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EUCLIDEAN TRAVELING SALESMAN ON ORIENTABLE SURFACES

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Abstract. Following closely Arora's technique in [1], we obtain a randomized PTAS to the traveling salesman problem on Euclidean graphs embedded on orientable surfaces.

1   Introduction

Our main goal is to give a randomized PTAS to the traveling salesman problem on Euclidean graphs embedded on orientable surfaces. We will do this by following closely Arora's technique introduced in [1]. The ideas about how to model an embedding of a graph in an orientable surface are based on [2]. The author would like to stress the fact that this note is an extension that mimics closely [1], therefore we encourage the reader to look into [1] for more details of the PTAS generation technique.

The problem to consider is: Given d pairs of handles and n points in the plane, find the shortest tour that visits all nodes. In order to make things easier we will restrict to instances of the problem that are well rounded.

Definition 1. *An instance of the TSP on an orientable surface is well rounded if*

1. *all nodes have integral coordinates,*

2. *all handle ends are triangles with sides of length less than 1,*

3. *all handle ends are positioned inside an unit square with integral corners,*

4. *each nonzero internode, (handle end)-node or (handle end)-(handle end) distance is at least 8, and*

5. *the maximum of all internode, (handle end)-node or (handle end)-(handle end) distance is $O(n)$.*

Given an instance I the *bounding box* of I is the smallest square on the plane that is aligned with the x-axis and contains inside it all the handles and points of I. The length of a side of the bounding box is known as the *size* of the bounding box and is denoted by L. We continue by defining some important concepts.

Definition 2. *A* dissection *of the bounding box is a recursive partitioning into smaller squares. We view it as a 4-ary tree whose root is the bounding box. Each square in the tree is partitioned into four equal squares, which are its children. We stop partitioning a square if it has size at most one.*

There are $O(L^2)$ squares in the dissection and its depth is $\log L$. Each square of the dissection either is empty, contains one node, or contains one handle end.

Definition 3. *An* extended quadtree *is defined similarly but we stop the recursive partitioning as soon as the square contains exactly one node, contains exactly one handle end, or is empty.*

The extended quadtree, in general, may have fewer squares than the dissection. In fact, since each leaf either contains a node, contains a handle end or is the sibling of such a square, the quadtree has $O(n + d)$ leaves and thus $O((n + d) \log L)$ squares in all.

Definition 4. *If a, b are integers in $[0, L]$, then the (a, b)-shift of the dissection is defined by shifting the x- and y- coordinates of all lines by a and b respectively, and then reducing modulo L.*

Definition 5. *The extended quadtree with shift (a, b) is obtained from the corresponding shifted dissection by cutting of the partitioning at squares that contain exactly one node, contains exactly one handle end, or are empty.*

Definition 6. *Let m be a positive integer. An m-regular set of portals for a shifted dissection is a set of points on the edges of the squares and the handle ends in it. Each square or handle end has a portal at each of its corners and m other equally-spaced portals on each of its edges.*

Definition 7. *A salesman path is a path in $\mathcal{S}$ that visits all the input nodes, and some subset of portals.*

Salesman paths may visit a portal more than once.
   Now we define some related problems that will allow us to simplify the original TSP.

Definition 8. *A salesman path P is (m, r, t)-light with respect to the shifted dissection if*

   1. *P crosses each edge of a square in the dissection at most r times,*

   2. *P crosses each edge of a handle end in the dissection at most t times, and*

   3. *all of the square and handle end crosses take place at portals.*

   Between visits to two successive input nodes i and j, the tour may cross region boundaries or handles and therefore has to pass through a sequence of portals $p_1, \ldots, p_{k-1}$. Thus the "edge" from i to j consists of line segments $P_1 = (i, p_1), \ldots, P_k = (p_{k-1}, j)$. This give us the notion of the (i, j)-edge being *bent* at the portals $p_1, \ldots, p_{k-1}$. We will assume w.l.o.g. that no node lies on the boundary of any region in the dissection.

1.1   Computing a Shifted Quadtree

The following steps are performed when computing a shifted quadtree.

