Rutcor Research Report

Chess-like games may have no uniform Nash equilibria even in mixed strategies

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Abstract. Recently, it was shown that Chess-like games may have no uniform (subgame perfect) Nash equilibria in pure positional strategies. Moreover, Nash equilibria may fail to exist already in two-person games in which all infinite plays are equivalent and ranked as the worst outcome by both players. In this paper, we extend this negative result further, providing examples that are uniform Nash equilibria free even in the (independently) mixed strategies. Given (independently) mixed strategies of all players, we consider two definitions of the corresponding mixed play and effective payoff: given by the Markov or a priori realization.

Keywords: pure, mixed, and independently mixed strategies; uniform (or subgame perfect) Nash equilibrium; Markov and a priori realizations.

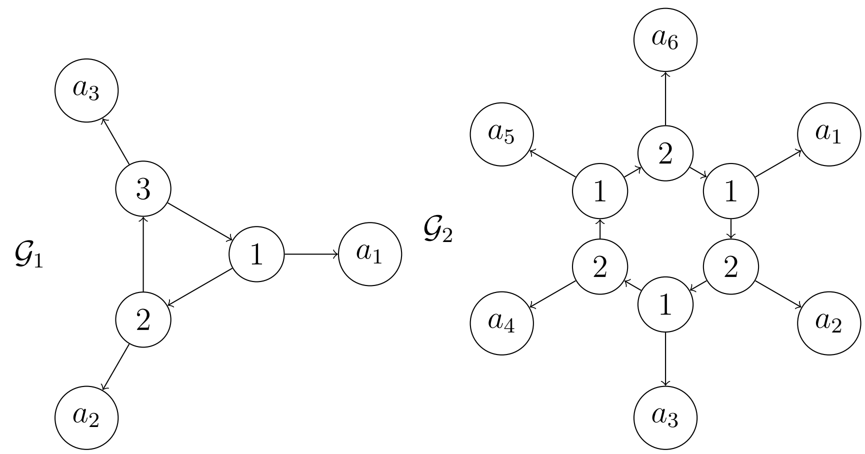


Figure 1: Two Chess-like game structures $\mathcal{G}_1$ and $\mathcal{G}_2$. In $\mathcal{G}_1$, there are 3 players controlling one position each (that is, $\mathcal{G}_1$ is play-once), while in $\mathcal{G}_2$ there are only two players who alternate turns; hence, each of them controls 3 positions: v_1, v_3, v_5 are controlled by player 1 and v_2, v_4, v_6 by player 2. In each position v_ℓ, the corresponding player has only two options: (p) to proceed to $v_{\ell+1}$ and (t) to terminate at a_ℓ, where $\ell \in \{1, 2, 3\}$ in $\mathcal{G}_1$ and $\ell \in \{1, \ldots, 6\}$ in $\mathcal{G}_2$; furthermore (where $v_4 = v_1$ in $\mathcal{G}_1$ and $v_7 = v_1$ in $\mathcal{G}_2$). To save space, we show only symbols a_ℓ, while the corresponding vertex names are omitted.

1 Introduction

1.1 Positional game structures

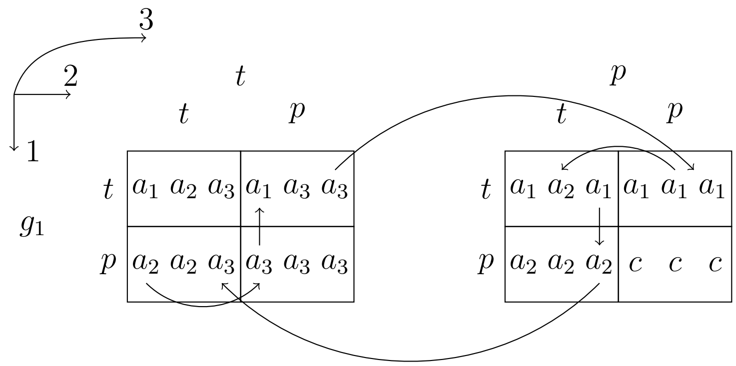
Given a finite directed graph (digraph) $G = (V, E)$ in which loops and multiple arcs are allowed, a vertex $v \in V$ is interpreted as a *position* and a directed edge (arc) $e = (v, v') \in E$ as a *move* from v to v'. A position of out-degree 0 (one with no moves) is called *terminal*. Let $V_T = \{a_1, \ldots, a_m\}$ be the set of all terminal positions. Let us also introduce a set of n players $I = \{1, \ldots, n\}$ and a partition $D : V = V_1 \cup \ldots \cup V_n \cup V_T$, assuming that each player $i \in I$ is in control of all positions of V_i.

An *initial* position $v_0 \in V$ may be fixed. The triplet (G, D, v_0) or pair $\mathcal{G} = (G, D)$ is called a *Chess-like positional game structure* (or just a *game structure*, for short), initialized or non-initialized, respectively. By default, we assume that it is not initialized.

Two examples of (non-initialized) game structures $\mathcal{G}_1$ and $\mathcal{G}_2$ are given in Figure 1.

1.2 Plays, outcomes, preferences, and payoffs

Given an initialized positional game structure (G, D, v_0), a *play* is defined as a directed path that begins in v_0 and either ends in a terminal position $a \in V_T$ or is infinite. In this article we assume that All Infinite Plays Form One Outcome a_∞ (or c), in addition to the standard Terminal outcomes of V_T. (In [4], this condition was referred to as AIPFOOT.)



$$u(1,a_2) > u(1,a_1) > u(1,a_3) > u(1,c)$$
$$u(2,a_3) > u(2,a_2) > u(2,a_1) > u(2,c)$$
$$u(3,a_1) > u(3,a_3) > u(3,a_2) > u(3,c)$$

Figure 2: The normal form g_1 of the positional game structures $\mathcal{G}_1$ from Figure 1.
Each player has only two strategies: to terminate (t) or proceed (p). Hence, g_1 is represented by a $2 \times 2 \times 2$ table, each entry of which contains 3 terminals corresponding to the 3 potential initial positions v_1, v_2, v_3 of $\mathcal{G}_1$. The rows and columns are the strategies of the players 1 and 2, while two strategies of the player 3 are the left and right 2×2 subtables.
The corresponding game (g_1, u) has no uniform NE whenever a utility function $u : I \times A \to \mathbb{R}$ satisfies the constraints U_1 specified in the figure.

