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STOCHASTIC NETWORK INTERDICTION

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Abstract. We introduce the network interdiction problem and show a reformulation of it in the form of a two-stage stochastic problem. We also use the theory of coherent risk measures to develop a risk-averse variant of the network interdiction problem.

1   Introduction:

The interdiction problem arises in a variety of areas including military logistics, national security, infectious disease control, and counter-terrorism. In the typical formulation of network interdiction, the task of the interdictor is to find a set of edges in a weighted network such that the removal of those edges would maximally increase the cost to an evader of traveling on a path through the network. In our version of the problem we play the role of the interdictor on a network with a given set of sources and destinations where each edge has assigned a probability of blocking and a cost of monitoring. Our objective is to block with high probability an unknown source-destination path traveled by the evader while satisfying certain budgetary constraints. This problem is known as the *shortest path interdiction problem.*

   The goal of this paper is to develop a 2-stage approach for the modeling and solution of the problem. Under this paradigm we select, before knowing the source and destination of the evader, a set of fixed monitoring stations and later, when the source and destination of the path are known, we assign a second set of monitoring stations. These two sets of monitoring stations might have different budget requirements. The purpose is to block the path with high probability while following some budgetary constraints. The first set of monitoring stations can be seen as fixed stations and the second can be interpreted as part of movable task force of monitoring stations which could be deployed in the event of a known threat (a known source-destination path).

2   Problem Definition

Let $G = (V, E)$ be a finite undirected network with vertex set V and edge set $E = \{e_1, \ldots, e_n\}$. Let $S, D \subseteq E$ be such that $S \cap D = \emptyset$. We call S the *set of sources* and D the *set of destinations* of the network G. The uncertainty of blocking the evader's path is represented by a probability vector $p \in \mathbb{R}^E$, where for each edge e, p_e is the probability of blocking the evader at e if the edge e is currently being monitored. A *monitoring policy* is a binary vector $x \in \{0, 1\}^E$ where $x_e = 1$ if and only if the edge e is being monitored. Let $c \in \mathbb{R}^E_{\geq 0}$ be such that c_e is the cost of setting up a monitoring station at the edge e.

   Then the *shortest path interdiction problem* is the problem of finding the policy that maximizes the probability of blocking the evader (assuming that he takes a shortest path) while satisfying a budget constraint, i.e.

Problem 1. (Shortest Path Interdiction Problem)

$$
\begin{aligned}
\underset{x}{\text{maximize}} \quad & P\,(\textit{blocking the evader under policy } x) \\
\text{subject to} \quad & c^\top x \leq b \\
& x \in \{0, 1\}^E,
\end{aligned}
$$

where b is the total budget alloted for the creation of monitoring stations. For the purpose of this paper we will focus on the *relaxed shortest path interdiction problem* which is the same as Problem 1 but with the integer constraint relaxed, i.e.

Problem 2. (Relaxed Shortest Path Interdiction Problem)

$$\begin{aligned}
\underset{x}{\text{maximize}} \quad & P\left(\text{blocking the evader under relaxed policy } x\right) \\
\text{subject to} \quad & c^\top x \le b \\
& 0 \le x_e \le 1, \ \forall e \in E.
\end{aligned}$$

We could interpret this problem as allowing partial allocation of resources per station with the draw back that the probability of blocking at the station will be affected by the partial resources (we will see later how too do this in a meaningful way). Because of this, we define a *relaxed monitoring policy* as a vector $x \in [0,1]^E$.

3 The Blocking Probability

In this section we show a closed formula for the probability of blocking the evader under a given policy.

For a pair of sets $A \subseteq S, B \subseteq D$ let $P_{A,B}(x)$ be the probability of blocking the evader under the monitoring policy x given that the evader takes an a-b shortest path for some $a \in A$ and $b \in B$. In this notation the probability of blocking the evader under policy x is simply denoted by $P_{S,D}(x)$. For every source and destination pair $s \in S, d \in D$ and s-d shortest path T, let $P_s(d)$ be the probability of the evader having d as destination given that his starting point is s and let $P_{s,d}(T)$ be the probability of the evader taking the path T if his source and destinations are s, d. Also, let $\mathcal{P}_{s,d}$ be the set of all s-d shortest paths.

