
$P_{50} = \{271a, 60 + 271a, 171 + 271a, 221 + 271a\}$,
$P_{51} = \{276a, 60 + 276a, 174 + 276a, 225 + 276a\}$,
$P_{52} = \{279a, 60 + 279a, 175 + 279a, 227 + 279a\}$,
$P_{53} = \{226a, 60 + 226a, 120 + 226a, 173 + 226a\}$,
$P_{54} = \{228a, 60 + 228a, 120 + 228a, 174 + 228a\}$;
$P_{117} = \{0, 258, 375 + 786a, 492 + 786a, 750 + 786a, 867 + 786a\}$,
$P_{118} = \{0, 261 + 775a, 379 + 775a, 497 + 775a, 876 + 775a\}$,
$P_{119} = \{0, 264+1012a, 383+1012a, 502+1012a, 766+1012a, 885+1012a, 1027+1012a, 1146+1012a\}$,
$P_{120} = \{0, 267+1267a, 387+1267a, 507+1267a, 774+1267a, 894+1267a, 1026+1267a, 1146+1267a, 1266 + 1267a\}$,
$P_{121} = \{512a, 270 + 512a, 391 + 512a\}$, $P_{122} = \{517a, 273 + 517a, 395 + 517a\}$,
$P_{123} = \{522a, 276 + 522a, 399 + 522a\}$, $P_{124} = \{525a, 277 + 525a, 401 + 525a\}$,
$P_{125} = \{530a, 280 + 530a, 405 + 530a\}$, $P_{126} = \{534a, 282 + 534a, 408 + 534a\}$,
$P_{127} = \{539a, 285 + 539a, 412 + 539a\}$, $P_{128} = \{542a, 286 + 542a, 414 + 542a\}, \ldots; a \in \mathbb{Z}_{\geq 0}$.

For $m = 4$ we have $P_k = \{0\}$ when $2 \leq k \leq 18$, $P_k = \{0, 21\}$ for $k = 19$ and $P_k = \{0, 21, 44\}$ for $k = 20$, while $P_k = \{ka\}$ for $k \in \{21, 23 - 37, 79 - 128\}$, $P_k = \{79a\}$ for $72 \leq k \leq 79$, and $P_k = \{0, 3(k - 12) + ka\}$ for $k \in \{38 - 41, 43\}$; furthermore

$P_{22} = \{68a, 23 + 68a, 46 + 68a\}$;
$P_{42} = \{0, 90 + 131a, 132 + 131a, 174 + 131a\}$;
$P_{44} = \{0, 96 + 139a, 140 + 139a, 184 + 139a\}$,
$P_{45} = \{0, 99 + 143a, 144 + 145a, 189 + 145a\}$,
$P_{46} = \{0, 102 + 147a, 148 + 147a, 194 + 147a\}$,
$P_{47} = \{0, 102 + 255a, 149 + 255a, 196 + 255a, 253 + 255a\}$,
$P_{48} = \{0, 161 + 262a, 209 + 262a, 257 + 262a, 316 + 262a\}$,
$P_{49} = \{0, 226 + 267a, 275 + 267a, 324 + 267a, 385 + 267a\}$,
$P_{50} = \{0, 199 + 274a, 249 + 274a, 299 + 274a, 362 + 274a\}$,
$P_{51} = \{0, 197 + 280a, 248 + 280a, 299 + 280a, 364 + 280a\}$,
$P_{52} = \{0, 138 + 287a, 190 + 287a, 242 + 287a, 309 + 287a\}$,
$P_{53} = \{0, 137 + 328a, 190 + 328a, 243 + 328a, 312 + 328a\}$,
$P_{54} = \{0, 136 + 363a, 190 + 363a, 244 + 363a, 315 + 363a, 370 + 363a, 424 + 363a\}$,
$P_{55} = \{0, 79 + 183a, 134 + 183a, 189 + 183a\}$,
$P_{56} = \{0, 79 + 187a, 135 + 187a, 191 + 187a\}$,
$P_{57} = \{0, 79 + 191a, 136 + 191a, 193 + 191a\}$,
$P_{58} = \{195a, 79 + 195a, 137 + 195a\}$,