A *utility (or payoff)* function is a mapping $u : I \times A \to \mathbb{R}$, whose value $u(i,a)$ is interpreted as a profit of the player $i \in I = \{1, \ldots, n\}$ in case of the outcome $a \in A = \{a_1, \ldots, a_m, a_\infty\}$.

A payoff is called *zero-sum* if $\sum_{i \in I} u(i,a) = 0$ for every $a \in A$. Two-person zero-sum games are important. For example, the standard Chess and Backgammon are two-person zero-sum games in which every infinite play is a draw, $u(1,a_\infty) = u(2,a_\infty)) = 0$. It is easy to realize that $u(i,a_\infty) = 0$ can be assumed for all players $i \in I$ without any loss of generality.

Another important class of payoffs is defined by the condition $u(i,a_\infty) < u(i,a)$ for all $i \in I$ and $a \in V_T$; in other words, the infinite outcome a_∞ is ranked as the worst one by all players. Several possible motivations for this assumption are discussed in [3] and [4].

A quadruple (G, D, v_0, u) and triplet (G, D, u) will be called a *Chess-like game*, initialized and non-initialized, respectively.

Remark 1 *From the other side, the Chess-like games can be viewed as the* transition-free deterministic stochastic games with perfect information*; see, for example, [8, 9, 10, 11].*

In these games, every non-terminal position is controlled by a player $i \in I$ and the local reward $r(i,e)$ is 0 for each player $i \in I$ and move e, unless $e = (v', v)$ is a terminal move, *that is, $v \in V_T$. Obviously, in the considered case all infinite plays are equivalent, since the effective payoff is 0 for every such play. Furthermore, obviously, a_∞ is the worst outcome for a player $i \in I$ if and only if $r(i,e) > 0$ for every terminal move e.*

g_2	*ttt*	*ttp*	*tpt*	*tpp*	*ptt*	*ptp*	*ppt*	*ppp*
ttt	$a_1a_2a_3a_4a_5a_6$	$a_1a_2a_3a_4a_5a_1$	$a_1a_2a_3a_5a_5a_6$	$a_1a_2a_3a_5a_5a_1$	$a_1a_3a_3a_4a_5a_6$ ○	$a_1a_3a_3a_4a_5a_1$ ■	$a_1a_3a_3a_5a_5a_6$	$a_1a_3a_3a_5a_5a_1$
ttp	$a_1a_2a_3a_4a_6a_6$	$a_1a_2a_3a_4a_1a_1$	$a_1a_2a_3a_6a_6a_6$	$a_1a_2a_3a_1a_1a_1$	$a_1a_3a_3a_4a_6a_6$ ■	$a_1a_3a_3a_4a_1a_1$	$a_1a_3a_3a_6a_6a_6$ ○	$a_1a_3a_3a_1a_1a_1$
tpt	$a_1a_2a_4a_4a_5a_6$ ○	$a_1a_2a_4a_4a_5a_1$	$a_1a_2a_5a_5a_5a_6$	$a_1a_2a_5a_5a_5a_1$	$a_1a_4a_4a_4a_5a_6$	$a_1a_4a_4a_4a_5a_1$	$a_1a_5a_5a_5a_5a_6$	$a_1a_5a_5a_5a_5a_1$
tpp	$a_1a_2a_4a_4a_6a_6$	$a_1a_2a_4a_4a_1a_1$	$a_1a_2a_6a_6a_6a_6$ ○	$a_1a_2a_1a_1a_1a_1$	$a_1a_4a_4a_4a_6a_6$	$a_1a_4a_4a_4a_1a_1$	$a_1a_6a_6a_6a_6a_6$	$a_1a_1a_1a_1a_1a_1$
ptt	$a_2a_2a_3a_4a_5a_6$	$a_2a_2a_3a_4a_5a_2$ ■	$a_2a_2a_3a_5a_5a_6$	$a_2a_2a_3a_5a_5a_2$	$a_3a_3a_3a_4a_5a_6$	$a_3a_3a_3a_4a_5a_3$ ○	$a_3a_3a_3a_5a_5a_6$	$a_3a_3a_3a_5a_5a_3$
ptp	$a_2a_2a_3a_4a_6a_6$ ■	$a_2a_2a_3a_4a_2a_2$	$a_2a_2a_3a_6a_6a_6$	$a_2a_2a_3a_2a_2a_2$	$a_3a_3a_3a_4a_6a_6$	$a_3a_3a_3a_4a_3a_3$	$a_3a_3a_3a_6a_6a_6$	$a_3a_3a_3a_3a_3a_3$ ○
ppt	$a_2a_2a_4a_4a_5a_6$	$a_2a_2a_4a_4a_5a_2$ ○	$a_2a_2a_5a_5a_5a_6$	$a_2a_2a_5a_5a_5a_2$ ■	$a_4a_4a_4a_4a_5a_6$	$a_4a_4a_4a_4a_5a_4$	$a_5a_5a_5a_5a_5a_6$	$a_5a_5a_5a_5a_5a_5$ ■
ppp	$a_2a_2a_4a_4a_6a_6$	$a_2a_2a_4a_4a_2a_2$	$a_2a_2a_6a_6a_6a_6$ ■	$a_2a_2a_2a_2a_2a_2$ ○	$a_4a_4a_4a_4a_6a_6$	$a_4a_4a_4a_4a_4a_4$	$a_6a_6a_6a_6a_6a_6$ ■	$c\ c\ c\ c\ c\ c$

$o_1 : a_6 > a_5 > a_2 > a_1 > a_3 > a_4 > c, \quad o_2 : a_3 > a_2 > a_6 > a_4 > a_5 > c; \quad a_6 > a_1 > c$

Figure 3: The normal form g_2 of the positional game structures $\mathcal{G}_2$ from Figure 1. There are two players controlling 3 positions each. Again, in every position there are only two options: to terminate (t) or proceed (p). Hence, in $\mathcal{G}_2$, each player has 8 strategies, which are naturally coded by the 3-letter words in the alphabet $\{t, p\}$. Respectively, g_2 is represented by the 8×8 table, each entry of which contains 6 terminals corresponding to the 6 (non-terminal) potential initial positions $v_1, \ldots, v_6$ of $\mathcal{G}_2$. Again, players 1 and 2 control the rows and columns, respectively. The corresponding game (g_1, u) has no uniform NE whenever a utility function $u : I \times A \to \mathbb{R}$ satisfies the constraints U_2 specified under the table. Indeed, a (unique) uniformly best response of the player 1 (respectively, 2) to each strategy of 2 (respectively, 1) are shown by the white discs (respectively, black squares). Since the obtained two sets are disjoint, no uniform NE exists in (g_1, u).