For every monitoring policy x

$$\begin{aligned}
P_{S,D}(x) &= \sum_{s \in S} P(s) P_{S,D}\left(x \mid s\right) \\
&= \sum_{s \in S} P(s) P_{s,D}\left(x\right) \\
&= \sum_{s \in S} P(s) \left(\sum_{d \in D} P\left(d \mid s\right) P_{s,D}\left(x \mid d\right) \right) \\
&= \sum_{s \in S} P(s) \left(\sum_{d \in D} P_s\left(d\right) P_{s,d}\left(x\right) \right) \\
&= \sum_{s \in S} P(s) \left(\sum_{d \in D} P_s\left(d\right) \left(\sum_{T \in \mathcal{P}_{s,d}} P_{s,d}(T) P_{s,d}\left(x \mid T\right) \right) \right).
\end{aligned} \tag{1}$$

We assume that all the monitoring station are independent from each other and that once the evader is detected at a station he will be blocked there and stopped from traversing any more edges. Let T be an s-d shortest path. Suppose for a moment that the policy x takes only one edge, say e_1 from the path T. Then the probability of blocking the evader under policy x given that he is going from s to d through T is p_{e_1}. Suppose now that the policy x takes two edges from T, say e_1, e_2, and these are traversed in this order from s to d. Then the probability of blocking the evader under policy x given that he is going through T is the probability of blocking the evader at e_1 plus the probability of the evader slipping through the monitoring station at e_1 but being caught at station e_2, namely: $p_{e_1} + (1 - p_{e_1})p_{e_2}$.

Continuing in this way we conclude that the probability of blocking the evader under policy x given that his source and destination are s, d and he is taking the shortest s-d path $T = (e_1, \ldots, e_n)$ is

$$P_{s,d}\left(x \mid T\right) = \sum_{i=1}^{n} p_{e_i} x_{e_i} \left(\prod_{k=0}^{i-1} (1 - p_{e_k} x_{e_k})\right), \tag{2}$$

where $p_{e_0} = x_{e_0} = 0$. This is a nonlinear function for which its analysis is outside the scope of this paper. Instead we will assume that the probabilities of blocking the evader are regular, i.e. there is a $p \in [0, 1]$ such that for every $e \in E$, $p_e = p$. In this case (2) is greatly simplified and we obtain

$$\begin{aligned} P_{s,d}\left(x \mid T\right) &= \sum_{i=1}^{n} p x_{e_i} \left(\prod_{k=0}^{i-1} (1 - p x_{e_k})\right) \\ &= \sum_{i=1}^{T^\top x} p \left(\prod_{k=0}^{i-1} (1 - p)\right) \\ &= \sum_{i=1}^{T^\top x} p (1 - p)^{i-1} \\ &= \frac{1 - (1 - p)^{T^\top x}}{p}, \end{aligned} \tag{3}$$

where T is the edge incidence vector of the path T and $T^\top x$ is just the number of edges from T that the policy x is monitoring. It is important to see that $P_{s,d}\left(x \mid T\right)$ relates directly to how many edges of the path T the policy x monitors, this will come handy later when we attempt to find a linear model for our problem.

Putting together (1) and (2) we obtain that

$$P_{S,D}(x) = \sum_{s \in S} P(s) \left(\sum_{d \in D} P_s\left(d\right) \left(\sum_{T \in \mathcal{P}_{s,d}} P_{s,d}(T) \frac{1 - (1 - p)^{T^\top x}}{p}\right)\right). \tag{4}$$

Clearly $P_{S,D}(x)$ is a concave function and Problem 1 is properly defined. We define the probability of blocking the evader under a relaxed monitoring policy $0 \le z \le 1$ as $P_{S,D}(z)$.

Then it is not difficult to see that $P_{S,D}(z)$ is a concave function and Problem 2 is a concave optimization problem with linear constraints. This problem could be solved by applying cutting plane methods since $P_{S,D}(z)$ is differentiable and calculating its derivative is straightforward (although not too computationally efficient). Our goal is to get to a two-stage linear stochastic model so we will do something else instead.