   1. Pick a shift (a, b) randomly.

   2. Compute a quadtree with these shifts. Can be done in $O((n + d) \log^2 n)$ time.

   3. The bounding box has size $L = O(n)$ (by property 5 of instance definition). So, the depth of tree is $O(\log n)$.

   4. Number of squares in the tree is

$$K = O(\# \text{ of leaves } \times \text{depth}) = O((n + d) \log n).$$

2   Dynamic Programming

The main part of our algorithm relies on a dynamic programming technique. In order to understand its correctness we have the following main observation.

MAIN OBSERVATION:

Suppose S is a square of the shifted quadtree containing d' handle ends and the optimal (m, r, t)-light path crosses the boundary and handles of S a total of $2p \leq 4r + 3td'$ times ($4r$ crossings on the exterior boundary of S and $3t$ crossings per handle end.) Let $a_1, \ldots, a_{2p}$ be the sequence of portals where these crossings occur in the order that they appear in the salesman path.

The portion of the optimal salesman path inside S is a sequence $P_1 \ldots, P_p$ of paths such that

1. P_i connects a_{2i-1} to a_{2i},

2. together the paths visit all nodes that lie inside S, and

3. the collection of paths cross each edge of each square in the quadtree restricted to S at most r times and it crosses each edge of each handle end in S at most t times.

Definition 9. *An instance of the (m, r, t)-multipath problem is given by*

1. *a nonempty square S in the shifted quadtree,*

2. *a multiset $\mathcal{R}$ of at most r portals on each of the four edges of S,*

3. *a multiset $\mathcal{T}$ of at most t portals on each of the three edges of every handle end in S,*

4. *the sum of the multisets is $2p \leq 4r + 3td'$, where d' is he number of handle ends in S, and*

5. *a pairing $\mathcal{P} = (a_1, a_2), (a_3, a_4) \ldots (a_{2p-1}, a_{2p})$ on the $2p$ portals specified in items 2 and 3.*

The goal is to find a minimum cost collection $P_1, \ldots, P_p$ of paths in S such that:

1. *P_i connects a_{2i-1} to a_{2i}.*

2. *Together the paths visit all nodes that lie inside S.*

3. *The collection of paths cross each edge of each square in the quadtree restricted to S at most r times and it crosses each edge of each handle end in S at most t times.*

4. *All the square or handle crosses happen exclusively at portals.*

The dynamic programming builds a lookup table containing the costs of the optimal solutions to all instances of the (m, r, t)-multipath problem arising form the quadtree. The solution is obtained from picking up the best from the entries that correspond to the root of the tree (i.e the bounding box) having the multiset $\mathcal{R}$ empty.

The number of entires in the lookup table is

$$O\left(K \cdot d(m + 3)^{4r+3t} \cdot (4r + 3t)!\right).$$

To see this we count: for each of K squares there are at most $(m + 3)^{4r}$ ways of selecting $\mathcal{R}$, there are at most $d(m + 3)^{3r}$ ways of selecting $\mathcal{T}$, and each multiset of portals can be ordered in at most $(4r + 3t)!$ ways.

The table is built from the bottom up. There are three different kind of instances at the leaves:

1. An empty square with at most $O(r)$ selected portals at the edges. Since the order of traveling the portals is predefined it takes $O(r)$ time to construct the unique optimal solution to this instance.

2. A square with one handle end inside with at most $O(t)$ selected portals at the handle and at most $O(r)$ selected portals at the square edges. Again, in this instance everything is prefixed and it takes $O(r + t)$ time to construct the unique optimal solution.

3. A square with one node inside and $O(r)$ selected portals at the edges. Then by trying all r ways of placing the single node in the specified $O(r)$ paths, the optimal solution can be found in $O(r)$ time.

So it takes $O(r + t)$ time to solve the leaves optimally.