If $|I| = n = m = |A| = 2$ then the zero-sum Chess-like games turn into a subclass of the so-called simple stochastic games*, which were introduced by Condon in [7].*

1.3 Pure positional strategies

Given game structure $\mathcal{G} = (G, D)$, a *(pure positional) strategy* x_i of a player $i \in I$ is a mapping $x_i : V_i \to E_i$ that assigns to each position $v \in V_i$ a move (v, v') from this position.

The concept of mixed strategies will be considered in Section 1.9; till then only pure strategies are considered. Moreover, in this article, we restrict the players to their *positional* (pure) strategies. In other words, the move (v, v') of a player $i \in I$ in a position $v \in V_i$ depends only on the position v itself, not on the preceding positions or moves.

Let X_i be the set of all strategies of a player $i \in I$ and $X = \prod_{i \in I} X_i$ be the direct product of these sets. An element $x = \{x_1, \ldots, x_n\} \in X$ is called a *strategy profile or situation.*

1.4 Normal forms

A positional game structure can be represented in the *normal (or strategic) form.*

Let us begin with the initialized case. Given a game structure $\mathcal{G} = (G, D, v_0)$ and a strategy profile $x \in X$, a play $p(x)$ is uniquely defined by the following rules: it begins in v_0 and in each position $v \in V_i$ proceeds with the arc (v, v') determined by the strategy x_i. Obviously, $p(x)$ either ends in a terminal position $a \in V_T$, or $p(x)$ is infinite. In the latter case $p(x)$ is a *lasso*, that is, it consists of an initial part and a directed cycle (dicycle) repeated infinitely. This holds, because all players are restricted to their positional strategies. Either case, an outcome $a = g(x) \in A = \{a_1, \ldots, a_m, a_\infty\}$ is assigned to each strategy profile $x \in X$. Thus, a game form $g_{v_0} : X \to A$ is defined. It is called the *normal form* of the initialized positional game structure $\mathcal{G}$.

If the game structure $\mathcal{G} = (G, D)$ is not initialized then we repeat the above construction for every initial position $v_0 \in V \setminus V_T$ to obtain a play $p = p(x, v_0)$, outcome $a = g(x, v_0)$, and mapping $g : X \times (V \setminus V_T) \to A$, which is the *normal form* of $\mathcal{G}$ in this case. In general we have $g(x, v_0) = g_{v_0}(x)$. For the (non-initialized) game structures in Figure 1 their normal forms are given in Figures 2 and 3.

Given also a payoff $u : I \times A \to \mathbb{R}$, the pairs (g_{v_0}, u) and (g, u) define the *games in the normal form*, for the above two cases.

Of course, these games can be also represented by the corresponding real-valued mappings

$$f_{v_0} : I \times X \to \mathbb{R} \quad \text{and} \quad f : I \times X \times (V \setminus V_T) \to \mathbb{R}$$

where $f_{v_0}(i, x) = f(i, x, v_0) = u(i, g_{v_0}(x)) = u(i, g(x, v_0)) \;\; \forall\, i \in I, x \in X, v_0 \in V \setminus V_T$.

Remark 2 *Yet, it seems convenient to separate the game form g and utility function u. By this approach, g "takes responsibility for structural properties" of the game (g, u), that is, the properties that hold for any u.*

1.5 Nash equilibria in pure strategies

The concept of Nash equilibria is defined standardly [14, 15] for the normal form games.

First, let us consider the initialized case. Given $g_{v_0} : X \to A$ and $u : I \times A \to \mathbb{R}$, a situation $x \in X$ is called a *Nash equilibrium* (NE) in the normal form game (g_{v_0}, u) if $f_{v_0}(i, x) \geq f_{v_0}(i, x')$ for each player $i \in I$ and every strategy profile $x' \in X$ that can differ from x only in the ith component. In other words, no player $i \in I$ can profit by choosing a new strategy if all opponents keep their old strategies.

In the non-initialized case, the similar property is required for each $v_0 \in V \setminus V_T$. Given $f : I \times X \times (V \setminus V_T) \to \mathbb{R}$, a strategy profile $x \in X$ is called a *uniform NE* if $f(i, x, v_0) \geq f(i, x', v_0)$ for each $i \in I$, every x' defined as above, and for all $v_0 \in V \setminus V_T$, too.

Remark 3 *In the literature, the last concept is frequently called "a* subgame perfect *NE". This name is justified when the digraph $G = (V, E)$ is acyclic and each vertex $v \in V$ can be reached from v_0. Indeed, in this case (G, D, v, u) is a subgame of (G, D, v_0, u) for each $v \in V$.*

However, if G has a dicycle then any two its vertices v' and v'' can be reached one from the other; in other words, (G, D, v', u) is a subgame of (G, D, v'', u) and vice versa. Thus, the name "uniform (or ergodic) *NE" is more accurate.*

1.6 Uniformly best responses

Again, let us start with the initialized case. Given the normal form $f_{v_0} : I \times X \to \mathbb{R}$ of an initialized Chess-like game, a player $i \in I$, and a pair of strategy profiles x, x' such that x' may differ from x only in the ith component, we say that x' *improves* x (for the player i) if $f_{v_0}(i, x) < f_{v_0}(i, x')$. Let us underline that the inequality is strict. Furthermore, by this definition, a situation $x \in X$ is a NE if and only if it can be improved by no player $i \in I$; in other words, any sequence of improvements either can be extended, or terminates in a NE.

Given a player $i \in I$ and situation $x = (x_i | i \in I)$, a strategy $x_i^* \in X_i$ is called a *best response (BR)* of i in x, if $f_{v_0}(i, x^*) \geq f_{v_0}(i, x')$ for any x', where x^* and x' are both obtained from x by replacement of its ith component x_i by x_i^* and x_i', respectively. A BR x_i^* is not necessarily unique but the corresponding best achievable value $f_{v_0}(i, x^*)$ is, of course, unique. Moreover, somewhat surprisingly, such best values can be achieved by a BR x_i^* simultaneously for all initial positions $v_0 \in V \setminus V_T$ (see e.g. [2] - [5]).