4 The Two-Stage Linear Stochastic Model

In order to obtain a linear model for our problem we will replace in (4) the factors $(1 - (1 - p)^{T^\top x})/p$ by $T^\top x$ and define the linear function

$$E_{S,D}(x) = \sum_{s \in S} P(s) \left(\sum_{d \in D} P_s(d) \left(\sum_{T \in \mathcal{P}_{s,d}} P_{s,d}(T) \, T^\top x \right) \right). \tag{5}$$

We define the *shortest path linear interdiction problem* as

Problem 3. (Shortest Path Linear Interdiction Problem)

$$\begin{aligned}
\underset{x}{\text{maximize}} \quad & E_{S,D}(x) \\
\text{subject to} \quad & c^\top x \le b \\
& x \in \{0,1\}^E
\end{aligned}$$

and its relaxed version as follows

Problem 4. (Relaxed Shortest Path Linear Interdiction Problem)

$$\begin{aligned}
\underset{x}{\text{maximize}} \quad & E_{S,D}(x) \\
\text{subject to} \quad & c^\top x \le b \\
& 0 \le x_e \le 1, \ \forall e \in E.
\end{aligned}$$

Clearly Problem 3 is a linear optimization problem with binary constraints and Problem 4 is an linear program. This time we have to work more to obtain a meaningful interpretation of Problem 3.

First we should remark that Problem 3 is equivalent to the following problem

Problem 5.

$$\begin{aligned}
\underset{x,y}{\text{maximize}} \quad & \sum_{s \in S} P(s) \left(\sum_{d \in D} P_s(d) \left(\sum_{T \in \mathcal{P}_{s,d}} P_{s,d}(T) \, T^\top (x + y) \right) \right) \\
\text{subject to} \quad & c^\top (x + y) \le b \\
& x_e + y_e \le 1, \ \forall e \in E \\
& x, y \in \{0,1\}^E.
\end{aligned}$$

For every policy z and $s \in S, d \in D$ define the random variable $\chi_{s,d} : \mathcal{P}_{s,d} \to \mathbb{R}$ by

$$\chi_{z,s,d}(T) = T^\top z,$$

with probability $P_{s,d}(T)$. In other words, the random variable $\chi_{z,s,d}(T)$ counts how many edges of T are monitored by the policy x (with assigned probability $P_{s,d}(T)$). Let $\mathcal{Q}(x,s,d)$ be the optimal value of the *second stage* problem

$$
\begin{aligned}
\underset{y}{\text{maximize}} \quad & \sum_{T \in \mathcal{P}_{s,d}} P_{s,d}(T)\, T^\top (x+y) \\
\text{subject to} \quad & c^\top (x+y) \le b \\
& x_e + y_e \le 1, \ \ \forall e \in E \\
& y \in \{0,1\}^E.
\end{aligned}
$$

We can also see $\mathcal{Q}(x,s,d)$ as the optimal value of the optimization problem

$$
\begin{aligned}
\underset{y}{\text{maximize}} \quad & \mathbb{E}\left[\chi(x+y,s,d)\right] \\
\text{subject to} \quad & c^\top (x+y) \le b \\
& x_e + y_e \le 1, \ \ \forall e \in E \\
& y \in \{0,1\}^E,
\end{aligned}
\tag{6}
$$

so an optimal solution y^* of $\mathcal{Q}(x,s,d)$ gives the maximum expected number of monitoring stations of policy $x + y$. Basically, y^* is the best way of extending an initial selection of monitoring stations x if what we care about is to maximize (in the average) the number of monitoring stations in each s-d shortest path. Then it is not difficult to see that our main Problem 5 is equivalent to

Problem 6.

$$
\begin{aligned}
\underset{x}{\text{maximize}} \quad & \sum_{s \in S} P(s) \left(\sum_{d \in D} P_s(d)\, \mathcal{Q}(x,s,d) \right) \\
\text{subject to} \quad & x \in \{0,1\}^E,
\end{aligned}
$$

and this finally lead us to the two-stage stochastic programing reformulation of Problem 3

Problem 7. (Shortest Path Two-Stage Linear Interdiction Problem)

$$
\begin{aligned}
\underset{x}{\text{maximize}} \quad & \mathbb{E}_{s \in S}\left[\mathbb{E}_{d \in D} \left[\mathcal{Q}(x,s,d) \right] \right] \\
\text{subject to} \quad & x \in \{0,1\}^E.
\end{aligned}
$$