Suppose the algorithm has solved all (m, r, t)-multipath problems for all squares at depth less that i. Let S be a square at depth i, let $\mathcal{R}, \mathcal{T}$ be multisets of portals in the edges of S and its handle ends, respectively, and let $\mathcal{P} = (a_1, a_2), (a_3, a_4) \ldots (a_{2p-1}, a_{2p})$ be a pairing of these selected portals. Furthermore, let S_1, S_2, S_3, S_4 be the four children of S in the quadtree. The algorithm enumerates all possible ways in which an (m, r, t)-multipath could cross the edges and handles of S_1, S_2, S_3, S_4. We do:

1. Choose a multiset of at most r portals on each of the four inner edges of the children, these are called the *new portals*. There are at most $(m + 3)^{4r}$ ways of doing this. Remember that the portals in the edges of S and its handle ends are fixed and so is their traversal order.

2. Choose an order to traverse the new portals:

    (a) First for each new portal we choose in which of the p paths defined by the pairing $\mathcal{P}$ they will lay. This is bounded by $(4r + 3td)^{4r}$.

    (b) Then choose the order in which the new portals appear in their respective path. There are ate most $(4r)!$ ways of doing this.

Each choice in (a) and (b) leads to a (m, r, t)-multipath problem in the four children, whose optimal solutions have already been calculated and stored in the lookup table. Thus, the running time of the algorithm is

$$O(K \cdot d(m + 3)^{8r+3t}(4r + 3d)^{4r}(4r)!(4r + 3t)!).$$

Now, choose $m = O(c \log L)$, $r = O(c)$, and $t = r$. We know that $K = O((n + d) \log n)$ and $L = O(n)$. Therefore the running time of the algorithm is

$$O\left((n + d) \left[\log (n + d)\right]^{O(c)}\right).$$

Th following theorem tell us that the structure of the (m, r, t)-light salesman tour is good enough to find a good approximation with probability $1/2$. Once we prove it we will have finalized our goal of obtaining a randomized PTAS for the well rounded TSP problem on surfaces.

Theorem 1 (Structure Theorem). *Let $c > 0$ be any constant. Let the minimum nonzero internode, (handle end)-node or (handle end)-(handle end) distance in a TSP instance be at least 8. Let L be the size of the bounding box and shifts $a, b \in [0, L]$ be picked uniformly at random. Then with probability at least $1/2$, there is a salesman path of cost at most $(1 + 1/c)OPT$ that is (m, r, t)-light with respect to the dissection with shift (a, b), where $m = O(c \log L)$ and $r = O(c)$.*

From now on, the rest of the paper is dedicated to the proof of the Structure Theorem.

3 The Structure Theorem

3.1 Patching Lemma

The following lemma is fundamental in the proof of the Structure Theorem. It will be stated without a proof.

Lemma 1 (Patching Lemma). *There is a constant $g > 0$ such that the following is true. Let S be any line segment of length s and π be a closed path that crosses S at least thrice. Then there exist line segments on S whose total length is at most $g \cdot s$ and whose addition to π changes it into a closed path π' that crosses S at most twice.*

In [1] it was shown that $g \leq 6$.

3.2 Approximation Lemma

Consider a grid G in the bounding box obtained by placing vertical and horizontal lines at unit distance from one another. If l is one of the lines in grid G and π is a salesman tour, then let $t(\pi, l)$ denote the number of times that π crosses l.

Lemma 2. *If the minimum internode, (handle end)-node or (handle end)-(handle end) distance is at least 4, and T is the length of π, then*

$$\sum_{l \ vertical} t(\pi, l) + \sum_{l' \ horizontal} t(\pi, l') \leq 2T. \tag{1}$$

Proof. Let e be an edge of π of length s. The edge e is possibly divided into segments of the form node→handle, handle→handle, or handle→node. Let $e_1, \ldots, e_k$ be the segments of e and let s_i be the length of e_i.