Theorem 1 *Let $f : I \times X \times (V \setminus V_T) \to \mathbb{R}$ be the normal form of a (non-initialized) Chess-like game (G, D, u). Given a player $i \in I$ and a situation $x \in X$, there is a (pure positional) strategy $x_i^* \in X_i$ which is a BR of i in x for* all *initial positions $v_0 \in V \setminus V_T$ simultaneously.*

We will call such a strategy x_i^* a *uniformly* BR of the player i in the situation x. Obviously, the non-strict inequality $f_v(i, x) \leq f_v(i, x^*)$ holds for each position $v \in V$. We will say that x_i^* improves x if this inequality is strict, $f_{v_0}(i, x) < f_{v_0}(i, x^*)$, for at least one $v_0 \in V$. This statement will serve as the definition of a uniform improvement for the non-initialized case. Let us remark that, by this definition, a situation $x \in X$ is a uniform NE if and only if x can be uniformly improved by no player $i \in I$; in other words, any sequence of uniform improvements either can be extended or terminates in a uniform NE.

For completeness, let us repeat here the simple proof of Theorem 1 suggested in [2].

Given a non-initialized Chess-like game $\mathcal{G} = (G, D, u)$, a player $i \in I$, and a strategy profile $x \in X$, in every position $v \in V \setminus (V_i \cup V_T)$ let us fix a move (v, v') in accordance with x and delete all other moves. Then, let us order A according to the preference $u_i = u(i, *)$. Let $a^1 \in A$ be a best outcome. (Note that there might be several such outcomes and also that $a^1 = c$ might hold.) Let V^1 denote the set of positions from which player i can reach a^1 (in particular, $a^1 \in V^1$). Let us fix corresponding moves in $V^1 \cap V_i$. Obviously, there is no move to a^1 from $V \setminus V^1$. Moreover, if $a^1 = c$ then player i cannot reach a dicycle beginning from $V \setminus V^1$; in particular, the induced digraph $G_1 = G[V \setminus V^1]$ contains no dicycle.

Then, let us consider an outcome a^2 that is the best for i in A, except maybe a^1, and repeat the same arguments as above for G_1 and a^2, etc. This procedure will result in a uniformly BR x_i^* of i in x, since the chosen moves of i are optimal independently of v_0. $\square$

1.7 Two open problems related to Nash-solvability of initialized Chess-like game structures

Given an initialized game structure $\mathcal{G} = (G, D, v_0)$, it is an open question whether a NE (in pure positional strategies) exists for every utility function u. In [3], the problem was raised and solved in the affirmative for two special cases: $|I| \leq 2$ or $|A| \leq 3$. The last result was strengthened to $|A| \leq 4$ in [6]. More details can be found in [2] and in the last section of [5].

In general the above problem is still open even if we assume that c is the worst outcome for all players.

Yet, if we additionally assume that $\mathcal{G}$ is play-once (that is, $|V_i| = 1$ for each $i \in I$) then the answer is positive [3]. However, in the next subsection we will show that it becomes negative if we ask for the existence of a *uniform* NE rather than an initialized one.

1.8 Chess-like games with a unique dicycle and without uniform Nash equilibria in pure positional strategies

Let us consider two non-initialized Chess-like positional game structures $\mathcal{G}_1$ and $\mathcal{G}_2$ given in Figure 1. For $j = 1, 2$, the corresponding digraph $G_j = (V_j, E_j)$ consists of a unique dicycle C_j of length $3j$ and a matching connecting each vertex v_ℓ^j of C_j to a terminal a_ℓ^j, where $\ell = 1, \ldots, 3j$ and $j = 1, 2$. The digraph G_2 is bipartite; respectively, $\mathcal{G}_2$ is a two-person game structures in which two players take turns; in other words, players 1 and 2 control positions v_1, v_3, v_5 and v_2, v_4, v_6, respectively. In contrast, $\mathcal{G}_1$ is a play-once three-person game structure, that is, each player controls a unique position. In every non-terminal position v_ℓ^j there are only two moves: one of them (t) immediately terminates in a_ℓ^j, while the other one (p) proceeds to $v_{\ell+1}^j$; by convention, we assume $3j + 1 = 1$.

Remark 4 *In Figure 1, the symbols a_ℓ^j for the terminal positions are shown but v_ℓ^j for the corresponding positions of the dicycle are skipped; moreover, in Figures 1-3, we omit the superscript j in a_ℓ^j, for simplicity and to save space.*

Thus, in $\mathcal{G}_1$ each player has two strategies coded by the letters t and p, while in $\mathcal{G}_2$ each player has 8 strategies coded by the 3-letter words in the alphabet $\{t, p\}$. For example, the strategy (tpt) of player 2 in $\mathcal{G}_2$ requires to proceed to v_5^2 from v_4^2 and to terminate in a_2^2 from v_2^2 and in a_6^2 from v_6^2.

The corresponding normal game forms g_1 and g_2 of size $2 \times 2 \times 2$ and 8×8 are shown in Figures 2 and 3, respectively. Since both game structures are non-initialized, each situation is a set of 2 and 6 terminals, respectively. These terminals correspond to the non-terminal positions of $\mathcal{G}_1$ and $\mathcal{G}_2$, each of which can serve as an initial position.

A uniform NE free example for $\mathcal{G}_1$ was suggested in [3]; see also [8] and [2].

Let us consider a family U_1 of the utility functions defined by the following constraints

$$u(1, a_2) > u(1, a_1) > u(1, a_3) > u(1, c),$$
$$u(2, a_3) > u(2, a_2) > u(2, a_1) > u(2, c),$$
$$u(3, a_1) > u(3, a_3) > u(3, a_2) > u(3, c).$$

In other words, for each player $i \in I = \{1, 2, 3\}$ to terminate is an average outcome; it is better (worse) when the next (previous) player terminates; finally, if nobody does then the dicycle c appears, which is the worst outcome for all. The considered game has an improvement cycle of length 6, which is shown in Figure 2. Indeed, let player 1 terminates at a_1, while 2 and 3 proceed. The corresponding situation (a_1, a_1, a_1) can be improved by 2 to (a_1, a_2, a_1), which in its turn can be improved by 1 to (a_2, a_2, a_2). Repeating the similar procedure two times more we obtain the improvement cycle shown in Figure 2.