Here we are selecting a first set of monitoring stations based on the probability distribution of the sources and later in the second stage we finalize add more monitoring stations based on the probabilities of the destinations. To obtain a model that solves the problem proposed in the introduction where the two sets of monitoring stations have different costs we might add some more budget constraints as follows:

Problem 8.

$$\underset{x}{\text{maximize}} \quad \underset{s \in S}{\mathbb{E}} \left[\underset{d \in D}{\mathbb{E}} \left[\overline{\mathcal{Q}}(x, s, d) \right] \right]$$
$$\text{subject to} \quad c_1^\top x \leq b_1$$
$$x \in \{0, 1\}^E,$$

where $\overline{\mathcal{Q}}(x, s, d)$ is the solution to the modified second stage problem

$$\underset{y}{\text{maximize}} \quad \mathbb{E} \left[\chi(x + y, s, d) \right]$$
$$\text{subject to} \quad c_1^\top x + c_2^\top y \leq b$$
$$x_e + y_e \leq 1, \ \forall e \in E \qquad (7)$$
$$c_2^\top y \leq b_2$$
$$y \in \{0, 1\}^E.$$

Since this is an integer stochastic model we will concentrate on its linear relaxation:

Problem 9. (Relaxed Shortest Path Two-Stage Linear Interdiction)

$$\underset{x}{\text{maximize}} \quad \underset{s \in S}{\mathbb{E}} \left[\underset{d \in D}{\mathbb{E}} \left[\overline{\mathcal{Q}}(x, s, d) \right] \right]$$
$$\text{subject to} \quad c_1^\top x \leq b_1$$
$$0 \leq x_e \leq 1, \ \forall e \in E$$

where $\overline{\mathcal{Q}}(x, s, d)$ is the solution to the relaxed second stage problem

$$\underset{y}{\text{maximize}} \quad \mathbb{E} \left[\chi(x + y, s, d) \right]$$
$$\text{subject to} \quad c_1^\top x + c_2^\top y \leq b$$
$$x_e + y_e \leq 1, \ \forall e \in E \qquad (8)$$
$$c_2^\top y \leq b_2$$
$$0 \leq y_e \leq 1, \ \forall e \in E.$$

Problem 10 is a two-stage linear stochastic problem which we could solve using classical techniques such as the cutting plane or proximal point methods. The development of such methods are outside the scope of this paper but we can remark that such applications are straightforward due to the fact that the functions involved are linear (and differentiable). As is usual in this type of problems if the size of the network is not too big we might use instead linear programming techniques directly to solve Problem 4, thus obtaining a solution without resorting to difficult to implement algorithms.

5   Risk-Averse Network Interdiction

The two-stage shortest path interdiction model that we presented has the disadvantage of requiring the knowledge of all the probabilities involved in the system. In particular we should know the probability of the evader coming from a particular source and the probability of the evader heading to a particular source. In practice these probabilities might be hard to obtain and too error prone. To alleviate this we propose the replacement of the expectation operators by coherent risk measures in Problem 10.

The coherent risk measures where introduced in the late 90's as an attempt to create a unified framework for the analysis and implementation of measures of risk and on its development it has effectively established a solid theoretical foundation of risk on which optimization models can be based on. By using this theory we can pick a particular risk measure and optimize for it, effectively minimizing the risk assessed by the selected measure. In Problem 10 this leads to the assessment of risk involved in the miscalculation of the probabilities mentioned before.

Let us first recall some basic concepts of the theory of coherent risk measures. For more details see [3, 4, 2, 5].

Let $(\Omega, \mathcal{F}, P)$ be a probability space with sigma algebra $\mathcal{F}$ and probability measure P. Also, let $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, where $p \in [1, +\infty)$. Each element $Z := Z(\omega)$ of $\mathcal{Z}$ is viewed as an uncertain outcome on $(\Omega, \mathcal{F})$ and it is by definition a random variable with finite p-th order moment. For $Z, Z' \in \mathcal{Z}$ we denote by $Z \preceq Z'$ the pointwise partial order meaning $Z(\omega) \geq Z'(\omega)$ for a.e. $\omega \in \Omega$. We also assume that the smaller the realizations of Z, the better; for example Z may represent a random cost.