Suppose that u_i, v_i are the lengths of the vertical and horizontal projections of e_i. Note that $u_i^2 + v_i^2 = s_i^2$. The segment e_i contributes to the left-hand side of (1) by at most $(u_i + 1) + (v_i + 1)$, and

$$u_i + v_i + 2 \leq \sqrt{2(u_i^2 + v_i^2)} + 2 \leq \sqrt{2s_i^2} + 2 \leq 2s_i,$$

where the last inequality is obtained because $s_i \geq 4$. Let $C(e)$ denote the contribution of e to the left-hand side of (1). Then

$$C(e) \leq \sum_{i=1}^{k}(u_i + v_i + 2) \leq \sum_{i=1}^{k} 2s_i = 2s$$

and the desired result is obtained. $\square$

3.3   Structure Theorem

Theorem 2 (Structure Theorem). *Let $c > 0$ be any constant. Let the minimum nonzero internode, (handle end)-node or (handle end)-(handle end) distance in a TSP instance be at least 8. Let L be the size of the bounding box and shifts $a, b \in [0, L]$ be picked uniformly at random. Then with probability at least $1/2$, there is a salesman path of cost at most $(1 + 1/c)OPT$ that is (m, r, t)-light with respect to the dissection with shift (a, b), where $m = O(c \log L)$ and $r = O(c)$.*

Proof. Let $s = 24gc$ where g is the constant appearing in the Patching Lemma and let $r = s + 4$, $u \geq 0$, $m \geq 2s \log L$.

Let π^* be an optimal salesman tour. We will apply a series of operations to π^* and obtain a modified salesman tour π that is $(m, r, t) - light$. Obviously, the total length of π is at least that of π^*, but in our procedure we will be able to bound in the expectation the increase in length. First we describe the procedure.

Place a grid of unit granularity in the bounding box and assume without loss of generality that the size of the bounding box L is a power of 2. Thus, all lines used in the dissection are grid lines. Since a, b are integers the lines of the shifted dissection are also gird lines.

Recall that the squares in the bounding box form a hierarchy, and have a natural notion of *level* (the bounding box is at level 0, its four children are at level 1, and so on.) A grid

line has *level i* in the shifted dissection if it contains the edge of some level i square. Note that the edges of level i get subdivided to yield the edges of level $i + 1$, so, a line that is at level i is also at level j for all $j > i$. For each $i \geq 0$ there are $2^i + 1$ horizontal lines and $2^i + 1$ vertical lines at level i. The vertical lines have x-coordinates $[a + p \cdot L/2^i] \bmod L$, where $p = 0, 1, \ldots, 2^i$ and the horizontal lines have y-coordinates $[b + p' \cdot L/2^i] \bmod L$, where $p' = 0, 1, \ldots, 2^i$. Define the *minimal level* of a line to be the lowest level it is at.

Since the horizontal shift a is chosen uniformly form $[0, L]$, we have for each vertical line l in the grid and each $i \leq \log L$ (recall that we have $\log L$ levels),

$$\Pr[l \text{ is at level } i] = \frac{2^i + 1}{L}.$$

The same is true for horizontal lines.

In terms of the grid a path ζ is (m, s, t)-light if:

(i) For each vertical grid line l, if i is the minimal level of l, then for $p = 0, 1, \ldots, 2^i$ the segment of l lying between the y-coordinates $[b + p \cdot L/2^i] \bmod L$ and $[b + (p + 1)L/2^i] \bmod L$ is crossed by ζ at most s times.

(ii) For each horizontal grid line l, if i is the minimal level of l, then for $p = 0, 1, \ldots, 2^i$ the segment of l lying between the x-coordinates $[a + p \cdot L/2^i] \bmod L$ and $[a + (p + 1)L/2^i] \bmod L$ is crossed by ζ at most s times.

(iii) ζ crosses each handle end edge at most t times.

(iv) All the grid crossings of ζ occur at portals.

First we will modify π^* to satisfy condition (i) by applying the procedure MODIFY(l, i, b) for each vertical line l.