There are two more situations, which result in (a_1, a_2, a_3) and (c, c, c). They appear when all three players terminate or proceed simultaneously. Yet, none of these two situations is a NE either. Moreover, each of them can be improved by every player $i \in I = \{1, 2, 3\}$.

Thus, the following negative result holds.

Theorem 2 *([2]). Game $(\mathcal{G}_1, u)$ has no uniform NE in pure strategies whenever $u \in U_1$.*

We note that each player has positive payoffs. This is without loss of generality as we can shift the payoffs by a positive constant without changing the game.

A similar two-person uniform NE free example was suggested in [2], for $\mathcal{G}_2$.

Let us consider a family U_2 of the utility functions defined by the following constraints

$$u(1, a_6) > u(1, a_5) > u(1, a_2) > u(1, a_1) > u(1, a_3) > u(1, a_4) > u(1, c),$$
$$u(2, a_3) > u(2, a_2) > u(2, a_6) > u(2, a_4) > u(2, a_5) > u(2, c), \quad u(2, a_6) > u(2, a_1) > u(2, c).$$

We claim that the Chess-like game $(\mathcal{G}_2, u)$ has no uniform NE whenever $u \in U_2$.

Let us remark that $|U_2| = 3$ and that c is the worst outcome for both players for all $u \in U_2$. To verify this, let us consider the normal form g_2 in Figure 3. By Theorem 1, there is a uniformly BR of player 2 to each strategy of player 1 and vice versa. It is not difficult to check that the obtained two sets of the BRs (which are denoted by the white discs and black squares in Figure 3) are disjoint. Hence, there is no uniform NE. Furthermore, it is not difficult to verify that the obtained 16 situations induce an improvement cycle of length 10 and two improvement paths of lengths 2 and 4 that end in this cycle.

Theorem 3 *([2]). Game $(\mathcal{G}_2, u)$ has no uniform NE in pure strategies whenever $u \in U_2$.*

The goal of the present article is to demonstrate that the above two game structures may have no uniform NE not only in pure but even in mixed strategies. Let us note that by Nash's theorem [14, 15] NE in mixed strategies exist in any initialized game structure. Yet, this result cannot be extended to the non-initialized game structure and uniform NE. In this research we are motivated by the results of [8, 11].

1.9 Mixed and independently mixed strategies

Standardly, a *mixed strategy* $y_i \in Y_i$ of a player $i \in I$ is defined as a probabilistic distribution over the set X_i of his pure (*positional*) strategies. Furthermore, y_i is called an *independently mixed* if i randomizes independently in the positions $v \in V_i$.

Let us recall that a game structure is called play-once if each player is in control of a unique position. For example, $\mathcal{G}_1$ is play-once. Obviously, the classes of mixed and independently mixed strategies coincide for a play-once game structure.

However, for $\mathcal{G}_2$ these two notion differ. Each player $i \in I = \{1, 2\}$ controls 3 positions and has 8 pure strategies. Hence, the set of mixed strategies Y_i is of dimension 7, while the set $Z_i \subseteq Y_i$ of the independently mixed strategies is only 3-dimensional.

2 Markovian and a priori realizations

Let us begin with independently mixed strategies. Being fixed for all players $i \in I$ they uniquely define probability distributions P_v for all non-terminal positions $v \in V \setminus V_T$, assigning a probability $p(v, v')$ to each move (v, v') from v so that $0 \leq p(v, v') \leq 1$ and $\sum_{v' \in V} p(v, v') = 1$; standardly, $p(v, v') = 0$ whenever $(v, v') \notin E$.

Now, the limit distributions of the terminals $A = \{a_1, \ldots, a_m, a_\infty\}$ can be defined in two ways, which we will call the *Markovian* and *a priori* realizations.

The first approach is classical; the limit distribution can be found by solving a $m \times m$ system of linear equations; see, for example, [12] and also [13].

For example, let us consider $\mathcal{G}_1$ and let p_j be the probability to proceed in v_j for $j = 1, 2, 3$. If $p_1 = p_2 = p_3 = 1$ then, obviously, the play will cycle with probability 1 resulting in the limit distribution $(0, 0, 0, 1)$ for (a_1, a_2, a_3, c).

Otherwise, assuming that v_1 is the initial position, we obtain the limit distribution:

$$\left(\frac{1 - p_1}{1 - p_1 p_2 p_3}, \frac{p_1(1 - p_2)}{1 - p_1 p_2 p_3}, \frac{p_1 p_2 (1 - p_3)}{1 - p_1 p_2 p_3}, 0 \right).$$

Indeed, positions v_1, v_2, v_3 are transient and the probability of cycling forever is 0 whenever $p_1 p_2 p_3 < 1$. Obviously, the sum of the above four probabilities is 1.

The Markovian approach assumes that for $t = 0, 1, \ldots$ the move $e(t) = (v(t), v(t+1))$ is chosen randomly, in accordance with the distribution $P_{v(t)}$, and *independently* for all t (furthermore, $v(0) = v_0$ is a fixed initial position). In particular, if the play comes to the same position again, that is, $v = v(t) = v(t')$ for some $t < t'$, then the moves $e(t)$ and $e(t')$ may be distinct, although they are chosen (independently) with the same distribution P_v.

The concept of *a priori realization* is based on the following alternative assumptions. A move (v, v') is chosen according to P_v independently for all $v \in V \setminus V_T$ *but only once*, before the game starts. Being chosen the move (v, v') is applied whenever the play comes at v. By these assumptions, each infinite play ℓ is a lasso, that is, it consists of an initial part (that might be empty) and an infinitely repeated dicycle c_ℓ. Alternatively, ℓ may be finite, that is, it terminates in a V_T. In both cases, ℓ begins in v_0 and the probability of ℓ is the product of the probabilities of all its moves, $P_\ell = \prod_{e \in \ell} p(e)$.

Let us also note that (in contrast to the Markovian case) the computation of limit distribution is not computationally efficient, since the set of plays may grow exponentially in the size of the digraph. No polynomial algorithm computing the limit distribution is known for a priori realizations.

Returning to our example $\mathcal{G}_1$, we obtain the following limit distribution:

$$(1-p_1, p_1(1-p_2), p_1p_2(1-p_3), p_1p_2p_3) \text{ for the outcomes } (a_1, a_2, a_3, c),$$

with initial position v_1. The probability of c is $p_1p_2p_3$; it is strictly positive whenever $p_i > 0$ for all $i \in I$. Indeed, in contrast to the Markovian realization, the cycle will be repeated infinitely whenever it appears once in a priori realization.