Let $\overline{\overline{\mathbb{R}}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. A *coherent risk measure* is a proper function $\rho : \mathcal{Z} \to \overline{\overline{\mathbb{R}}}$ satisfying the following axioms:

(A1) *Convexity*: $\rho\left(tZ + (1-t)Z'\right) \leq t\rho(Z) + (1-t)\rho(Z')$, for all $Z, Z' \in \mathcal{Z}$ and all $t \in [0, 1]$;

(A2) *Monotonicity*: If $Z, Z' \in \mathcal{Z}$ and $Z \preceq Z'$, then $\rho(Z) \leq \rho(Z')$;

(A3) *Translation Equivalence*: If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z + a) = \rho(Z) + a$;

(A4) *Positive Homogeneity*: If $t > 0$ and $Z \in \mathcal{Z}$, then $\rho(tZ) = t\rho(Z)$.

We are interested in a *two stage risk-averse problem* of the form

$$\underset{x \in X}{\text{minimize}} \ \rho_1\left(c^\top x + \rho_2\left[\mathcal{Q}(x, \xi)\right]\right), \tag{9}$$

where ρ_1, ρ_2 are coherent risk measures, $X \subseteq \mathbb{R}^n$ is compact and polyhedral, and $\mathcal{Q}(x, \xi)$ is the optimal value of the *second stage problem*

$$\underset{y \in \mathbb{R}^m}{\text{minimize}} \ q^\top y$$
$$\text{s.t. } Tx + Wy = h, \ y \geq 0. \tag{10}$$

Here $\xi := (q, h, T, W)$ is the data of the second stage problem. We view some or all elements of the vector ξ as random and the ρ_1 operator at the first stage problem (9) is taken with respect to the probability distribution of $c^\top x + \rho_2 \left[\mathcal{Q}(x, \xi) \right]$.

If for some x and ξ the second stage problem (10) is infeasible, then by definition $\mathcal{Q}(x, \xi) = +\infty$. It could also happen that the second stage problem is unbounded from below and hence $\mathcal{Q}(x, \xi) = -\infty$. This is somewhat pathological situation, meaning that for some value of the first stage decision vector and a realization of the random data, the value of the second stage problem can be improved indefinitely. Models exhibiting such properties should be avoided.

We assume that the distribution of ξ has finite support. That is, ξ has a finite number of realizations (called *scenarios*) $\xi_k = (q_k, h_k, T_k, W_k)$ with respective probabilities $p_k, k = 1, \ldots, N$. In this case we will let $\mathcal{Z} := \mathcal{L}_1(\Omega, \mathcal{F}, P)$ which we will just identify with the space $\mathbb{R}^N$.

In [1] we analyze the two-stage risk averse problem and give dual type cutting plane algorithms for its solution. The relaxed shortest path two-stage linear interdiction model can easily be adapted to fit this risk-averse form. Namely,

Problem 10. (Risk-Averse Relaxed Shortest Path Two-Stage Linear Interdiction Problem) *Let ρ_1, ρ_2 be coherent risk measures.*

$$\begin{aligned}
\underset{x}{\text{maximize}} \quad & - \rho_1 \left[\underset{d}{\rho_2} \left[\overline{\mathcal{Q}}(x, s, d) \right] \right] \\
\text{subject to} \quad & c_1^\top x \le b_1 \\
& 0 \le x_e \le 1, \ \forall e \in E
\end{aligned}$$

where $\overline{\mathcal{Q}}(x, s, d)$ is the solution to the relaxed second stage problem

$$\begin{aligned}
\underset{y}{\text{maximize}} \quad & \mathbb{E}\left[\chi(x + y, s, d) \right] \\
\text{subject to} \quad & c_1^\top x + c_2^\top y \le b \\
& x_e + y_e \le 1, \ \forall e \in E \\
& c_2^\top y \le b_2 \\
& 0 \le y_e \le 1, \ \forall e \in E.
\end{aligned} \tag{11}$$

The techniques and methods developed in [1] can be directly applied to Problem 10, thus allowing us to obtain risk-averse solution to the two-stage network interdiction problem.

6 Conclusions

We defined the network interdiction problem and trough a series of reformulations arrived at a meaningful two-stage stochastic version of the problem. In order to avoid integer programing complications we considered a relaxed version of the problem and gave a proper interpretation in the sense of the general interdiction ideas. Finally, by using coherent risk measure we were able to define a risk-averse network interdiction problem that addresses the fundamental problem of calculating errors in the probabilities of the stochastic model.

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