MODIFY(l, i, b):
   Input: l is a vertical line, b is the vertical shift of the dissection, and i is the minimal level of i.
   for $j = \log L$ down to i **do**
      for all $p = 0, 1, \ldots, 2^j$ **do**
         if the segment of l between the y coordinates $[b + p \cdot L/2^i] \bmod L$ and $[b + (p + 1)L/2^i] \bmod L$ is crossed by the current salesman path more than s times, then use the Patching Lemma to reduce the number of crossings to 4.
      end for
   end for
Remark:

1. *The Patching Lemma reduces the number of crossings per segment to 4 because if the segment is "wrapped around" the patching has to be done separately for its two parts.*

2. *The patching on a vertical line adds to the cost of the salesman path and could increase the number of times the path crosses a horizontal line. We ignore this effect for now and explain later in the proof how to deal with this.*

To modify π^* to satisfy condition (ii) we apply an analogous procedure to each horizontal line l' (we omit the details.) In order to modify π^* to satisfy (iii) we apply the Patching Lemma to each edge of a handle that is crossed by the current salesman path more than 2 times. In these edges the number of crossings will be reduced to 2.

Call π' the salesman path obtained at this stage. The only missing property from π' is that of all crosses occurring only at portals. Before describing how to fix this we will analyze the cost of increase so far.

If $j \geq i$, let $c_{l,j}(b)$ be the number of segments to which we apply the patching lemma in the iteration corresponding to j in the outermost **for** loop in MODIFY(l, i, b). It is important to notice that $c_{l,j}(b)$ is independent of i because of the order of computation in the loop. Let l be a line of the unit grid. The optimal salesman tour crossed l, $t(\pi^*, l)$ times and each application of the Patching Lemma to l replaced at least $s + 1$ crossings by at most 4. Therefore $(s - 3) \sum_{j \geq 1} c_{l,j}(b) \leq t(\pi^*, l)$ and so

$$\sum_{j \geq 1} c_{l,j}(b) \leq \frac{t(\pi^*, l)}{s - 3}.$$

The increase in length due to the application of MODIFY(l, i, b) can be estimated using the Patching Lemma as follows:

$$\text{Increase in length due to MODIFY}(l, i, b) \leq \sum_{j \geq 1} c_{l,j}(b) \cdot g \cdot \frac{L}{2^i},$$

where g is the constant appearing in the Patching Lemma. We will "charge" this cost to l and later get the total cost increase by adding the charges of all lines l to the increase due to the handles.

Whether or not we actually incur in the cost charged to l depends on i being the minimal level of l, which happens with probability $(2^i + 1)/L$ over the choice of the horizontal shift

a. Thus, for every vertical line l and every $0 \leq b \leq L$,

$$E\left[\text{charge to } l \text{ when horizontal shift is } a\right]$$

$$= \sum_{i \geq 1} \frac{2^i + 1}{L} \cdot \text{cost increase due to } \text{MODIFY}(l, i, b)$$

$$\leq \sum_{i \geq 1} \left[\frac{2^{i+1}}{L} \cdot \sum_{j \geq i} \left(c_{l,j}(b) \cdot g \cdot \frac{L}{2^j} \right) \right]$$

$$= 2g \cdot \sum_{j \geq 1} \left[\frac{c_{l,j}(b)}{2^j} \cdot \sum_{i \leq j} 2^i \right]$$

$$\leq 2g \cdot \sum_{j \geq 1} 2 \cdot c_{l,j}(b)$$

$$\leq \frac{4g\, t(\pi^*, l)}{s - 3}.$$

We do similar for the horizontal lines l' and get,

$$E\left[\text{charge to } l' \text{ when vertical shift is } b\right] \leq \frac{4g\, t(\pi^*, l')}{s - 3}.$$

Clearly the increase in length due to the application of the Patching Lemma to the handles is independent of the (a, b)-shift. Remember that we have d pairs of handle ends, each having 3 edges of length at most one. Then

$$E\left[\text{increase in length due to handles}\right] \leq 3dg.$$

The next modification to the current salesman tour π' consists in moving each crossing to its nearest portal. If a line l has minimal level i, then each of the $t(\pi', l)$ crossings might have to be displaced by $L/2^{i+1}m$ (i.e. half the interportal distance) to reach the nearest portal. We break the edge at he point where it crosses l and add to it two line segments (one on each side of l) of length at most $L/2^i m$, so that the new edge crosses l at a portal. We do similarly for crossings at the handles but taking care that now the interportal distance is at most $1/(m+1)$ and the two "sides" of an edge are the two corresponding identified edges in the identified handle ends.