$P_{59} = \{0, 79 + 195a, 138 + 195a, 197 + 195a\}$,
$P_{60} = \{199a, 79 + 199a, 139 + 199a\}$,
$P_{61} = \{201a, 79 + 201a, 140 + 201a\}$,
$P_{62} = \{204a, 79 + 204a, 142 + 204a\}$,
$P_{63} = \{205a, 79 + 205a, 142 + 205a\}$,
$P_{64} = \{630a, 79 + 630a, 222 + 630a, 286 + 630a, 359 + 630a, 502 + 630a, 566 + 630a\}$,
$P_{65} = \{568a, 79 + 568a, 223 + 568a, 288 + 568a, 359 + 568a, 438 + 568a, 503 + 568a\}$,
$P_{66} = \{358a, 79 + 358a, 226 + 358a, 292 + 358a\}$,
$P_{67} = \{362a, 79 + 362a, 228 + 362a, 295 + 362a\}$,
$P_{68} = \{367a, 79 + 367a, 231 + 367a, 299 + 367a\}$,
$P_{69} = \{296a, 79 + 296a, 158 + 296a, 227 + 296a\}$,
$P_{70} = \{298a, 79 + 298a, 158 + 298a, 228 + 298a\}$,
$P_{71} = \{300a, 79 + 300, 158 + 300a, 229 + 298a\}, \ldots$, where $a \in \mathbb{Z}_{\geq 0}$.

One can also introduce another parameter $\ell \in \mathbb{Z}_{\geq 1}$ and require $|n' - n''| < \ell$.

In particular, $\ell = 1$ means that in the beginning the Next player can take any number of stones $(s \mid 0 < s < k)$, yet, after this both players are allowed to take only the same number of stones s. Obviously, in this case, n is a P-position if and only if $\lfloor n/s \rfloor$ is even for all $\{s \mid 0 < s < k\}$. For example, if $k = 4$ or $k = 5$, the P-positions are formed by the union of two arithmetic progressions: $\{12a, 8 + 12a\}$ or, respectively, $\{24a, 8 + 24a\}$, where $a \in \mathbb{Z}_{\geq 0}$. If $k = 6$ then the P-positions are formed by the union of six arithmetic progressions: $\{120a,\ 24+120a,\ 32+120a,\ 72+120a,\ 80+120a,\ 104+120a \mid a \in \mathbb{Z}_{\geq 0}\}$ with the same difference 120.

For $\ell = 2$ and $4 \leq k \leq 8$ computations show that the P-positions are respectively:
$P_4 = \{5a\}$, $P_5 = \{8a\}$, $P_6 = \{0, 21 + 18a\}$, $P_7 = \{0, 26 + 23a\}$, $P_8 = \{26a\}$, where $a \in \mathbb{Z}_{\geq 0}$.

For $\ell = 3$ and $k \in \{4 - 10, 14\}$ computations show that the P-positions are respectively:
$P_4 = \{4a\}$, $P_5 = \{6a\}$, $P_6 = \{9a\}$, $P_7 = \{0, 13+10a\}$, $P_8 = \{49a, 13+49a, 29+49a, 39+49a\}$,
$P_9 = \{0, 19 + 71a, 29 + 71a, 45 + 71a, 61 + 71a\}$, $P_{10} = \{19, 39, 49, 67a, 114 + 67a\}$,
$P_{14} = \{0, 121 + 146a, 169 + 146a, 217 + 146a\}$, where $a \in \mathbb{Z}_{\geq 0}$.

For $\ell = 4$ and $4 \leq k \leq 17$ computations show that the P-positions are respectively:
$P_4 = \{4a\}$, $P_5 = \{5a\}$, $P_6 = \{7a\}$, $P_7 = \{10a\}$, $P_8 = \{24a, 14 + 24a\}$, $P_9 = \{14a\}$,
$P_{10} = \{46a, 35 + 46a\}$, $P_{11} = \{87a, 35 + 87a, 73 + 87a\}$,
$P_{12} = \{136a, 40 + 136a, 58 + 136a, 118 + 136a\}$, $P_{13} = \{0, 28, 91, 108 + 85a, 176 + 85a\}$,
$P_{14} = \{0, 28, 100, 142, 189, 264 + 47a\}$,
$P_{15} = \{0, 28, 228, 334 + 430a, 382 + 430a, 530 + 430a, 658 + 430a\}$,
$P_{16} = \{0, 28, 190, 274, 302, 629 + 301a, 827 + 301a\}$,
$P_{17} = \{0, 324 + 819a, 822 + 819a\}$, where $a \in \mathbb{Z}_{\geq 0}$.

Finally, one can consider all three parameters k, ℓ, m such that $m < l < k$, simultaneously.

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References

[1] E.R. Berlekamp, J.H. Conway, and R.K. Guy, Winning ways for your mathematical plays, vol.1-4, second edition, A.K. Peters, Natick, MA, 2001–2004.

[2] T.S. Ferguson, On sums of graph games with last player loosing, Int. J. of Game Theory 3 (1974), 159-167.

[3] A.S. Fraenkel, The vile, dopy, evil and odious game players, Discrete Math. 312 (2012) 42–46.

[4] V. Gurvich, Miserable and strongly miserable impartial games, RUTCOR Research Report 18-2011, Rutgers University.

[5] J. von Neumann and O. Morgenstern, Theory of games and economic behavior, Princeton University Press, Princeton, NJ, 1944.