Let us proceed with the (not necessarily independently) mixed strategies. In this case, each player $i \in I$ chooses a probability distribution $y_i \in Y_i$ over all (pure positional) strategies X_i. Still, a probability distribution P_v is well defined for every position $v \in V_i$: the probability of a move (v, v') equals the sum of probabilities of all strategies $x_i \in X_i$ that choose this move. Then, the Markovian and a priori realizations are defined as above.

Let us remark that in this case the latter look more natural than the former. Indeed, since a mixed strategy is not related to a position, a player can choose a realization once, in the beginning, for the whole play, rather than trying a new realization in every new position.

Remark 5 *Thus, solving the Chess-like games in mixed strategies looks more natural under a priori (rather than Markovian) realizations. Unfortunately, it is not easy to suggest more applications of a priori realizations and we have to acknowledge that the concept of the Markovian realization is much more fruitful.*

Let us also note that playing in pure strategies can be viewed as a special case of both Markovian and a priori realizations with degenerate probability distributions.

As we already mentioned, the mixed and independently mixed strategies coincide for $\mathcal{G}_1$, since it is play-once. Yet, these two classes of strategies differ in $\mathcal{G}_2$. They will be compared in Section 3.2 for the Markovian and a priori realizations.

3 Chess-like games with no uniform NE

In the present paper, we will strengthen Theorems 2 and 3, showing that games $(\mathcal{G}_1, u)$ and $(\mathcal{G}_2, u)$ may have no NE in the mixed strategies either. Let $J = \{1, \ldots, m\}$ be the set of indices of non-terminal positions where m is the number of non-terminal positions. Let us recall the definition of payoff function $f_{v_0}(i, x)$ of player i from initial position v_0 with x strategy profile which was given by Theorem 1. We extend this definition to $F_{v_0}(i, p)$ for (independently) mixed strategies, where p is an m-vector such that p_j denotes the probability of proceeding at position $j \in J$.

Remark 6 *Let us observe that in both $\mathcal{G}_1$ and $\mathcal{G}_2$, the payoff functions $F_{v_0}(i,p)$ are continuously derivable functions of p_i, for $0 < p_j < 1$ for all $j \in J$, for all players $i \in I$ and all initial positions $v_0 \in V$. Hence, if p is a uniform NE (with respect to either a priori or Markovian realization) then*

$$\frac{\partial F_{v_0}(i,p)}{\partial p_i} = 0, \quad \forall \ i \in I, j \in J, v_0 \in V. \tag{1}$$

In the next two sections, we construct games that have no uniform NE under both a priori and Markovian realizations. We will show that if $0 < p_j < 1$ for all positions $j \in V$ whenever p is a uniform mixed NE. Then, using (1), we will derive a contradiction.

3.1 $(\mathcal{G}_1, u)$ examples

The next Lemma will be instrumental in the proofs of the following two theorems.

Lemma 1 *The proceeding probabilities satisfy $0 < p_j < 1$ for all $j \in J = \{1, 2, 3\}$ in any (independently) mixed uniform NE for game-form $(\mathcal{G}_1, u)$, where $u \in U_1$, and under both a priori and Markovian realizations.*

Proof Let us assume indirectly that there is an (independently) mixed uniform NE under a priori realization with $p_j = 0$ for some $j \in J$. This would imply the existence of an acyclic game with uniform NE, in contradiction with Theorem 2. Now let us consider the case $p_j = 1$. Due to the circular symmetry of $(\mathcal{G}_1, u)$, we can choose any player, say, $j = 1$. The preference list of player 3 is $u(3, a_1) > u(3, a_3) > u(3, a_2) > u(3, c)$. His most favorable outcome, a_1, is not achievable, since $p_1 = 1$. Hence, $p_3 = 0$, because his second best outcome is a_3. Thus, the game is reduced to an acyclic one, in contradiction with Theorem 2, again.

Theorem 4 *Game $(\mathcal{G}_1, u)$ has no uniform NE in (independently) mixed strategies and under a priori realization, whenever $u \in U_1$.*

Proof Assume indirectly that (p_1, p_2, p_3) form a uniform NE and consider the effective payoff of player 1:

$$F_j(1,p) = (1-p_j)u(1,a_j) + p_j(1-p_{j_+})u(1,a_{j_+}) + p_j p_{j_+}(1-p_{j_-})u(1,a_{j_-}) + p_j p_{j_+} p_{j_-} u(1,c),$$

where j is the initial position, while j_+ and j_- denote the positions succeeding and preceding j, respectively.

By Lemma 1, we have $0 < p_j < 1$ for $j \in J = \{1, 2, 3\}$. Therefore equations (1) must hold. Hence, $\frac{\partial F_j(1,p)}{\partial p_{j_-}} = p_j p_{j_+}(u(1,c) - u(1,a_{j_-})) = 0$ and $p_j p_{j_+} = 0$ follows, since $u(1, a_{j_-}) > u(1, c)$. Thus, $p_1 p_2 p_3 = 0$, in contradiction with our assumption. □

Remark 7 *Let us underline here that Nash's results [14, 15] guarantee the existence of a NE does not extend to the case of uniform NE; in contrast to the zero sum case.*

Now, let us consider the Markovian realization. Game $(\mathcal{G}_1, u)$ may have no NE in mixed strategies under Markov's realization either, yet, only for some special payoffs $u \in U_1$.

Theorem 5 *Given a game $(\mathcal{G}_1, u)$ with $u \in U_1$, let us define*

$$\mu_1 = \frac{u(1,a_2) - u(1,a_1)}{u(1,a_1) - u(1,a_3)},\ \mu_2 = \frac{u(2,a_3) - u(2,a_2)}{u(2,a_2) - u(2,a_1)},\ \mu_3 = \frac{u(3,a_1) - u(3,a_3)}{u(3,a_3) - u(3,a_2)}.$$

Then, $(\mathcal{G}_1, u)$ has no uniform NE in (independently) mixed strategies and under Markovian realization if and only if and $\mu_1\mu_2\mu_3 \geq 1$.

It is easy to verify that $\mu_i > 0$ for $i = 1, 2, 3$ whenever $u \in U_1$. Let us also note that in the symmetric case $\mu_1 = \mu_2 = \mu_3 = \mu$ the above condition $\mu_1\mu_2\mu_3 \geq 1$ turns into $\mu \geq 1$.