Call π the resulting salesman tour. Then

$$E[\text{increase by moving crossings in } l \text{ to portals}]$$

$$= \sum_{i=1}^{\log L} \frac{2^i + 1}{L} \cdot t(\pi', l) \cdot \frac{L}{2^i m}$$

$$\leq \sum_{i=1}^{\log L} \frac{2^{i+1}}{L} \cdot t(\pi', l) \cdot \frac{L}{2^i m}$$

$$\leq 2 \sum_{i=1}^{\log L} \frac{2^i}{L} \cdot t(\pi^*, l) \cdot \frac{L}{2^i m}$$

$$= 2 \frac{t(\pi^*, l) \log L}{m}$$

$$\leq \frac{t(\pi^*, l)}{s},$$

if $m \geq 2s \log L$. Thus the expected cost (over the random shift a) of transforming π^* into an (m, s, u)-light salesman path charged at line l is

$$\frac{4gt(\pi^*, l)}{s - 3} + \frac{t(\pi^*, l)}{s} \leq \frac{6gt(\pi^*, l)}{s},$$

where we assume that $s > 15$.

There are d handles, each handle edge has at most 2 crossings, and the portals in each handle edge are at distance at most $1/(m + 1)$. Therefore,

$$E[\text{increase by moving crossings in handles to portals}]$$

$$\leq 3d \cdot 2 \cdot \frac{1}{(m + 1)} = \frac{6\,d}{(m + 1)} \leq 6d.$$

At this point we will explain the remark about the procedure MODIFY(l, i, b). Whenever we apply MODIFY on a vertical line l, we use the Patching Lemma and augment the salesman path with some segments lying on l. These segments could cause the path to cross some horizontal line l' more than the $t(\pi^*, l')$ times it was crossed earlier. However, our analysis assumed that the number of crossings remains constant at $t(\pi^*, l')$ throughout the modification.

To solve this problem we call the Patching Lemma on those segments of l where the increase in crossings of l' is more than 2 as a result of our previous procedure. This will reduce the number of crossings of l' (in those segments) to 2 and since the Patching lemma is being invoked for segments lying on l, these have zero horizontal separation and so the tour cost does not increase. Also, we apply the patching separately on both sides of l, so the number of crossings on l does not change. Doing this for all pairs of grid lines, we can ensure that at the end of the modifications, each side of each square in the shifted dissection

is crossed by the modified tour up to $s + 4$ times. So, we obtained an (m, r, t)-light salesman path having the same tour length as π. From now on we call this modification π.

Now we analyze the total expected increase. We just saw that

$$E[\text{charge to line } l \text{ when shift is } (a, b)] \leq \frac{6gt(\pi^*, l)}{s}.$$

and

$$E[\text{increase due to portals when shift is } (a, b)] \leq 3dg + 6d$$

Let $k = 3dg + 6d$. By linearity of expectation, it then follows that the expected increase in length of π^* is at most

$$\sum_{l \text{ vertical}} \frac{6gt(\pi^*, l)}{s} + \sum_{l' \text{ horizontal}} \frac{6gt(\pi^*, l')}{s} + k,$$

which is at most $12g\,OPT/s + k$ by Lemma 2. Since $s \geq 24gc$, the expected increase in the tour cost is at most $OPT/2c + k$. Let X denote the random variable that counts the increase in length. Then Markov's inequality gives

$$\begin{aligned}
Pr\left[|X| \leq \frac{OPT}{c}\right] &\geq \frac{E[X]}{OPT/c} \\
&\geq \frac{OPT/2c + k}{OPT/c} \\
&\geq \frac{1}{2} + \frac{c\,k}{OPT} \\
&\geq \frac{1}{2},
\end{aligned}$$

i.e. with probability at least $1/2$ the increase is no more than OPT/c. We conclude that with probability at least $1/2$ the cost of the best (m, r, t)-light salesman path for the shifted dissection is at most $(1 + 1/c)OPT$. $\qquad\square$

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