Proof Let $p = (p_1, p_2, p_3)$ be a uniform NE in the game $(\mathcal{G}_1, u)$ under Markovian realization. Then, by Lemma 1, $0 < p_i < 1$ for $i \in I = \{1, 2, 3\}$. The payoff of a player, with respect to the initial position that this player controls, is given by one of the next three formulas:

$$F_1(1,p) = \frac{(1-p_1)u(1,a_1) + p_1(1-p_2)u(1,a_2) + p_1p_2(1-p_3)u(1,a_3)}{1 - p_1p_2p_3},$$
$$F_2(2,p) = \frac{(1-p_2)u(2,a_2) + p_2(1-p_3)u(2,a_3) + p_2p_3(1-p_1)u(2,a_1)}{1 - p_1p_2p_3},$$
$$F_3(3,p) = \frac{(1-p_3)u(3,a_3) + p_3(1-p_1)u(3,a_1) + p_3p_1(1-p_2)u(3,a_2)}{1 - p_1p_2p_3}.$$

By Lemma 1, equations (1) hold at any uniform NE. Therefore we have

$$(1 - p_1p_2p_3)\frac{\partial F_1(1,p)}{\partial p_1} = p_2(1-p_3)u(1,a_3) + (p_2p_3 - 1)u(1,a_1) + (1-p_2)u(1,a_2) = 0,$$
$$(1 - p_1p_2p_3)\frac{\partial F_2(2,p)}{\partial p_2} = p_3(1-p_1)u(2,a_1) + (p_1p_3 - 1)u(2,a_2) + (1-p_3)u(2,a_3) = 0,$$
$$(1 - p_1p_2p_3)\frac{\partial F_3(3,p)}{\partial p_3} = p_1(1-p_2)u(3,a_2) + (p_1p_2 - 1)u(3,a_3) + (1-p_1)u(3,a_1) = 0.$$

Then, setting $\lambda_i = \mu_i + 1$ for $i = 1, 2, 3$, we transform the above equations as follows:

$$\lambda_1(1-p_2) = 1 - p_2p_3,$$
$$\lambda_2(1-p_3) = 1 - p_1p_3,$$
$$\lambda_3(1-p_1) = 1 - p_1p_2.$$

Under the constraints $0 < p_j < 1$, $j \in J$, we can express p via λ in a unique way getting:

$$0 < p_1 = \frac{\lambda_2 + \lambda_3 - \lambda_1\lambda_2 - \lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_3 - 1}{\lambda_1\lambda_3 - \lambda_1 + 1} < 1,$$
$$0 < p_2 = \frac{\lambda_1 + \lambda_3 - \lambda_1\lambda_3 - \lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_3 - 1}{\lambda_1\lambda_2 - \lambda_2 + 1} < 1,$$
$$0 < p_3 = \frac{\lambda_1 + \lambda_2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 + \lambda_1\lambda_2\lambda_3 - 1}{\lambda_2\lambda_3 - \lambda_3 + 1} < 1.$$

Interestingly, all three $p_j < 1$ inequalities are equivalent, by Lemma 1, with the condition $(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1) < 1$, that is, $\mu_1\mu_2\mu_3 < 1$, which completes the proof. □

3.2 $(\mathcal{G}_2, u)$ examples

Here we will show that $(\mathcal{G}_2, u)$ may have no uniform NE for both Markovian and a priori realizations, in the independently mixed strategies, whenever $u \in U_2$. As for the mixed (unlike the independently mixed) strategies, we obtain NE-free examples only for some (not for all) $u \in U_2$ and a priori realization, leaving open (a) whether such examples exist for all $u \in U_2$ and (b) case of the Markov realization.

Remark 8 *Let us note, however, that in case of the mixed (unlike the independently mixed) strategies the case of a priory realization looks more natural than the Markovian one. Indeed, it would be difficult to motivate why the players should choose a new realization of their mixed strategies in every new position and, in particular, when a position is repeated.*

We begin with extending Lemma 1 to game $(\mathcal{G}_2, u)$ and $u \in U_2$ as follows.

Lemma 2 *The proceeding probabilities satisfy $0 < p_j < 1$ for all $j \in J = \{1, 2, 3, 4, 5, 6\}$ in any (independently) mixed uniform NE for game-form $(\mathcal{G}_2, u)$, where $u \in U_2$, and under both a priori and Markovian realizations.*

Proof To prove that $p_j < 1$ for all $j \in J$ let us consider the following six cases:
(i) If $p_1 = 1$, then player 2 will proceed at position 6, as $a_2 > a_6$ in U_2, implying $p_6 = 1$.
(ii) If $p_2 = 1$, then either $p_1 = 0$ or $p_3 = 1$, as player 1 prefers a_1 to a_3.
(iii) If $p_3 = 1$, then $p_2 = 0$, as player 2 cannot achieve his best outcome of a_3, while a_2 is his second best.
(iv) If $p_4 = 1$, then $p_3 = 1$, as player 1's worst outcome is a_3 in the current situation.
(v) If $p_5 = 1$, then $p_4 = 1$, as player 2 prefers a_6 to a_4.
(vi) If $p_6 = 1$, then $p_5 = 0$, as player 1's best outcome is a_5 now.
It is easy to verify that, by the above implications, in all six cases at least one of the proceeding probabilities should be 0, in contradiction to Theorem 3. □

Let us show that the game $(\mathcal{G}_2, u)$ might have no NE in independently mixed strategies under both the Markov and a priori realizations. Let us consider the Markov one first.

Theorem 6 *Game $(\mathcal{G}_2, u)$ has no uniform NE in the independently mixed strategies under Markov's realization for all $u \in U_2$.*

Proof Let us consider the uniform NE conditions for player 2. Lemma 2 implies that (1) must be satisfied. Applying it to the partial derivatives with respect to p_4 and p_6 we obtain:

$$\begin{aligned}\frac{(1-p_1p_2p_3p_4p_5p_6)^2}{p_1p_2p_3p_4p_5}\frac{\partial F_1(2,p)}{\partial p_6} =&((1-p_1)u(2,a_1)+p_1(1-p_2)u(2,a_2)+p_1p_2(1-p_3)u(2,a_3)+\\ &p_1p_2p_3(1-p_4)u(2,a_4)+p_1p_2p_3p_4(1-p_5)u(2,a_5)-(1-p_1p_2p_3p_4p_5)u(2,a_6))=0,\\ \frac{(1-p_1p_2p_3p_4p_5p_6)^2}{p_1p_2p_3p_5p_6}\frac{\partial F_5(2,p)}{\partial p_4} =&(1-p_5)u(2,a_5)+p_5(1-p_6)u(2,a_6)+p_5p_6(1-p_1)u(2,a_1)+\\ &p_5p_6p_1(1-p_2)u(2,a_2)+p_5p_6p_1p_2(1-p_3)u(2,a_3)-(1-p_1p_2p_3p_5p_6)u(2,a_4)=0,\end{aligned}$$

Let us multiply the first equation by p_5p_6 and subtract it from the second one, yielding

$$(1-p_1p_2p_3p_4p_5p_6)\left[-u(2,a_4)+(1-p_5)u(2,a_5)+p_5u(2,a_6)\right]=0,$$

or equivalently, $u(2,a_4)-(1-p_5)u(2,a_5)-p_5u(2,a_6)=0$. From this equation, we find

$$p_5=\frac{u(2,a_4)-u(2,a_5)}{u(2,a_6)-u(2,a_5)}.$$

Furthermore, the condition $0<p_5<1$ implies that either $u(2,a_5)<u(2,a_4)<u(2,a_6)$ or $u(2,a_5)>u(2,a_4)>u(2,a_6)$. Both orders contradict U_2, which completes the proof. □

Remark 9 *In this case, unlike in Theorem 5, uniform NE exist for no $u \in U_2$.*

Now let us consider the case of a priori realization.

Theorem 7 *Game $(\mathcal{G}_2, u)$ has no uniform NE in independently mixed strategies under a priori realization for all $u \in U_2$.*

Proof Let us assume indirectly that $p=(p_1,p_2,p_3,p_4,p_5,p_6)$ form a uniform NE. Let us consider the effective payoff of the player 1 with respect to the initial position 2:

$$\begin{aligned}F_2(1,p)=&(1-p_2)u(1,a_2)+p_2(1-p_3)u(1,a_3)+p_2p_3(1-p_4)u(1,a_4)+p_2p_3p_4(1-p_5)u(1,a_5)\\ &+p_2p_3p_4p_5(1-p_6)u(1,a_6)+p_2p_3p_4p_5p_6(1-p_1)u(1,a_1).\end{aligned}$$

By Lemma 2, we have $0<p_j<1$ for $j\in J=\{1,2,3,4,5,6\}$. Hence, equations (1) must hold; in particular, $\frac{\partial F_2(1,p)}{\partial p_1}=0$. Furthermore, since $u\in U_2$ is positive, we obtain $p_2p_3p_4p_5p_6=0$, in contradiction with our assumption. □

The last result can be extended from the independently mixed to the mixed strategies. However, the corresponding example is constructed not for all but only for some $u \in U_2$.

Theorem 8 *Under a priori realization, the game $(\mathcal{G}_2, u)$ has no uniform NE in the mixed strategies, at least for some $u \in U_2$.*

Proof Let us recall that there are two players in $\mathcal{G}_2$ controlling three positions each and two possible moves in every position. Thus, each player has eight pure strategies. Standardly, the mixed strategies are defined as probability distributions on the set of the pure strategies, that is, $x, y \in \mathcal{S}_8$, where $z = (z_1, \ldots, z_8) \in \mathcal{S}_8$ if and only if $\sum_{i=1}^{8} z_i = 1$ and $z \geq 0$.

Furthermore, let us denote by $a_{kl}(v_0)$ the outcome of the game beginning in the initial position $v_0 \in V$ in case when player 1 chooses his pure strategy k and player 2 chooses her pure strategy l, where $k, l \in \{1, \ldots, 8\}$. Given a utility function $u : I \times A \to \mathbb{R}$, if a pair of mixed strategies $x, y \in \mathcal{S}_8$ form a uniform NE then

$$\sum_{k=1}^{8} x_k u(2, a_{kl}(v_0)) \begin{cases} = z_{v_0}, & \text{if } y_l > 0 \\ \leq z_{v_0}, & \text{otherwise} \end{cases}$$

must hold for some z_{v_0} value for all initial positions $v_0 \in V$. Indeed, otherwise player 2 would change the probability distribution y to get a better value. Let $S = \{i | y_i > 0\}$ denote the set of indices of all positive components of $y \in \mathcal{S}_8$. By (2) , there exists a subset $S \subseteq \{1, \ldots, n\}$ such that the next system is feasible:

$$\begin{array}{rll} \sum_{k=1}^{8} x_k u(2, a_{kl}(v_0)) & = z_{v_0}, & \forall l \in S \\ \sum_{k=1}^{8} x_k u(2, a_{kl}(v_0)) & \leq z_{v_0}, & \forall l \notin S \\ \sum_{k=1}^{8} x_k & = 1, & \\ x_k & \geq 0, & \forall k = 1, \ldots, 8 \\ z_{v_0} & \text{unrestricted}, & \forall v_0 \in V. \end{array} \tag{2}$$

Then, let us consider a utility function $u \in U_2$ with the following payoffs of player 2:

$$u(2, a_1) = 43, u(2, a_2) = 81, u(2, a_3) = 93, u(2, a_4) = 50, u(2, a_5) = 15, u(2, a_1) = 80, u(2, c) = 0.$$

We verified that (2) is infeasible for all subsets $S \subseteq \{1, \ldots, 8\}$ such that $|S| \geq 2$. Since for any $u \in U_2$ there is no pure strategy NE either, we obtain a contradiction. □

Remark 10 *It seems that the same holds for all $u \in U_2$. We tested (2) for many randomly chosen $u \in U_2$ and encountered infeasibility for all $S \subseteq \{1, \ldots, 8\}$ such that $|S| \geq 2$. Yet, we have no proof and it still remains open whether for* any $u \in U_2$ *there is no NE in mixed strategies under a priori realization.*

Remark 11 *Finally, let us note that for an arbitrary Chess-like game structure (not only for $\mathcal{G}_1$ and $\mathcal{G}_2$) in independently mixed strategies under both the Markov and a priori realizations for any $i \in I$ and $k, l \in J$, the ratio $\frac{\partial F_l(i,p)}{\partial p_i} / \frac{\partial F_k(i,p)}{\partial p_i} = P(i, k, l)$ is a positive constant.*

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